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On Long-Time Decay for Magnetic Schrödinger and Klein–Gordon Equations[1]

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We obtain a dispersive long-time decay in weighted energy norms for the solutions of the 3D Schrödinger and Klein–Gordon equations with small magnetic potentials.

1. INTRODUCTION

In this paper, we establish a dispersive long-time decay in weighted energy norms for the solutions to the 3D magnetic Schrödinger equation

$$i\dot\psi(x,t) = H\psi := \big(i\nabla - A(x)\big)^2\psi(x,t) + B(x)\psi(x,t) \tag{1.1}$$

and the 3D magnetic Klein–Gordon equation

$$\ddot\psi(x,t) = \big(\nabla + iA(x)\big)^2\psi(x,t) - m^2\psi(x,t) - B(x)\psi(x,t), \qquad m>0. \tag{1.2}$$

Here $x\in\mathbb{R}^3$ and $(\nabla+iA(x))^2=\sum_1^3(\nabla_j+iA_j(x))^2$. In the vector form, equation (1.2) reads

$$i\dot\Psi(t) = \mathcal{H}\Psi(t) \tag{1.3}$$

where

$$\Psi(t)=\begin{pmatrix}\psi(t)\\ \dot\psi(t)\end{pmatrix}, \qquad \mathcal{H}=\begin{pmatrix}0 & i\\ i\big((\nabla+iA(x))^2-m^2-B(x)\big) & 0\end{pmatrix}.$$

For $s,\sigma\in\mathbb{R}$, let us denote by $H^s_\sigma=H^s_\sigma(\mathbb{R}^3)$ the weighted Sobolev spaces with the finite norms

$$\|\psi\|_{H^s_\sigma}=\|\langle x\rangle^\sigma\langle\nabla\rangle^s\psi\|_{L^2(\mathbb{R}^3)}<\infty, \qquad \langle x\rangle=(1+|x|^2)^{1/2}.$$

Denote $\mathcal{F}_\sigma=H^1_\sigma\oplus H^0_\sigma$.

We assume that $A_j\in C^1(\mathbb{R}^3)$ and $B\in C(\mathbb{R}^3)$ are real functions, and

$$|B(x)|+\sum_{j=1}^3\big(|A_j(x)|+|\nabla A_j(x)|\big)\le d\langle x\rangle^{-\beta}, \qquad x\in\mathbb{R}^3, \tag{1.4}$$

for some $d>0$ and $\beta>3$.

Our main results are the following:

I. For solutions $\psi(t)$ to the Schrödinger equation (1.1), the decay

$$\|\psi(t)\|_{H^0_{-\sigma}}=\mathcal{O}(|t|^{-3/2}), \qquad t\to\pm\infty, \tag{1.5}$$

holds for initial data $\psi_0=\psi(0)\in H^0_\sigma$ if $d>0$ is sufficiently small, $\beta>7/2$ and $\sigma>5/2$.

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II. For solutions $\Psi(t)=(\psi(t),\dot\psi(t))$ to the Klein–Gordon equation (1.3), the decay

$$\|\Psi(t)\|_{\mathcal{F}_{-\sigma}}=\mathcal{O}(|t|^{-3/2}),\qquad t\to\pm\infty, \tag{1.6}$$

holds for initial data $\Psi_0=\Psi(0)\in\mathcal{F}_\sigma$ if $d>0$ is sufficiently small, $\beta>3$ and $\sigma>3/2$.

Let us comment on previous results in this direction. For the wave and Klein–Gordon equations with magnetic potentials, the decay $\sim t^{-3/2}$ was established primarily by Vainberg [10, 11] in local energy norms for initial data with compact support. The decay of type (1.5) in weighted norms was established by Jensen and Kato [7] for the Schrödinger equation with generic scalar potentials, and by Murata [9] for more general PDEs of the Schrödinger type.

For the 3D Klein–Gordon equation with generic scalar potentials, the decay of type (1.6) in weighted energy norms was proved in [8]. The Strichartz estimates for the magnetic Schrödinger, wave, Klein–Gordon and Dirac equations with smallness conditions on potentials were obtained by D'Ancona and Fanelli [3, 4]. The decay $\sim t^{-1}$ in the L^∞ norm was established in [2] for the wave equation with small magnetic and scalar potentials.

Let us comment on our techniques. To prove the decay (1.5), we extend the approach of [7] to the magnetic Schrödinger equation with small potentials using the limiting absorption principle and the high energy decay of the resolvent obtained in [2]. Further, to prove the decay (1.6), we express the magnetic Klein–Gordon resolvent in terms of the magnetic Schrödinger resolvent. Since the high energy decay for the magnetic Klein–Gordon resolvent is absent, we apply the infinite Born series and convolutions.

2. FREE SCHRÖDINGER EQUATION

The free Schrödinger equation with $A\equiv 0$ and $B\equiv 0$ reads

$$i\dot\psi(x,t)=H_0\psi:=-\Delta\psi(x,t). \tag{2.1}$$

The resolvent $R_0(\zeta)=(-\Delta-\zeta)^{-1}$ of the free Schrödinger operator is an integral operator with the integral kernel

$$R_0(\zeta,x-y)=\exp\bigl(i\zeta^{1/2}|x-y|\bigr)/4\pi|x-y|,\qquad \zeta\in\mathbb{C}\setminus[0,\infty),\quad \operatorname{Im}\zeta^{1/2}>0. \tag{2.2}$$

We denote by $\mathcal{L}(B_1,B_2)$ the Banach space of bounded linear operators from a Banach space B_1 to a Banach space B_2.

The explicit formula (2.2) implies the following properties of $R_0(\zeta)$, which are obtained in [1, 7]:

(i) $R_0(\zeta)$ is an analytic function of $\zeta\in\mathbb{C}\setminus[0,\infty)$ with values in $\mathcal{L}(H^0_0,H^2_0)$;

(ii) the limiting absorption principle holds:

$$R_0(\lambda\pm i\varepsilon)\to R_0(\lambda\pm i0),\qquad \varepsilon\to 0+,\quad \lambda>0, \tag{2.3}$$

in $\mathcal{L}(H^0_\sigma,H^2_{-\sigma})$ with $\sigma>1/2$;

(iii) the asymptotic relations

$$\|R_0(\zeta)\|_{\mathcal{L}(H^0_\sigma,H^2_{-\sigma'})}=\mathcal{O}(1),\qquad \zeta\to 0,\quad \sigma,\sigma'>\frac12,\quad \sigma+\sigma'>2, \tag{2.4}$$

$$\bigl\|R_0^{(k)}(\zeta)\bigr\|_{\mathcal{L}(H^0_\sigma,H^2_{-\sigma})}=\mathcal{O}\bigl(\zeta^{\frac12-k}\bigr),\qquad \zeta\to 0,\quad \sigma>\frac12+k,\quad k=1,2,\dots, \tag{2.5}$$

hold for $\zeta\in\mathbb{C}\setminus[0,\infty)$;

(iv) the asymptotic relation

$$\bigl\|R_0^{(k)}(\zeta)\bigr\|_{\mathcal{L}(H^m_\sigma,H^{m+l}_{-\sigma})}=\mathcal{O}\bigl(|\zeta|^{-\frac{1-l+k}{2}}\bigr),\qquad |\zeta|\to\infty,\quad \zeta\in\mathbb{C}\setminus[0,\infty),\quad l=0,1,2,\dots, \tag{2.6}$$

holds for $k=0,1,2,\dots$ and any $\sigma>k+1/2$.

Corollary 2.1. *For $t \in \mathbb{R}$ and $\psi_0 \in H^0_\sigma$ with $\sigma > 1$, the dynamical group $\mathcal{U}_0(t)$ of equation* (2.1) *admits the integral representation*

$$\mathcal{U}_0(t)\psi_0 = \frac{1}{2\pi i}\int\limits_0^\infty e^{-i\zeta t}\big[\mathcal{R}_0(\zeta + i0) - \mathcal{R}_0(\zeta - i0)\big]\psi_0\, d\zeta \tag{2.7}$$

where the integral converges in the sense of distributions of $t \in \mathbb{R}$ with values in $H^2_{-\sigma}$.

The decay (1.5) for the free Schrödinger equation can be deduced from an explicit formula. Namely, $\mathcal{U}_0(t)$ is the integral operator with the integral kernel

$$\mathcal{U}_0(x,y,t) = (4\pi i t)^{-3/2} e^{i|x-y|^2/4t}.$$

For small $|t| \le 1$ we apply the charge conservation for the Schrödinger equation and obtain

$$\|\mathcal{U}_0(t)\psi_0\|_{L^2_{-\sigma}} \le \|\mathcal{U}_0(t)\psi_0\|_{L^2_0} = \|\psi_0\|_{L^2_0} \le \|\psi_0\|_{L^2_\sigma}$$

for any $\sigma > 0$ and $\psi_0 \in L^2_\sigma$. For large $|t| \ge 1$ and $\sigma > 3/2$ the operators

$$\langle x\rangle^{-\sigma}\mathcal{U}_0(t)\langle y\rangle^{-\sigma}\colon\ L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

are Hilbert–Schmidt operators and their Hilbert–Schmidt norms do not exceed $C|t|^{-3/2}$. Finally,

$$\|\mathcal{U}_0(t)\psi_0\|_{L^2_{-\sigma}} \le \frac{C\|\psi_0\|_{L^2_\sigma}}{(1+|t|)^{3/2}}, \qquad t \in \mathbb{R}.$$

3. FREE KLEIN–GORDON EQUATION

In the vector form, the free Klein–Gordon equation reads

$$i\dot\Psi(t) = \mathcal{H}_0\Psi(t) \tag{3.1}$$

where

$$\Psi(t) = \begin{pmatrix}\psi(t)\\ \dot\psi(t)\end{pmatrix}, \qquad \mathcal{H}_0 = \begin{pmatrix}0 & i\\ i(\Delta - m^2) & 0\end{pmatrix}.$$

The resolvent $\mathcal{R}_0(\omega) = (\mathcal{H}_0 - \omega)^{-1}$ can be expressed in terms of the resolvent $R_0(\zeta) = (-\Delta - \zeta)^{-1}$:

$$\mathcal{R}_0(\omega) = \begin{pmatrix}\omega R_0(\omega^2 - m^2) & iR_0(\omega^2 - m^2)\\ -i(1 + \omega^2 R_0(\omega^2 - m^2)) & \omega R_0(\omega^2 - m^2)\end{pmatrix}. \tag{3.2}$$

Let us denote $\Gamma := (-\infty, -m) \cup (m, \infty)$. Then properties (i)–(iv) of $R_0(\zeta)$ and formula (3.2) imply the following lemma.

Lemma 3.1. (i) *The resolvent $\mathcal{R}_0(\omega)$ is an analytic function of $\omega \in \mathbb{C}\setminus\overline{\Gamma}$ with values in $\mathcal{L}(\mathcal{F}_0, \mathcal{F}_0)$.*

(ii) *The limiting absorption principle holds*:

$$\mathcal{R}_0(\omega \pm i\varepsilon) \to \mathcal{R}_0(\omega \pm i0), \qquad \varepsilon \to 0+, \quad \omega \in \Gamma,$$

in $\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$ with $\sigma > 1/2$.

(iii) *The asymptotic relations*

$$\|\mathcal{R}_0(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(1), \qquad \omega \pm m \to 0, \quad \sigma > 1, \tag{3.3}$$

$$\big\|\mathcal{R}_0^{(k)}(\omega)\big\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}\big(|\omega \pm m|^{\frac12 - k}\big), \qquad \omega \pm m \to 0, \quad \sigma > \frac12 + k, \quad k = 1, 2, \dots, \tag{3.4}$$

hold for $\omega \in \mathbb{C}\setminus\overline{\Gamma}$.

(iv) *The asymptotic relation*

$$\big\|\mathcal{R}_0^{(k)}(\omega)\big\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})}=\mathcal{O}(1),\qquad |\omega|\to\infty,\quad \omega\in\mathbb{C}\setminus\Gamma, \tag{3.5}$$

holds for $k=0,1,2,\dots$ *and any* $\sigma>k+\frac{1}{2}$.

Corollary 3.2. *For* $t\in\mathbb{R}$ *and* $\Psi_0\in\mathcal{F}_\sigma$ *with* $\sigma>1$, *the dynamical group* $\mathcal{G}_0(t)$ *of equation* (3.1) *admits the integral representation*

$$\mathcal{G}_0(t)\Psi_0=\frac{1}{2\pi i}\int\limits_{\Gamma} e^{-i\omega t}\big[\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\big]\Psi_0\,d\omega \tag{3.6}$$

where the integral converges in the sense of distributions of $t\in\mathbb{R}$ *with values in* $\mathcal{F}_{-\sigma}$.

Estimates (3.5) do not allow us to obtain the decay (1.6) for the free Klein–Gordon equation by the partial integration in (3.6). This is why we deduce the decay in [8] (see also [6, Lemma 18.2]) from explicit formulas. Namely, for $\sigma>3/2$ and for $\Psi_0\in\mathcal{F}_\sigma$ the following bound holds:

$$\|\mathcal{G}_0(t)\Psi_0\|_{\mathcal{F}_{-\sigma}}\le\frac{C_0\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}},\qquad t\in\mathbb{R}, \tag{3.7}$$

where $C_0>0$ does not depend on σ.

4. MAGNETIC SCHRÖDINGER EQUATION

Here we obtain spectral properties of the magnetic Schrödinger resolvent $R(\zeta)=(H-\zeta)^{-1}$. Note that under condition (1.4) with sufficiently small $d>0$ and $\beta>3$, the resolvent $R(\zeta)$ is continuous at $\zeta=0$, i.e., the point $\zeta=0$ is a regular point of H in the sense of [7]. Then properties (i) and (iii) of the free resolvent imply

Lemma 4.1 (cf. [7, Theorem 5.1]). *Let condition* (1.4) *with sufficiently small* $d>0$ *and* $\beta>3$ *hold. Then*

(i) $R(\zeta)$ *is an analytic function of* $\zeta\in\mathbb{C}\setminus[0,\infty)$ *with values in* $\mathcal{L}(H_0^0,H_0^2)$;

(ii) *the asymptotic relations*

$$\begin{aligned}\|R(\zeta)\|_{\mathcal{L}(H^0_\sigma,H^2_{-\sigma})}&=\mathcal{O}(1),&&\zeta\to0,\quad \sigma>\frac{3}{2},\\ \big\|R^{(k)}(\zeta)\big\|_{\mathcal{L}(H^0_\sigma,H^2_{-\sigma})}&=\mathcal{O}(\zeta^{1/2-k}),&&\zeta\to0,\quad k=1,2,\quad \sigma>\frac{1}{2}+k,\end{aligned} \tag{4.1}$$

hold for $\zeta\in\mathbb{C}\setminus[0,\infty)$.

The next lemma follows from the results of [2].

Lemma 4.2 (cf. [2, Proposition 3.4]). *Let condition* (1.4) *with sufficiently small* $d>0$ *and* $\beta>2$ *hold. Then*

(i) *The limiting absorption principle holds*:

$$R(\lambda\pm i\varepsilon)\to R(\lambda\pm i0),\qquad \varepsilon\to0+,\quad \lambda>0, \tag{4.2}$$

in $\mathcal{L}(H^0_\sigma,H^2_{-\sigma})$ *with* $\sigma>1$;

(ii) *the asymptotic relation*

$$\|R(\zeta)\|_{\mathcal{L}(H^0_\sigma,H^l_{-\sigma})}=\mathcal{O}\big(|\zeta|^{-\frac{1-l}{2}}\big),\qquad |\zeta|\to\infty,\quad \zeta\in\mathbb{C}\setminus[0,\infty),\quad l=0,1, \tag{4.3}$$

holds for any $\sigma>1$.

Now we derive asymptotics for the derivatives of $R(\zeta)$ for large ζ.

Lemma 4.3. *Assume condition* (1.4) *with sufficiently small* $d>0$ *and* $\beta>7/2$ *holds. Then the asymptotic relation*

$$\|R^{(k)}(\zeta)\|_{\mathcal{L}(H^0_\sigma,H^l_{-\sigma})}=\mathcal{O}\big(|\zeta|^{-\frac{1-l+k}{2}}\big),\qquad \zeta\to\infty,\quad \zeta\in\mathbb{C}\setminus[0,\infty),\quad l=1,0, \tag{4.4}$$

holds for $k=1,2$ *and* $\sigma>5/2$.

Proof. *Step* (i). In the case $k=1$ we use the identity

$$R'=(1-RW)R_0'(1-WR)=R_0'-RWR_0'-R_0'WR+RWR_0'WR \tag{4.5}$$

where

$$W=-2iA\cdot\nabla-i\nabla\cdot A+|A|^2+B.$$

The relation implies the asymptotics (4.4) with $k=1$ and $\sigma>3/2$ by (4.3) and (2.6) with $k=1$. Namely, for the first term on the right-hand side of (4.5) this is obvious. Next let us consider the second term. Choosing $\sigma'\in(3/2,\beta-3/2)$, for large $\omega\in\mathbb{C}\setminus[0,\infty)$ we obtain

$$\|RWR_0'\psi\|_{H^l_{-\sigma}}\le C|\zeta|^{-\frac{1-l}{2}}\|WR_0'\psi\|_{H^0_{\sigma'}}\le C_1|\zeta|^{-\frac{1-l}{2}}\|R_0'\psi\|_{H^1_{\sigma'-\beta}}\le C_2|\zeta|^{-\frac{2-l}{2}}\|\psi\|_{H^0_\sigma},\qquad l=0,1. \tag{4.6}$$

The remaining terms on the right-hand side of (4.5) can be estimated similarly. Hence, (4.4) with $k=1$ and $\sigma>3/2$ is proved.

Step (ii). In the case $k=2$ we apply the formula

$$R''=R_0''-RWR_0''-R_0''WR+RWR_0''WR-2R'WR_0'+2R'WR_0'WR. \tag{4.7}$$

The asymptotics (4.4) with $k=2$ and $\sigma>5/2$ for the first term on the right-hand side follows from (2.6) with $k=2$. The last two terms can be estimated similarly to (4.6) by using (2.6) and (4.4) with $k=1$ and (4.3).

Let us estimate the remaining terms. Recall that $\beta>7/2$, and we consider $\sigma>5/2$ and large $\zeta\in\mathbb{C}\setminus[0,\infty)$. Using (4.3) and (2.6) with $k=2$ and $l=0,1$, we find that

(a) for $\sigma'\in(5/2,\beta-1)$ the following bounds hold:

$$\begin{aligned}
&\|RWR_0''\psi\|_{H^l_{-\sigma}}\le C|\zeta|^{-\frac{1-l}{2}}\|WR_0''\psi\|_{H^0_{-\sigma'+\beta}}\le C_1|\zeta|^{-\frac{1-l}{2}}\|R_0''\psi\|_{H^1_{-\sigma'}}\le C_2|\zeta|^{-\frac{3-l}{2}}\|\psi\|_{H^0_\sigma},\\
&\|R_0''WR\psi\|_{H^l_{-\sigma}}\le C|\zeta|^{-\frac{3-l}{2}}\|WR\psi\|_{H^0_{\sigma'}}\le C_1|\zeta|^{-\frac{3-l}{2}}\|R\psi\|_{H^1_{\sigma'-\beta}}\le C_2|\zeta|^{-\frac{3-l}{2}}\|\psi\|_{H^0_\sigma};
\end{aligned} \tag{4.8}$$

(b) for $\sigma'\in(1,\beta-5/2)$ the following bound holds:

$$\begin{aligned}
\|RWR_0''WR\psi\|_{H^l_{-\sigma}}&\le C|\zeta|^{-\frac{1-l}{2}}\|WR_0''WR\psi\|_{H^0_{\sigma'}}\le C_1|\zeta|^{-\frac{1-l}{2}}\|R_0''WR\psi\|_{H^1_{\sigma'-\beta}}\\
&\le C_2|\zeta|^{-\frac{3-l}{2}}\|WR\psi\|_{H^0_{-\sigma'+\beta}}\le C_3|\zeta|^{-\frac{3-l}{2}}\|R\psi\|_{H^1_{-\sigma'}}\le C_4|\zeta|^{-\frac{3-l}{2}}\|\psi\|_{H^0_\sigma}.
\end{aligned} \tag{4.9}$$

Hence, (4.4) with $k=2$ and $\sigma>5/2$ is proved. □

Finally, we obtain the decay (1.5) for the perturbed Schrödinger equation.

Theorem 4.4. *Let condition* (1.4) *with sufficiently small* $d>0$ *and* $\beta>7/2$ *hold. Then*

$$\|e^{-itH}\|_{\mathcal{L}(L^2_\sigma,L^2_{-\sigma})}=\mathcal{O}(|t|^{-3/2}),\qquad t\to\pm\infty, \tag{4.10}$$

with $\sigma>5/2$.

Proof. Since $\mathrm{Sp}\, H = \mathrm{Sp}_{\mathrm{ac}}\, H = [0,\infty)$ for sufficiently small $d>0$, Lemma 4.2 and the asymptotics (4.1) and (4.4) with $k=l=0$ imply similarly to (2.7) that

$$\psi(t) = \frac{1}{2\pi i}\int_0^\infty e^{-i\zeta t}\big[R(\zeta+i0)-R(\zeta-i0)\big]\psi_0\,d\zeta.$$

Then we obtain the decay (4.10) using the asymptotics (4.1) and (4.4) by the methods of [7]. □

5. MAGNETIC KLEIN–GORDON EQUATION

Denote by $\mathcal{R}(\omega)=(\mathcal{H}-\omega)^{-1}$ the resolvent of the magnetic Klein–Gordon equation (1.2). The resolvent $\mathcal{R}(\omega)$ can be expressed in terms of $R(\zeta)$ similarly to (3.2):

$$\mathcal{R}(\omega) = \begin{pmatrix} \omega R(\omega^2-m^2) & iR(\omega^2-m^2) \\ -i(1+\omega^2 R(\omega^2-m^2)) & \omega R(\omega^2-m^2)\end{pmatrix}. \tag{5.1}$$

Representation (5.1) and Lemmas 4.1–4.3 imply

Lemma 5.1. *Let condition* (1.4) *with sufficiently small $d>0$ hold. Then*

(i) *$\mathcal{R}(\omega)$ is an analytic function of $\zeta\in\mathbb{C}\setminus\overline{\Gamma}$ with values in $\mathcal{L}(\mathcal{F}_0,\mathcal{F}_0)$;*

(ii) *the asymptotic relation*

$$\|\mathcal{R}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(1), \qquad \omega\pm m\to 0,$$

holds for $\omega\in\mathbb{C}\setminus\overline{\Gamma}$ and $\sigma>3/2$;

(iii) *the limiting absorption principle holds*:

$$\mathcal{R}(\lambda\pm i\varepsilon)\to\mathcal{R}(\lambda\pm i0), \qquad \varepsilon\to 0+, \quad \lambda\in\Gamma,$$

in $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ with $\sigma>1$;

(iv) *for $\sigma>1$ the following bound holds*:

$$\|\mathcal{R}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(1), \qquad |\omega|\to\infty, \quad \zeta\in\mathbb{C}\setminus\Gamma.$$

Now we obtain the decay (1.6) for the perturbed Klein–Gordon equation.

Theorem 5.2. *Let condition* (1.4) *with sufficiently small $d>0$ and $\beta>3$ hold. Then*

$$\|e^{-it\mathcal{H}}\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(|t|^{-3/2}), \qquad t\to\pm\infty,$$

with $\sigma>3/2$.

Proof. *Step* (i). Lemma 5.1 implies, similarly to (3.6), that

$$\Psi(t) = \frac{1}{2\pi i}\int_\Gamma e^{-i\omega t}\big[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\big]\Psi_0\,d\omega \tag{5.2}$$

for the initial state $\Psi(0)\in\mathcal{F}_\sigma$ with $\sigma>3/2$. Now we combine the spectral properties of the perturbed resolvent and the time decay for the unperturbed dynamics using the following Born perturbation series:

$$\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{W}\mathcal{R}_0(\omega) + \mathcal{R}_0(\omega)\mathcal{W}\mathcal{R}_0(\omega)\mathcal{W}\mathcal{R}_0(\omega) - \dots, \qquad \omega\in\mathbb{C}\setminus\Gamma, \tag{5.3}$$

which follows by the iteration of $\mathcal{R}(\omega)=\mathcal{R}_0(\omega)-\mathcal{R}_0(\omega)\mathcal{W}\mathcal{R}(\omega)$. Here

$$\mathcal{W} = \begin{pmatrix} 0 & 0 \\ -iW & 0\end{pmatrix}.$$

We substitute the series (5.3) into the spectral representation (5.2) and obtain

$$\Psi(t)=\frac{(-1)^{k-1}}{2\pi i}\sum_{k=1}^{\infty}\int\limits_{\Gamma} e^{-i\omega t}\Big[(\mathcal{R}_0(\omega+i0)\mathcal{W})^{k-1}\mathcal{R}_0(\omega+i0)-(\mathcal{R}_0(\omega-i0)\mathcal{W})^{k-1}\mathcal{R}_0(\omega+i0)\Big]\Psi_0\,d\omega$$
$$=\sum_{k=1}^{\infty}(-1)^{k-1}\Psi_k(t),\qquad t\in\mathbb{R}. \tag{5.4}$$

Further, we analyze each term Ψ_k separately.

Step (ii). The first term $\Psi_1(t)=\mathcal{G}_0(t)\Psi_0$ by (3.6). Hence, (3.7) implies that

$$\|\Psi_1(t)\|_{\mathcal{F}_{-\sigma}}\le\frac{C_0\|\psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}},\qquad t\in\mathbb{R},\quad \sigma>3/2. \tag{5.5}$$

Step (iii). The other terms $\Psi_k(t)$ can be rewritten as a convolution.

Lemma 5.3 (cf. [8, Lemma 3.8]). *Suppose that*

$$\|\Psi_{k-1}(t)\|_{\mathcal{F}_{-\sigma}}\le C_k(1+|t|)^{-3/2},\qquad t\in\mathbb{R},\quad\sigma>3/2,\quad k=2,3,\dots. \tag{5.6}$$

Then the convolution representation

$$\Psi_k(t)=i\int\limits_0^t\mathcal{G}_0(t-\tau)\mathcal{W}\Psi_{k-1}(\tau)\,d\tau,\qquad t\in\mathbb{R},\quad k=2,3,\dots, \tag{5.7}$$

holds where the integral converges in $\mathcal{F}_{-\sigma}$ *with* $\sigma>3/2$.

Proof. We have

$$\Psi_k(t)=\frac{1}{2\pi i}\int\limits_{\mathbb{R}}\Big[e^{-i\omega t}\mathcal{R}_0(\omega+i0)\mathcal{W}(\mathcal{R}_0(\omega+i0)\mathcal{W})^{k-2}\mathcal{R}_0(\omega+i0)$$
$$-e^{-i\omega t}\mathcal{R}_0(\omega-i0)\mathcal{W}(\mathcal{R}_0(\omega-i0)\mathcal{W})^{k-2}\mathcal{R}_0(\omega-i0)\Big]\Psi_0\,d\omega. \tag{5.8}$$

Let us denote

$$\mathcal{G}_0^{\pm}(t):=\theta(\pm t)\mathcal{G}_0(t),\qquad \Psi_{k-1}^{\pm}(t):=\theta(\pm t)\Psi_{k-1}(t),\qquad t\in\mathbb{R}.$$

We know that

$$(\mathcal{R}_0(\omega+i0)\mathcal{W})^{k-2}\mathcal{R}_0(\omega+i0)\Psi_0=i\widetilde{\Psi}_{k-1}^{+}(\omega);$$

hence the first term on the right-hand side of (5.8) reads

$$\Psi_{k1}(t)=\frac{1}{2\pi}\int\limits_{\mathbb{R}}e^{-i\omega t}\mathcal{R}_0(\omega+i0)\mathcal{W}\widetilde{\Psi}_{k-1}^{+}(\omega)\,d\omega=\frac{1}{2\pi}\int\limits_{\mathbb{R}}e^{-i\omega t}\mathcal{R}_0(\omega+i0)\mathcal{W}\left[\int\limits_{\mathbb{R}}e^{i\omega\tau}\Psi_{k-1}^{+}(\tau)\,d\tau\right]d\omega$$
$$=\frac{1}{2\pi}(i\partial_t+i)^2\int\limits_{\mathbb{R}}\frac{e^{-i\omega t}}{(\omega+i)^2}\mathcal{R}_0(\omega+i0)\mathcal{W}\left[\int\limits_{\mathbb{R}}e^{i\omega\tau}\Psi_{k-1}^{+}(\tau)\,d\tau\right]d\omega. \tag{5.9}$$

The last double integral converges in $\mathcal{F}_{-\sigma}$ with $\sigma>3/2$ by (5.6) and (3.5) with $k=0$. Hence, we can change the order of integration by the Fubini theorem. Then we obtain

$$\Psi_{k1}(t)=i\int\limits_{\mathbb{R}}\mathcal{G}_0^{+}(t-\tau)\mathcal{W}\Psi_{k-1}^{+}(\tau)\,d\tau=\begin{cases} i\displaystyle\int\limits_0^t\mathcal{G}_0(t-\tau)\mathcal{W}\Psi_{k-1}(\tau)\,d\tau, & t>0,\\ 0, & t<0,\end{cases} \tag{5.10}$$

since

$$\mathcal{G}_0^+(t-\tau)=\frac{1}{2\pi i}(i\partial_t+i)^2\int\limits_{\mathbb{R}}\frac{e^{-i\omega(t-\tau)}}{(\omega+i)^2}\mathcal{R}_0(\omega+i0)\,d\omega$$

by (2.7). Similarly, integrating the second term on the right-hand side of (5.8), we obtain

$$\Psi_{k2}(t)=i\int\limits_{\mathbb{R}}\mathcal{G}_0^-(t-\tau)\mathcal{W}\Psi_{k-1}^-(\tau)\,d\tau=\begin{cases}0, & t>0,\\ i\int\limits_0^t\mathcal{G}_0(t-\tau)\mathcal{W}\Psi_{k-1}(\tau)\,d\tau, & t<0.\end{cases}\tag{5.11}$$

Now (5.7) follows since $\Psi_k(t)$ is the sum of the two expressions (5.10) and (5.11). □

Now we estimate $\Psi_k(t)$ by induction.

Lemma 5.4. *There exists a constant $M>0$ such that*

$$\|\Psi_k(t)\|_{\mathcal{F}_{-\sigma}}\le\frac{C_0M^kd^k\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}},\qquad t\in\mathbb{R},\quad\sigma>3/2,\quad k=2,3,\dots,\tag{5.12}$$

with C_0 from (5.5).

Proof. First, let us consider the case $k=2$. For any $\sigma'\in(3/2,\min\{\sigma,\beta/2\})$, applying the bounds (3.7) and (5.5) to the integrand in (5.7), we obtain

$$\begin{aligned}\|\mathcal{G}_0(t-\tau)\mathcal{W}\Psi_1(\tau)\|_{\mathcal{F}_{-\sigma}}&\le\frac{C_0\|\mathcal{W}\Psi_1(\tau)\|_{\mathcal{F}_{\sigma'}}}{(1+|t-\tau|)^{3/2}}=\frac{C_0\|W\psi_1(\tau)\|_{L^2_{\sigma'}}}{(1+|t-\tau|)^{3/2}}\le\frac{C_0d\|\psi_1(\tau)\|_{H^1_{-\sigma'}}}{(1+|t-\tau|)^{3/2}}\\&\le\frac{C_0^2d\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t-\tau|)^{3/2}(1+|\tau|)^{3/2}}\end{aligned}\tag{5.13}$$

where ψ_1 is the first component of Ψ_1. Integrating with respect to τ, we obtain

$$\|\Psi_2(t)\|_{\mathcal{F}_{-\sigma}}\le\frac{C_1C_0^2d\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}}=\frac{C_0Md\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}},\qquad t\in\mathbb{R},\quad\sigma>3/2,\tag{5.14}$$

with $M=C_0C_1$. Suppose that

$$\|\Psi_{k-1}(t)\|_{\mathcal{F}_{-\sigma}}\le\frac{C_0M^{k-1}d^{k-1}\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}},\qquad t\in\mathbb{R},\quad\sigma>3/2.\tag{5.15}$$

Then (5.3) holds with $C_k=C_0M^{k-1}d^{k-1}\|\Psi_0\|_{\mathcal{F}_\sigma}$ and hence representation (5.7) for Ψ_k holds by Lemma 5.3. Similarly to (5.13), we find from (5.15) that

$$\begin{aligned}\|\mathcal{G}_0(t-\tau)\mathcal{W}\Psi_{k-1}(\tau)\|_{\mathcal{F}_{-\sigma}}&\le\frac{C_0\|\mathcal{W}\Psi_{k-1}(\tau)\|_{\mathcal{F}_{\sigma'}}}{(1+|t-\tau|)^{3/2}}=\frac{C_0\|W\psi_{k-1}(\tau)\|_{L^2_{\sigma'}}}{(1+|t-\tau|)^{3/2}}\le\frac{C_0d\|\psi_{k-1}(\tau)\|_{H^1_{-\sigma'}}}{(1+|t-\tau|)^{3/2}}\\&\le\frac{C_0^2M^{k-1}d^k\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t-\tau|)^{3/2}(1+|\tau|)^{3/2}}\end{aligned}$$

where ψ_{k-1} is the first component of Ψ_{k-1}. Integrating with respect to τ, we obtain (5.12). □

Finally, we choose a small $d>0$ so that $Md<1$. By (5.4) and Lemma 5.4, we see that

$$\|\Psi\|_{\mathcal{F}_{-\sigma}}\le\frac{C_0\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|\tau|)^{3/2}}\frac{1}{1-Md},\qquad t\in\mathbb{R},\quad\sigma>3/2.$$

Theorem 5.2 is proved. □

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