Communications in **M**athematical **A**nalysis
Conference (2012), pp. 1–8
Proceedings of "*Analysis, Operator Theory, and Mathematical Physics*"
held in Ixtapa, Mexico, January 23-27, 2012 www.math-res-pub.org/cma
ISSN 1938-9787

ON THE SPREADING RATE OF THE SOLITON PERTURBATION FOR RELATIVISTIC NONLINEAR WAVE EQUATIONS

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Abstract. We consider nonlinear relativistic wave equations in one space dimension and prove the spreading rate estimates for a general class of potentials. Such estimates play an important role in studying the asymptotic stability of solitons.
Keywords. Relativistic nonlinear wave equation; soliton; weighted norms.
AMS (MOS) subject classification: Primary 35Q51; Secondary 37K40.

1 Introduction

Since pioneering works of Buslaev and Perelman [1] the questions of the stability of solitons for nonlinear Schrödinger equations attracts a broad attention. However, an extension to relativistic wave equations remained an open question until 2011 when the problem was solved in [5, 6] for the kink solutions of relativistic Ginzburg-Landau equations. Next step would be the study of soliton solutions for the relativistic wave equations. Here we obtain some useful estimates

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similar to those used in [1, 2] and [5, 6]. We prove the estimates for a new class of relativistic nonlinear wave equations with potentials different from those in [5, 6].

We consider the nonlinear wave equations

$$\ddot{\psi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \quad x \in \mathbb{R}, \tag{1.1}$$

where $\psi(x,t)$ is a complex-valued solution. We identify a complex number $\psi = \psi_1 + i\psi_2$ with the real two-dimensional vector $\psi = (\psi_1, \psi_2) \in \mathbb{R}^2$ and assume that the $\mathbb{R}^2$- version $\mathbf{F}$ of the force F admits a real-valued potential,

$$\mathbf{F}(\psi) = -\nabla U(\psi), \quad \psi \in \mathbb{R}^2, \quad U \in C^\infty(\mathbb{R}^2). \tag{1.2}$$

Then (1.1) is formally a Hamilton system with Hamilton functional

$$\mathcal{H}(\psi,\pi) = \int \Big[\frac{|\pi(x)|^2}{2} + \frac{|\psi'(x)|^2}{2} + U(\psi(x))\Big]dx, \tag{1.3}$$

where π is the momentum canonically conjugate to ψ. The functional is conserved for sufficiently regular finite energy solutions.

We assume that $U(\psi) = u(|\psi|^2)$. Then by (1.2),

$$F(\psi) = a(|\psi|^2)\psi, \quad \psi \in \mathbb{C}, \quad a \in C^\infty(\mathbb{R}), \tag{1.4}$$

where $a(|\psi|^2)$ is real. Therefore, $F(e^{i\theta}\psi) = e^{i\theta}F(\psi)$, $\theta \in [0, 2\pi]$ and $F(0) = 0$. The symmetry implies that if $\psi(x,t)$ is a solution to (1.1) then $e^{i\theta}\psi(x,t)$ is also a solution. Hence the equation is $U(1)$-invariant in the sense of [3].

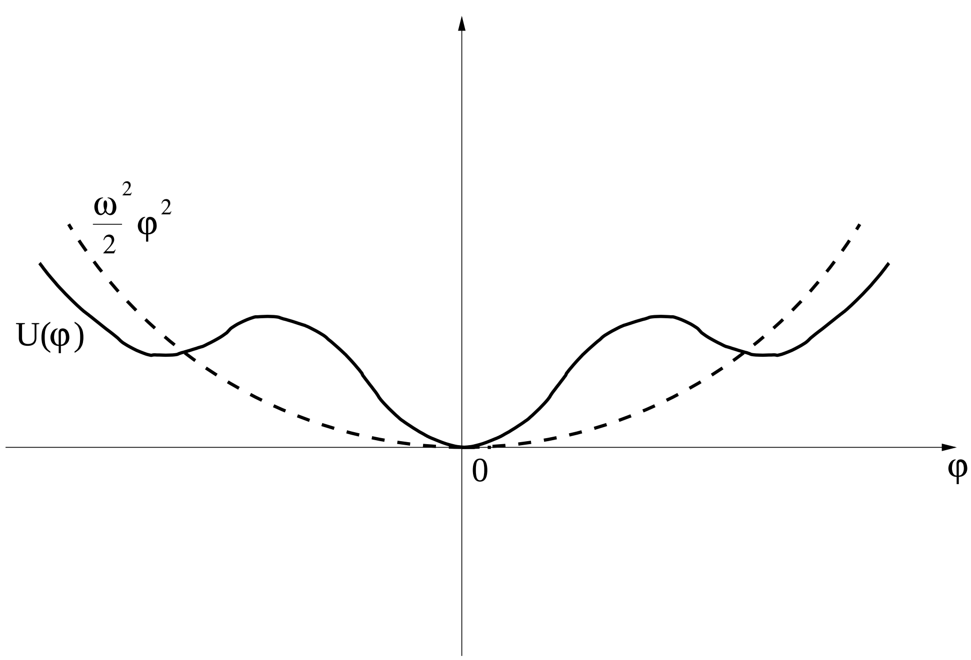

Figure 1. Potential U

We are interested in the solitary wave solutions of the form

$$\psi(x,t) = e^{i(\omega t+\theta)}\varphi_\omega(x), \quad \theta \in [0, 2\pi],$$

where φ_ω is the positive solution of the equation

$$\varphi''_\omega(x)+\omega^2\varphi_\omega(x)-U'(\varphi_\omega(x))=0. \tag{1.5}$$

We formulate our basic conditions on the potential $U(\psi)=u(|\psi|^2)$ (see Figure 1):

Condition U1. *$u(s)$ is a real smooth function and the following conditions hold with some $m>0$*

$$u(0)=0,\quad u'(0)=m^2/2. \tag{1.6}$$

Condition U2. *There exists a number $\alpha>0$ such that*

$$u(s)\ge\alpha s,\quad s\ge 0. \tag{1.7}$$

Further assumption is made in terms of

$$U_\omega(\varphi)=-\frac{\omega^2}{2}\varphi^2+U(\varphi),\quad \varphi\in\mathbb{R}. \tag{1.8}$$

Condition U3. *For some $|\omega_0|<m$ and for all ω from an interval $[\omega_0-\delta,\omega_0+\delta]$ with $\delta<m-|\omega_0|$, the mapping $\varphi\to U_\omega(\varphi)$ has a positive root and the smallest positive root φ_0 is simple, i.e. $U'_\omega(\varphi_0)\ne 0$.*

Under these assumptions there exists a unique, positive even solution $\varphi_\omega(x)$ of the equation

$$\varphi''_\omega-U'_\omega(\varphi_\omega)=0 \tag{1.9}$$

decreasing like $C(\omega)e^{-\sqrt{m^2-\omega^2}|x|}$ as $x\to\pm\infty$.

Let us illustrate the existence of the soliton solution graphically (see Figure 2).

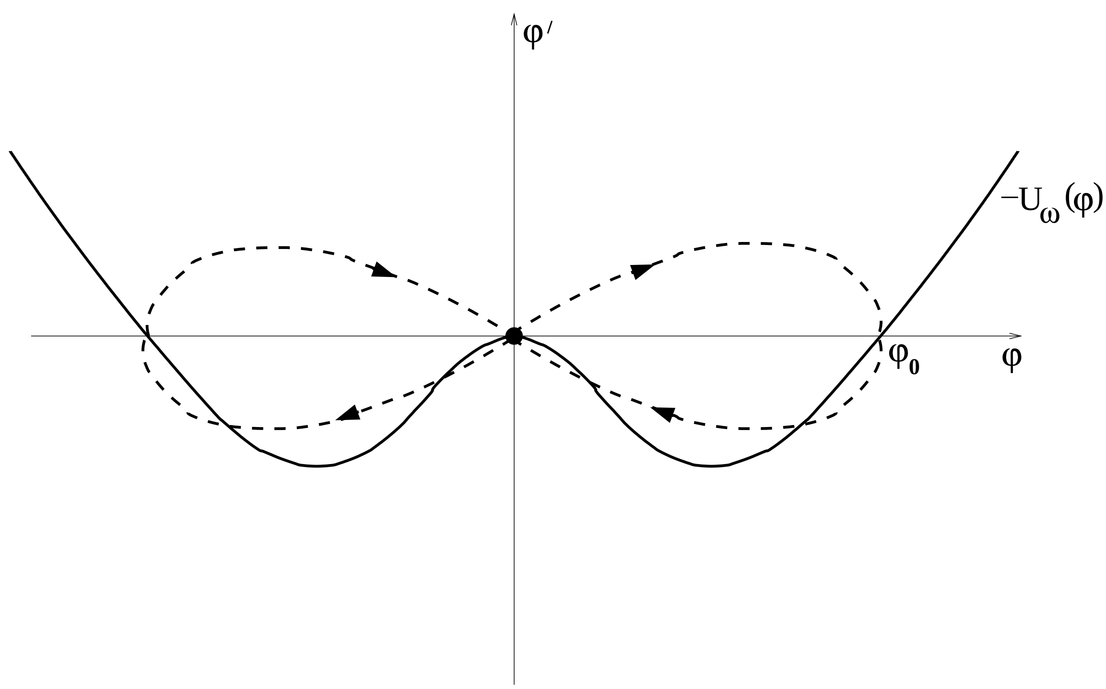

Figure 2. Phase diagram of the soliton

Multiplying (1.9) by $\varphi'_\omega(x)$ and taking a primitive of the result, we obtain

$$(\varphi'_\omega(x))^2/2 - U_\omega(\varphi_\omega(x)) = C, \tag{1.10}$$

where $C \in \mathbb{R}$ is a constant. It is easy to see that $C = 0$ gives the soliton solution under the conditions **U1 - U3**.

Soliton-like solutions are of special importance not only because they are simple and sometimes explicit solutions of evolution equations, but also because of the distinguished role they appear to play in the long time asymptotics of the solution of initial value problem. Numerical experiments [4] have shown that solutions, in general, eventually resolved themselves into an approximate superposition of weakly interacting solitary waves and decaying dispersive waves.

We split the solution to (1.1) as

$$(\psi(x,t), \pi(x,t)) = e^{i\gamma(t)}[\phi_{\omega(t)}(x) + (\Psi(x,t), \Pi(x,t))],$$

where

$$\phi_\omega(x) = (\varphi_\omega(x), i\omega\varphi_\omega(x)).$$

Our main result is the bound

$$\int (1+|x|)^{2\sigma}\Big[|\Psi(x,t)|^2 + |\Psi'(x,t)|^2 + |\Pi(x,t)|^2\Big]dx \le C(1+t)^{2\sigma+1}, \quad t > 0 \tag{1.11}$$

for any $0 < \sigma \le \sigma_0$, where the constants C and σ_0 depend on the initial conditions. This bound characterizes the rate of spreading of the initial wave packet. Similar bounds play an important role in the study of asymptotic stability of soliton solutions in [1, 2] and kink solutions in [5, 6].

2 Existence of dynamics

In the vector form, equation (1.1) reads

$$\begin{cases} \dot\psi(x,t) = \pi(x,t), \\ \dot\pi(x,t) = \psi''(x,t) + F(\psi(x,t)), \end{cases} \tag{2.1}$$

where $x, t \in \mathbb{R}$. We consider the Cauchy problem for the Hamilton system (2.1) which we write as

$$\dot Y(t) = \mathcal{F}(Y(t)), \quad t \in \mathbb{R}: \quad Y(0) = Y_0. \tag{2.2}$$

Here $Y(t) = (\psi(t), \pi(t))$, $Y_0 = (\psi_0, \pi_0)$, and all derivatives are understood in the sense of distributions. Now the soliton solutions become

$$Y_{\omega,\theta}(t) = e^{i(\omega t+\theta)}\phi_\omega(x), \ \ |\omega - \omega_0| \le \delta, \ \ \theta \in [0, 2\pi]. \tag{2.3}$$

The states $S_{\omega,\theta} := Y_{\omega,\theta}(0)$ form the solitary manifold

$$\mathcal{S} := \{S_{\omega,\theta} : \ \ |\omega - \omega_0| \le \delta, \ \ \theta \in [0, 2\pi]\}. \tag{2.4}$$

To formulate our results precisely, let us introduce a suitable phase space for the Cauchy problem (2.2). For $s, \sigma \in \mathbb{R}$, let us denote by $H^s_\sigma = H^s_\sigma(\mathbb{R})$ the weighted Sobolev spaces with the finite norms

$$\|\psi\|_{H^s_\sigma} = \|\langle x\rangle^\sigma \langle\nabla\rangle^s \psi\|_{L^2} < \infty, \qquad \langle x\rangle = (1+|x|^2)^{1/2}.$$

Denote $L^2_\alpha := H^0_\alpha$.

Definition 2.1. $E_\alpha := H^1_\alpha \oplus L^2_\alpha$ is the space of the states $Y = (\psi, \pi)$ with finite norm

$$\|Y\|_{E_\alpha} = \|\psi\|_{H^1_\alpha} + \|\pi\|_{L^2_\alpha} < \infty. \tag{2.5}$$

Denote $E = E_0$. Obviously, the Hamilton functional (1.3) is continuous on the phase space E. The existence and uniqueness of the solutions to the Cauchy problem (2.2) follows by methods [7, 8, 9]:

Proposition 2.2. *i) For any initial data $Y_0 \in E$ there exists the unique solution $Y(t) \in C(\mathbb{R}, E)$ to the problem (2.2).*
(ii) For every $t \in \mathbb{R}$, the map $U(t) : Y_0 \mapsto Y(t)$ is continuous in E.
(iii) The energy is conserved, i.e.

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y_0), \quad t \in \mathbb{R}. \tag{2.6}$$

3 Main results

Definition 3.1. A soliton state is $S(\omega, \gamma) := e^{i\gamma}(\varphi_\omega(x), i\omega\varphi_\omega(x))$, where $\gamma \in \mathbb{R}$ and $|\omega - \omega_0| \le \delta$.

Obviously, the soliton solution (2.3) admits the representation $S(\omega(t), \gamma(t))$, where

$$\gamma(t) = \omega t + \theta, \quad \omega(t) = \omega. \tag{3.1}$$

Let us consider a solution to (2.1) with initial data

$$Y_0 = S_{\omega_0, \theta_0} + X_0, \tag{3.2}$$

where $X_0 \in E_{\sigma_0}$ with some $\sigma_0 > 0$:

$$\|X_0\|_{E_{\sigma_0}} = d_0 < \infty. \tag{3.3}$$

Each solution $Y(t) \in C(\mathbb{R}, E)$ can be splitted as the sum

$$Y(t) = S(\omega(t), \gamma(t)) + X(t) \tag{3.4}$$

where $X(t) \in E$ and $\omega(t)$ and $\gamma(t)$ are arbitrary real smooth functions of t such that

$$|\omega(t) - \omega_0| \le \delta. \tag{3.5}$$

In detail, for $Y = (\psi, \pi)$ and $X = e^{i\gamma(t)}(\Psi, \Pi)$ representation (3.4) means that

$$\begin{cases} \psi(x,t) = e^{i\gamma(t)}[\varphi_{\omega(t)}(x) + \Psi(x,t)], \\ \pi(x,t) = e^{i\gamma(t)}[i\omega(t)\varphi_{\omega(t)}(x) + \Pi(x,t)]. \end{cases} \tag{3.6}$$

We now formulate the main result of our paper.

Theorem 3.2. *Let the potential U satisfy conditions* **U1 - U3**, *and let $Y(t) = (\psi(t), \pi(t))$ be the solution to* (2.2) *with initial data Y_0 satisfying* (3.2)–(3.3). *Let condition* (3.5) *holds for splitting* (3.6). *Then the following bounds hold (cf. (1.11))*

$$\|X(t)\|_E \le C(\delta), \quad t > 0, \tag{3.7}$$

$$\|X(t)\|_{E_\sigma} \le C(\delta)(1+t)^{\sigma+1/2}, \quad t > 0, \quad 0 < \sigma \le \sigma_0. \tag{3.8}$$

4 Energy propagation

We will deduce the theorem from the following two lemmas. Denote

$$e(x,t) = \frac{|\pi(x,t)|^2}{2} + \frac{|\psi'(x,t)|^2}{2} + U(\psi(x,t)).$$

Lemma 4.1. *For the solution $\psi(x,t)$ to equation (1.1) the local energy estimate holds*

$$\int_{a_1}^{a_2} e(x,t)\,dx \le \int_{a_1-t}^{a_2+t} e(x,0)\,dx, \quad a_1 < a_2, \quad t > 0. \tag{4.1}$$

Proof. We identify a complex number $\psi = \psi_1 + i\psi_2$ with the real two-dimensional vector $\psi = (\psi_1, \psi_2) \in \mathbb{R}^2$ and consider a vector version of equation (1.1). Denote $A = (a_1 - t, 0)$, $B = (a_1, t)$, $C = (a_2, t)$, $D = (a_2 + t, 0)$. Taking the scalar product of the vector version of (1.1) and vector $\dot\psi$ and integrating over the trapezium *ABCD* we get:

$$\int_{ABCD} \Big[\frac{d}{dt}\frac{|\dot\psi|^2}{2} - \psi'' \cdot \dot\psi + \frac{d}{dt}U(\psi)\Big]dxdt = 0.$$

Using the identity $-\psi'' \cdot \dot\psi = -(\psi' \cdot \dot\psi)' + \dfrac{d}{dt}\dfrac{|\psi'|^2}{2}$, and applying Green's theorem, we obtain

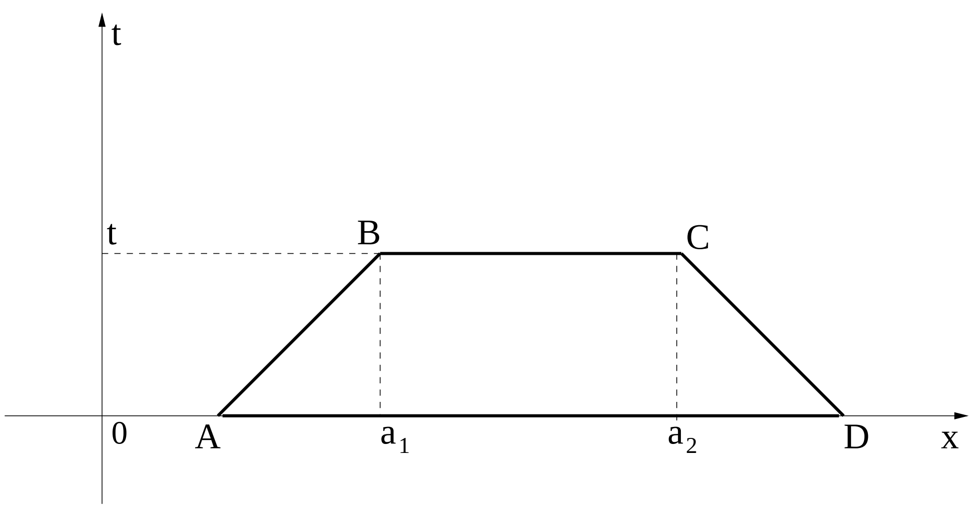

Figure 3. Trapezium ABCD

$$\int_{BC} \Big[\frac{|\dot\psi|^2}{2} + \frac{|\psi'|^2}{2} + U(\psi)\Big]dx \;-\; \int_{AD} \Big[\frac{|\dot\psi|^2}{2} + \frac{|\psi'|^2}{2} + U(\psi)\Big]dx$$
$$+ \int_{AB\cup CD} \Big[\frac{|\dot\psi|^2}{2} + \frac{|\psi'|^2}{2} + U(\psi)\Big]dx + \int_{AB\cup CD} \psi' \cdot \dot\psi dt = 0,$$

which implies (4.1) since $|\psi' \cdot \dot\psi| \le \frac{1}{2}(|\dot\psi|^2 + |\psi'|^2)$ and $U(\psi) \ge 0$. □

Lemma 4.2. *For any* $\nu > 0$,

$$\int (1+|x|)^{\nu} e(x,t)dx \le C(\nu)(1+t)^{\nu+1} \int (1+|x|)^{\nu} e(x,0)dx. \tag{4.2}$$

Proof. By (4.1),

$$\int (1+|y|)^{\nu} \Big(\int\limits_{y-1}^{y} e(x,t)dx \Big) dy \le \int (1+|y|)^{\nu} \Big(\int\limits_{y-1-t}^{y+t} e(x,0)dx \Big) dy.$$

Hence,

$$\int e(x,t) \Big(\int\limits_{x}^{x+1} (1+|y|)^{\nu} dy \Big) dx \le \int e(x,0) \Big(\int\limits_{x-t}^{x+1+t} (1+|y|)^{\nu} dy \Big) dx. \tag{4.3}$$

Applying the mean value theorem, we obtain

$$\int\limits_{x}^{x+1} (1+|y|)^{\nu} dy \ge c(\nu)(1+|x|)^{\nu} \tag{4.4}$$

with some $c(\nu) > 0$. On the other hand, for $\nu > 0$

$$\int\limits_{x-t}^{x+1+t} (1+|y|)^{\nu} dy \le (2t+1)(1+t+|x|)^{\nu} \le C(1+t)^{\nu+1}(1+|x|)^{\nu} \tag{4.5}$$

since

$$(1+t+|x|)^{\nu} \le (1+t)^{\nu}(1+|x|)^{\nu}.$$

Finally, (4.3) - (4.5) imply (4.2). □

5 Spreading rate

Here we prove Theorem 3.2.
Step i) First, we verify that

$$U_0 = \int (1+|x|)^{2\sigma} U(\psi_0(x))dx < C(\sigma), \quad \psi_0(x) = \psi(x,0), \quad 0 \le \sigma \le \sigma_0. \tag{5.1}$$

Indeed, $\psi_0(x) = e^{i\theta_0}[\varphi_{\omega_0}(x) + \Psi_0(x)] \in H^1(\mathbb{R}) \subset C_b(\mathbb{R})$ by (3.2) - (3.3). Hence

$$\sup_{x\in\mathbb{R}} |\psi_0(x)| \le K_0 < \infty,$$

and (1.6) implies that

$$|U(\psi_0(x))| = |u(|\psi_0(x)|^2)| \le C(K_0)|\varphi_{\omega_0}(x) + \Psi_0(x)|^2 \le C_1(K_0)\Big(|\varphi_{\omega_0}(x)|^2 + |\Psi_0(x)|^2\Big).$$

Then (5.1) follows by (3.3).

Step ii) Bound (3.7) follows now from energy conservation (2.6) and splitting (3.6) since

$$\alpha|\psi(x,t)|^2 \le U(\psi(x,t)) \le e(x,t) \tag{5.2}$$

by condition **U2**.

Step iii) Let us prove bound (3.8). First, (3.6), (4.2) and (5.1) imply

$$\begin{aligned}\|\Psi'(t)\|^2_{L^2_\sigma}+\|\Pi(t)\|^2_{L^2_\sigma} &\le 4\int(1+|x|)^{2\sigma}e(x,t)dx+2\|\varphi'_{\omega(t)}\|^2_{L^2_\sigma}+C(\delta)\|\varphi_{\omega(t)}\|^2_{L^2_\sigma}\\ &\le C(1+t)^{2\sigma+1}\int(1+|x|)^{2\sigma}e(x,0)dx+C_1(\delta)\\ &\le C_2(\delta,K_0)(1+t)^{2\sigma+1}.\end{aligned}$$

Similarly, (3.6), (4.2), (5.1) and (5.2) imply

$$\begin{aligned}\|\Psi(t)\|^2_{L^2_\sigma} &\le 2\int(1+|x|)^{2\sigma}|\psi(x,t)|^2dx+2\|\varphi_{\omega(t)}\|^2_{L^2_\sigma}\\ &\le \frac{2}{\alpha}\int(1+|x|)^{2\sigma}e(x,t)dx+2\|\varphi_{\omega(t)}\|^2_{L^2_\sigma}\le C(\delta,K_0)(1+t)^{2\sigma+1}.\end{aligned}$$

Hence, bound (3.8) follows. Theorem 3.2 is proved.

Acknowledgments

The first author was partially supported by the Austrian Science Fund (FWF): M1329-N13 and by RFBR grants; the second author was partially supported by the SEP-CONACyT project 168104 (México); the third author was partially supported by the Alexander von Humboldt Research Award and by the Austrian Science Fund (FWF): P22198-N13; the fourth author was partially supported by the CONACyT (México). All the authors were also supported by PROMEP (México) via “Proyecto de Redes”.

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