Dispersion estimates for 2D Dirac equation

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Abstract

We obtain a dispersion long-time decay in weighted norms for solutions of the 2D Dirac equation with generic potentials.

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1 Introduction

In this paper, we establish a dispersive long time decay for solutions to 2D linear Dirac equation with the Maxwell potentials:

$$\left\{\begin{array}{l} i\dot\psi(x,t)=\mathcal{H}\psi(x,t):=[-i\alpha\cdot\nabla+\beta m+\mathcal{V}(x)]\psi(x,t)\\ \psi(x,0)=\psi_0(x)\end{array}\right| \quad x\in\mathbb{R}^2 \tag{1.1}$$

where $\psi(x,t)\in\mathbb{C}^2$, $m>0$ and $\alpha=(\alpha_1,\alpha_2)$. The Hermitian matrices $\beta=\alpha_0$ and α_k satisfy the following relations:

$$\left\{\begin{array}{l} \alpha_k^*=\alpha_k,\\ \alpha_k\alpha_l+\alpha_l\alpha_k=2\delta_{kl}I\end{array}\right| \quad k,l=0,1,2. \tag{1.2}$$

The standard form of the Dirac matrices α_k and β is

$$\beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \alpha_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \alpha_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}. \tag{1.3}$$

We assume the following conditions:

E1. The potential $\mathcal{V}\in C^\infty(\mathbb{R}^2)$ is a Hermitian 2×2 matrix function such that

$$|\partial^\alpha\mathcal{V}(x)|\le C(\alpha)\langle x\rangle^{-\rho-|\alpha|},\quad \langle x\rangle^\sigma=(1+|x|^2)^{\sigma/2} \tag{1.4}$$

with some $\rho>5$.

E2. The operator $\mathcal{H}$ presents neither resonance nor eigenvalue at the thresholds.

Note that condition **E2** corresponds to the "regular case" in the terminology of [6] (or "non-singular case" in [13]). Under condition **E2** the operator $\mathcal{H}$ has a finite set of eigenvalues $\omega_j\in(-m,m)$, $j=1,...,N$. Denote by P_j the Riesz projection onto the corresponding eigenspaces and by

$$P_d=\sum_j P_j \quad\text{and}\quad P_c:=1-P_d \tag{1.5}$$

the projections onto the discrete and continuous spectral spaces of $\mathcal{H}$.

For $s,\sigma\in\mathbb{R}$, denote by $H^s_\sigma(\mathbb{R}^2)$ the weighted Sobolev spaces (see [1]) with the finite norms

$$\|u\|_{H^s_\sigma(\mathbb{R}^2)}=\|\langle x\rangle^\sigma\langle\nabla x\rangle^s u\|_{L^2(\mathbb{R}^2)}<\infty.$$

Denote $H^s_\sigma:=H^s_\sigma(\mathbb{R}^2)\otimes\mathbb{C}^2$, $L^2_\sigma:=H^0_\sigma$. Our main result is the following long time decay of the solutions to (1.1):

$$\|P_c\psi(t)\|_{L^2_{-\sigma}}=\mathcal{O}(|t|^{-1}\log^{-2}|t|),\quad t\to\pm\infty \tag{1.6}$$

for initial data $\psi_0=\psi(0)\in L^2_\sigma$ with $\sigma>5/2$.

The decay in weighted norms has been established first by Jensen and Kato [6] for the Schrödinger equation in the dimension $n=3$. The result was extended to all other dimensions by Jensen and Nenciu [4, 5, 7], and to more general PDEs of the Schrödinger type by Murata [13]. For the Klein-Gordon equations the dispersive decay has been proved in [9]-[11], for 3D Dirac equation in [3], and for 1D Dirac equation in [12]. For 2D Dirac equation the decay was not obtained before.

Let us comment on our techniques. We extend our approach [11] to the Dirac equation. Note that decay (1.6) violates for the free 2D Dirac equation corresponding to $\mathcal{V}(x) = 0$ when the solutions slow decay, like $\sim t^{-1}$. The slow decay is caused by the " resonance functions" $\psi_m(x) = (1, 0)$ and $\psi_{-m}(x) = (0, 1)$ corresponding to the edge points $\pm m$ of the continuous spectrum of the free Dirac operator $\mathcal{H}_0 = -i\alpha \cdot \nabla + m\beta$ (see Remark 3.1). Hence, decay (1.6) cannot be deduced by perturbation arguments from the corresponding estimate for the free equation.

Following [6], we consider low energy and high energy components of solution to (1.1) separately. The high energy component decays like t^{-N} with any $N > 0$. The proof relies on the Mourre estimates and the minimal escape velocity estimates obtained by Boussaid [3] in the context of the 3D Dirac operator.

For the low energy component, the decay $\sim t^{-1} \log^{-2} t$ follows by a suitable modification of the methods [6, 13, 15]. Our main novelties are the asymptotics (3.2) of the perturbed resolvent near the points $\pm m$ in the nonsingular case (i.e., under Condition **E2**) and "2D version" of Jensen-Kato-Zygmund lemma (Lemma 3.7) on "one-and-half partial integration".

Our paper is organized as follows. In Section 2 we obtain the time decay for the solution to the free Dirac equation and state the spectral properties of the free resolvent. In Section 3 we obtain spectral properties of the perturbed resolvent and prove decay (1.6).

2 Free Dirac equation

First, we consider the free Dirac equation:

$$i\dot{\psi}(x, t) = \mathcal{H}_0\psi(x, t) := -i\alpha \cdot \nabla\psi(x, t) + m\beta\psi(x, t). \qquad (2.1)$$

Denote by $\mathcal{U}_0(t) : \psi(\cdot, 0) \to \psi(\cdot, t)$ the dynamical group of equation (2.1). It is a strongly continuous group in $L^2 := L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$. The group is unitary due to the charge conservation.

2.1 Spectral properties

We state spectral properties of $\mathcal{U}_0(t)$ applying known results [1, 13] on spectral properties of the free Schrödinger dynamical group. For $t > 0$ and $\psi_0 = \psi(0) \in L^2$, the solution $\psi(t)$ to free equation (2.1) admits the spectral Fourier-Laplace representation

$$\theta(t)\psi(t) = \frac{1}{2\pi i} \int\limits_{\mathbb{R}} e^{-i(\omega+i\varepsilon)t} \mathcal{R}_0(\omega + i\varepsilon)\psi_0 \, d\omega, \qquad t \in \mathbb{R} \qquad (2.2)$$

with any $\varepsilon > 0$ where $\theta(t)$ is the Heaviside function, $\mathcal{R}_0(\omega) = (\mathcal{H}_0 - \omega)^{-1}$ is the resolvent of the operator $\mathcal{H}_0$. The representation follows from the stationary equation $\omega\tilde{\psi}^+(\omega) = \mathcal{H}_0\tilde{\psi}^+(\omega) + i\psi_0$ for the Fourier-Laplace transform $\tilde{\psi}^+(\omega) := \int_{\mathbb{R}} \theta(t)e^{i\omega t}\psi(t)dt$, where $\omega \in \mathbb{C}^+ := \{\mathrm{Im}\,\omega > 0\}$. The solution $\psi(t)$ is a continuous bounded function of $t \in \mathbb{R}$ with the values in L^2. Hence, $\tilde{\psi}^+(\omega) = -i\mathcal{R}_0(\omega)\psi_0$ is an analytic function of $\omega \in \mathbb{C}^+$ with the values in L^2, bounded for $\omega \in \mathbb{R} + i\varepsilon$. Therefore, integral (2.2) converges in the sense of distributions of $t \in \mathbb{R}$ with the values in L^2. Similarly to (2.2),

$$\theta(-t)\psi(t) = -\frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i(\omega - i\varepsilon)t}\mathcal{R}_0(\omega - i\varepsilon)\psi_0\, d\omega, \qquad t\in\mathbb{R}. \tag{2.3}$$

The resolvent $\mathcal{R}_0(\omega)$ can be expressed in terms of the resolvent $R_0(\zeta) = (-\partial_x^2 - \zeta)^{-1}$ of the free Schrödinger operator. Indeed, (1.2) implies

$$(\mathcal{H}_0 - \omega)(\mathcal{H}_0 + \omega) = (-i\alpha\cdot\nabla + m\beta - \omega)(-i\alpha\cdot\nabla + m\beta + \omega) = -\Delta + m^2 - \omega^2. \tag{2.4}$$

Therefore,

$$\mathcal{R}_0(\omega) = (-i\alpha\cdot\nabla + m\beta + \omega)R_0(\omega^2 - m^2) \tag{2.5}$$

where $R_0(\zeta)$ is the operator with the integral kernel

$$R_0(\zeta, x-y) = \frac{i}{4}H_0^{(1)}(\zeta^{1/2}|x-y|) = \frac{1}{2\pi}K_0(-i\zeta^{1/2}|x-y|), \quad \zeta\in\mathbb{C}^+, \quad \operatorname{Im}\zeta^{1/2} > 0. \tag{2.6}$$

Here $H_0^{(1)}$ is the modified Hankel function, and K_0 is the Macdonald's function. Denote by $\mathcal{L}(B_1, B_2)$ the Banach space of bounded linear operators from a Banach space B_1 to a Banach space B_2. Explicit formula (2.6) obviously implies the properties of $R_0(\zeta)$ (cf. [1, 11, 13]):

Lemma 2.1. *i) $R_0(\zeta)$ is an analytic function of $\zeta\in\mathbb{C}\setminus[0,\infty)$ with the values in $\mathcal{L}(H^0, H^2)$;*
ii) For $\zeta > 0$, the convergence (limiting absorption principle) holds

$$R_0(\zeta \pm i\varepsilon) \to R_0(\zeta \pm i0), \quad \varepsilon\to 0+ \tag{2.7}$$

in $\mathcal{L}(H^0_\sigma, H^2_{-\sigma})$ with $\sigma > 1/2$.
iii) The asymptotic expansion holds

$$R_0(\zeta) = A_0\log\zeta + B_0 + \mathcal{O}(\zeta^{3/4}), \quad \zeta\to 0, \quad \zeta\in\mathbb{C}\setminus[0,\infty) \tag{2.8}$$

in $\mathcal{L}(H^0_\sigma; H^2_{-\sigma})$ with $\sigma > 5/2$. Here $A_0, B_0\in\mathcal{L}(H^0_\sigma; H^2_{-\sigma})$, with $\sigma > 1$, are the operators with the integral kernels $A_0(x-y), B_0(x-y)$ respectively, where

$$A_0(x-y) = -\frac{1}{4\pi}, \quad B_0(x-y) = -\frac{\gamma}{2\pi} - \frac{1}{2\pi}\log\frac{|x-y|}{2} + \frac{i}{4}, \quad x,y\in\mathbb{R}^2. \tag{2.9}$$

Furthermore,

$$R_0'(\zeta) = A_0\zeta^{-1} + \mathcal{O}(\zeta^{-1/4}), \quad R_0''(\zeta) = -A_0\zeta^{-2} + \mathcal{O}(\zeta^{-5/4}), \quad \zeta\to 0,\ \zeta\in\mathbb{C}\setminus[0,\infty) \tag{2.10}$$

in $\mathcal{L}(H^0_\sigma; H^2_{-\sigma})$ with $\sigma > 5/2$.
iv) For any $s\in\mathbb{R}$, $l = -1, 0, 1, 2$ and $k = 0, 1, 2...$ the asymptotics hold

$$R_0^{(k)}(\zeta) = \mathcal{O}(|\zeta|^{-\frac{1-l+k}{2}}), \quad |\zeta|\to\infty, \quad \zeta\in\mathbb{C}\setminus[0,\infty)$$

in $\mathcal{L}(H^s_\sigma; H^{s+l}_{-\sigma})$ with $\sigma > 1/2 + k$.

Let us denote $\Gamma := (-\infty, -m)\cup(m,\infty)$. Lemma 2.1 and formula (2.5) imply

sp **Lemma 2.2.** *i) The resolvent $\mathcal{R}_0(\omega)$ is an analytic function of $\omega\in\mathbb{C}\setminus\overline{\Gamma}$ with the values in $\mathcal{L}(L^2,L^2)$.*
ii) For $\omega\in\Gamma$, the limiting absorption principle holds

$$\mathcal{R}_0(\omega\pm i\varepsilon)\to\mathcal{R}_0(\omega\pm i0),\quad \varepsilon\to 0+ \tag{2.11}$$ lap1

in $\mathcal{L}(L^2_\sigma,L^2_{-\sigma})$ with $\sigma>1/2$.
iii) The asymptotics hold

$$\left.\begin{array}{l}\mathcal{R}_0(\omega)=\mathcal{A}_0^\pm\log(\omega\mp m)+\mathcal{B}_0^\pm+\mathcal{O}((\omega\mp m)^{3/4})\\[2mm] \mathcal{R}_0'(\omega)=\mathcal{A}_0^\pm(\omega\mp m)^{-1}+\mathcal{O}((\omega\mp m)^{-1/4})\\[2mm] \mathcal{R}_0''(\omega)=-\mathcal{A}_0^\pm(\omega\mp m)^{-2}+\mathcal{O}((\omega\mp m)^{-5/4})\end{array}\right|\quad \omega\to\pm m,\quad \omega\in\mathbb{C}\setminus\overline{\Gamma}. \tag{2.12}$$ expR0

in $\mathcal{L}(L^2_\sigma;L^2_{-\sigma})$ with $\sigma>5/2$. Here the operators

$$\begin{array}{l}\mathcal{A}_0^\pm=\mathrm{Op}\Big[(m\beta\pm m)A_0\Big]\\[2mm] \mathcal{B}_0^\pm=\mathrm{Op}\Big[(m\beta\pm m)B_0+\log(\pm 2m)+\dfrac{i}{2\pi|x-y|}\alpha\cdot\nabla|x-y|\Big]\end{array} \tag{2.13}$$ cAA

belong to $\mathcal{L}(L^2_\sigma;L^2_{-\sigma})$ with $\sigma>1$.
iv) For $k=0,1,2,...$ and $\sigma>1/2+k$ the asymptotics hold

$$\|\mathcal{R}_0^{(k)}(\omega)\|_{\mathcal{L}(L^2_\sigma,L^2_{-\sigma})}=\mathcal{O}(1),\quad \omega\to\infty,\quad \omega\in\mathbb{C}\setminus\Gamma. \tag{2.14}$$ bR0

irep **Corollary 2.3.** *For $t\in\mathbb{R}$ and $\psi_0\in L^2_\sigma$ with $\sigma>1$, the group $\mathcal{U}_0(t)$ admits the integral representation*

$$\mathcal{U}_0(t)\psi_0=\frac{1}{2\pi i}\int\limits_\Gamma e^{-i\omega t}\Big[\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\Big]\psi_0\,d\omega \tag{2.15}$$ Gint1

where the integral converges in the sense of distributions of $t\in\mathbb{R}$ with the values in $L^2_{-\sigma}$.

Proof. Summing up representations (2.2) and (2.3), and sending $\varepsilon\to 0+$, we obtain (2.15) by the Cauchy theorem and lemma 2.2. □

2.2 Time decay

Tdec

Here we prove time decay for the free Dirac equation (2.1). Estimates (2.14) do not allow to obtain the decay of $\mathcal{U}_0(t)\psi_0$ by partial integration in (2.15). We deduce the decay from explicit formulas. We have

$$(\partial_t+\alpha\cdot\nabla-im\beta)(\partial_t-\alpha\cdot\nabla+im\beta)=\partial_t^2-\Delta+m^2.$$

Then

$$\mathcal{U}_0(t)=(\partial_t+\alpha\cdot\nabla-im\beta)G(t), \tag{2.16}$$ UG

where $G(t) = \mathrm{Op}[G(x-y,t)]$, and $G(z,t)$ is the retarded fundamental solution to the Klein-Gordon operator $\partial_t^2 - \Delta + m^2$:

$$G(z,t) = \frac{1}{2\pi}\theta(|t|-|z|)\frac{\cos m\sqrt{t^2-|z|^2}}{\sqrt{t^2-|z|^2}}. \qquad (2.17)$$

We derive a time decay of $\mathcal{U}(t)$ from the time decay of dynamical group $\mathcal{G}(t)$ of the free Klein-Gordon equation. The matrix kernel $\mathcal{G}(t,x-y)$ of the group $\mathcal{G}(t)$ can be written as

$$\mathcal{G}(t,x-y) = \begin{pmatrix} \dot G(t,x-y) & G(t,x-y) \\ \ddot G(t,x-y) & \dot G(t,x-y) \end{pmatrix}, \quad x,y\in\mathbb{R}^2. \qquad (2.18)$$

For any $s\in\mathbb{R}$, denote $\mathcal{F}_s = H^1_s \oplus L^2_s$.

Proposition 2.4. *The asymptotics hold*

$$\mathcal{G}(t) = \mathcal{O}(t^{-1}), \quad t\to\infty \qquad (2.19)$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma;\mathcal{F}_{-\sigma})$ with $\sigma>1$.

Proof. We apply the method from the proof of Lemma 18.2 from [8]. We split the function $\Psi\in\mathcal{F}_\sigma$ in two terms, $\Psi = \Psi'_t + \Psi''_t$ such that

$$\|\Psi'_t\|_{\mathcal{F}_\sigma} + \|\Psi''_t\|_{\mathcal{F}_\sigma} \le C\|\Psi\|_{\mathcal{F}_\sigma}, \quad t\ge 1, \qquad (2.20)$$

$$\Psi'_t(x) = 0 \quad \text{for} \quad |x| > \frac{t}{3}, \quad \text{and} \quad \Psi''_t(x) = 0 \quad \text{for} \quad |x| < \frac{t}{4}. \qquad (2.21)$$

We estimate $\mathcal{G}(t)\Psi'_t$ and $\mathcal{G}(t)\Psi''_t$ separately.

Step i) First we estimate $\mathcal{G}(t)\Psi''_t$ using the energy conservation for the Klein-Gordon equation, and properties (2.21) and (2.20):

$$\|\mathcal{G}(t)\Psi''_t\|_{\mathcal{F}_{-\sigma}} \le \|\mathcal{G}(t)\Psi''_t\|_{\mathcal{F}_0} \le C\|\Psi''_t\|_{\mathcal{F}_0} \le C_1 t^{-\sigma}\|\Psi''_t\|_{\mathcal{F}_\sigma} \le C_2 t^{-1}\|\Psi\|_{\mathcal{F}_\sigma}, \quad t\ge 1, \qquad (2.22)$$

since $\sigma > 1$.
Step ii) Now we consider $\mathcal{G}(t)\Psi'_t$. We split the operator $\mathcal{G}(t)$ in two terms:

$$\mathcal{G}(t) = (1-\zeta)\mathcal{G}(t) + \zeta\mathcal{G}(t), \quad t\ge 1,$$

where ζ is the operator of multiplication by the function $\zeta(|x|/t)$ such that $\zeta=\zeta(s)\in C_0^\infty(\mathbb{R})$, $\zeta(s)=1$ for $|s|<1/4$, and $\zeta(s)=0$ for $|s|>1/3$. Obviously, for any α, we have

$$|\partial_x^\alpha \zeta(|x|/t)| \le C(\alpha) < \infty, \qquad t\ge 1.$$

Furthermore, $1-\zeta(|x|/t) = 0$ for $|x|<t/4$. Then

$$||(1-\zeta)\mathcal{G}(t)\Psi'_t||_{\mathcal{F}_{-\sigma}} \le Ct^{-\sigma}||(1-\zeta)\mathcal{G}(t)\Psi'_t||_{\mathcal{F}_0} \le C_1 t^{-\sigma}||\mathcal{G}(t)\Psi'_t||_{\mathcal{F}_0}.$$

Hence, by the energy conservation and (2.20), we obtain

$$||(1-\zeta)\mathcal{G}(t)\Psi'_t||_{\mathcal{F}_{-\sigma}} \le C_2 t^{-\sigma}||\Psi'_t||_{\mathcal{F}_0} \le C_3 t^{-\sigma}||\Psi'_t||_{\mathcal{F}_\sigma} \le C_3 t^{-1}||\Psi||_{\mathcal{F}_\sigma}, \qquad t\ge 1, \qquad (2.23)$$

since $\sigma > 1$.
Step iii) Finally, let us estimate $\zeta\mathcal{G}(t)\Psi'_t$. Denote χ_t the characteristic function of the ball $|x| \le t/3$. We will use the same notation for the operator of multiplication by this characteristic function. By (2.21), we have

$$\zeta\mathcal{G}(t)\Psi'_t = \zeta\mathcal{G}(t)\chi_t\Psi'_t. \tag{2.24}$$

Formulas (2.17) - (2.18) imply that for any α the bounds hold

$$|\partial_z^\alpha \mathcal{G}(z,t)| \le C(\alpha,\varepsilon)t^{-1}, \quad |z| \le \varepsilon t, \quad t \ge 1. \tag{2.25}$$

Therefore,

$$|\partial_x^\alpha[\zeta(|x|/t)\mathcal{G}(x-y,t)\chi_t(y)]| \le Ct^{-1}, \quad |\alpha| \le 1, \quad t \ge 1 \tag{2.26}$$

since $\zeta(|x|/t) = 0$ for $|x| > t/3$ and $\chi_t(y) = 0$ for $|y| > t/3$. The norm of the operator $\zeta\mathcal{G}(t)\chi_t : \mathcal{F}_\sigma \to \mathcal{F}_{-\sigma}$ is equivalent to the norm of the operator

$$\langle x\rangle^{-\sigma}\zeta\mathcal{G}(t)\chi_t\langle y\rangle^{-\sigma} : \mathcal{F}_0 \to \mathcal{F}_0.$$

The norm of the later operator does not exceed the sum in α, $|\alpha| \le 1$ of the norms of operators

$$\partial_x^\alpha[\langle x\rangle^{-\sigma}\zeta\mathcal{G}(t)\chi_t\langle y\rangle^{-\sigma}] : L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2). \tag{2.27}$$

Estimates (2.26) imply that operators (2.27) are the Hilbert-Schmidt operators since $\sigma > 1$, and their Hilbert-Schmidt norms do not exceed Ct^{-1}. Hence, (2.20) and (2.24) imply that

$$||\zeta\mathcal{G}(t)\Psi'_t||_{\mathcal{F}_{-\sigma}} \le Ct^{-1}||\Psi'_t||_{\mathcal{F}_\sigma} \le C_1t^{-1}||\Psi||_{\mathcal{F}_\sigma}, \quad t \ge 1. \tag{2.28}$$

Finally, estimates (2.23) and (2.28) yield

$$||\mathcal{G}(t)\Psi'_t||_{\mathcal{F}_{-\sigma}} \le Ct^{-1}||\Psi_0||_{\mathcal{F}_\sigma}, \quad t \ge 1. \tag{2.29}$$

□

Formula (2.16) and decay (2.19) imply that

Corollary 2.5. *The asymptotics hold*

$$\mathcal{U}_0(t) = \mathcal{O}(t^{-1}), \quad t \to \infty \tag{2.30}$$

in the norm of $\mathcal{L}(L^2_\sigma; L^2_{-\sigma})$ *with* $\sigma > 1$.

Therefore, the decay of the free Dirac group $\mathcal{U}_0(t)$ is weaker than (1.6).

3 Perturbed equation

To prove a long time decay for the perturbed equation, we first establish the spectral properties of its generator $\mathcal{H}$.

3.1 Spectral properties

Similarly to [13, Formula (3.1)], let us introduce a generalized eigenspaces $\mathbf{M}^{\pm}$ of the operator $\mathcal{H}$:

$$\mathbf{M}^{\pm}=\{\psi\in H^1_{-1-0}:\ \ (1+\mathcal{B}_0^{\pm}\mathcal{V})\psi\in\Re(\mathcal{A}_0^{\pm}),\quad \mathcal{A}_0^{\pm}\mathcal{V}\psi=0\},$$

where $\mathcal{A}_0^{\pm}$ and $\mathcal{B}_0^{\pm}$ are defined in (2.13), and $\Re(\mathcal{A}_0^{\pm})$ denotes the range of $\mathcal{A}_0^{\pm}$. Below we assume that

$$\mathbf{M}^{\pm}=0. \tag{3.1}$$ SC

res-df **Remark 3.1.** *i) Note that $\mathcal{N}^{\pm}(\mathcal{H})\subset\mathbf{M}^{\pm}$, where $\mathcal{N}^{\pm}(\mathcal{H})$ are the eigenspaces of the operator $\mathcal{H}$ corresponding to the eigenvalues $\pm m$. Similar embedding is obtained in [13, Lemma 3.2] in the context of the Schrödinger operator. The functions from $\mathbf{M}^{\pm}\setminus\mathcal{N}^{\pm}(\mathcal{H})$ are called resonance functions. Hence, condition (3.1) means that $\lambda=\pm m$ are neither eigenvalues nor resonances for the operator $\mathcal{H}$.*
ii) In the case $V(x)=0$ we have $\mathbf{M}^{+}=\{(1,0)\}$ and $\mathbf{M}^{-}=\{(0,1)\}$, and hence the edge points $\lambda=\pm m$ are resonances for the operator $\mathcal{H}_0$.
iii) Conditions (3.1) hold for "generic" potentials. (cf. [9, 10]).

Denote by $\mathcal{R}(\omega)=(\mathcal{H}-\omega)^{-1}$, $\omega\in\mathbb{C}\setminus\Gamma$, the resolvent of the operator $\mathcal{H}$. The next lemma is the vector version of [13, Theorem 7.2].

R-b **Lemma 3.2.** *Let conditions (1.4) and (3.1) hold. Then for any $\sigma>1$ and for sufficiently small $\delta>0$ the families $\{\mathcal{R}(\pm m+\varepsilon):\pm m+\varepsilon\in\mathbb{C}\setminus\overline{\Gamma},|\varepsilon|<\delta\}$ are bounded in the operator norm of $\mathcal{L}(L^2_{\sigma},L^2_{-\sigma})$.*

Asymptotics (2.12) and lemma 3.2 imply

R-exp **Proposition 3.3.** *Let conditions (1.4) and (3.1) hold. Then for $\omega\in\mathbb{C}\setminus\overline{\Gamma}$ the asymptotics hold*

$$\left.\begin{aligned}
&\mathcal{R}(\omega)=\mathcal{A}^{\pm}+\mathcal{B}^{\pm}\log^{-1}(\omega\mp m)+\mathcal{O}(\log^{-2}(\omega\mp m))\\
&\mathcal{R}'(\omega)=-\mathcal{B}^{\pm}(\omega\mp m)^{-1}\log^{-2}(\omega\mp m)+\mathcal{O}(\omega\mp m)^{-1}\log^{-3}(\omega\mp m))\\
&\mathcal{R}''(\omega)=\mathcal{O}((\omega\mp m)^{-2}\log^{-2}(\omega\mp m))
\end{aligned}\right|\quad \omega\to\pm m \tag{3.2}$$ Rexp

in $\mathcal{L}(L^2_{\sigma},L^2_{-\sigma})$ with $\sigma>5/2$. Here $\mathcal{A}^{\pm}$, $\mathcal{B}^{\pm}\in\mathcal{L}(L^2_{\sigma},L^2_{-\sigma})$, $\sigma>5/2$, are the integral operators.

Proof. For concreteness we consider the sign "+". Lemma 3.2 implies that for any $1<\sigma<\rho/2$ the operators $(1+\mathcal{R}_0(\omega)\mathcal{V})^{-1}=1-\mathcal{R}(\omega)\mathcal{V}$ and $(1+\mathcal{V}\mathcal{R}_0(\omega))^{-1}=1-\mathcal{V}\mathcal{R}(\omega)$ are bounded in $\mathcal{L}(L^2_{-\sigma},L^2_{-\sigma})$ and in $\mathcal{L}(L^2_{\sigma},L^2_{\sigma})$ respectively for $|\omega-m|<\delta$, $\omega\in\mathbb{C}\setminus\overline{\Gamma}$ with sufficiently small $\delta>0$. Further, asymptotics (2.12) yield

$$\left.\begin{aligned}
&\mathcal{R}(\omega)=\big(1+\mathcal{R}_0(\omega)\mathcal{V}\big)^{-1}\mathcal{R}_0(\omega)=\big(1+\mathcal{R}_0(\omega)\mathcal{V}\big)^{-1}\big(\mathcal{A}_0^{+}\log(\omega-m)+\mathcal{B}_0^{+}+\mathcal{O}(\omega-m)^{\frac34}\big)\\
&\mathcal{R}(\omega)=\mathcal{R}_0(\omega)\big(1+\mathcal{V}\mathcal{R}_0(\omega)\big)^{-1}=\big(A_0^{+}\log(\omega-m)+\mathcal{B}_0^{+}+\mathcal{O}(\omega-m)^{\frac34}\big)\big(1+\mathcal{V}\mathcal{R}_0(\omega)\big)^{-1}
\end{aligned}\right| \tag{3.3}$$ RAR

as $\omega\to m$ in $\mathcal{L}(L^2_\sigma,L^2_{-\sigma})$ with $\sigma>5/2$. Hence, the boundedness of $\mathcal{R}(\omega)$, $(1+\mathcal{R}_0(\omega)\mathcal{V})^{-1}$, and $(1+\mathcal{V}\mathcal{R}_0(\omega))^{-1}$ at the point $\omega=m$ in the corresponding norms imply that

$$\left.\begin{array}{l}(1+\mathcal{R}_0(\omega)\mathcal{V})^{-1}\mathcal{A}_0^+=\mathcal{O}(\log^{-1}(\omega-m))\\[2ex] \mathcal{A}_0^\pm(1+\mathcal{V}\mathcal{R}_0(\omega))^{-1}=\mathcal{O}(\log^{-1}(\omega-m))\end{array}\right|\quad \omega\to\pm m,\quad \omega\in\mathbb{C}\setminus\overline{\Gamma} \tag{3.4}$$

in $\mathcal{L}(L^2_\sigma,L^2_{-\sigma})$ with $\sigma>5/2$. Therefore,

$$\|(1+\mathcal{R}_0(\omega)\mathcal{V})^{-1}[1]\|_{L^2_{-\sigma}}=\mathcal{O}(\log^{-1}(\omega-m)),\quad \omega\to\pm m,\quad \omega\in\mathbb{C}\setminus\overline{\Gamma},\quad \sigma>5/2, \tag{3.5}$$

where 1 stands for the constant vector function $f(x)\equiv(1,1)$, and for any $f\in L^2_\sigma$ with $\sigma>5/2$

$$\int[(1+\mathcal{V}\mathcal{R}_0(\omega))^{-1}f](x)dx=\mathcal{O}(\log^{-1}(\omega-m)),\quad \omega\to\pm m,\quad \omega\in\mathbb{C}\setminus\overline{\Gamma}. \tag{3.6}$$

Taking into account the identities

$$\mathcal{R}'=(1+\mathcal{R}_0\mathcal{V})^{-1}\mathcal{R}_0'(1+\mathcal{V}\mathcal{R}_0)^{-1},\quad \mathcal{R}''=\Big[(1+\mathcal{R}_0\mathcal{V})^{-1}\mathcal{R}_0''-2\mathcal{R}'\mathcal{V}\mathcal{R}_0'\Big](1+\mathcal{V}\mathcal{R}_0)^{-1} \tag{3.7}$$

we obtain from (3.5) - (3.6) the asymptotics

$$\mathcal{R}'(\omega)=\mathcal{O}((\omega-m)^{-1}\log^{-2}(\omega-m)),\quad \omega\to m,\quad \omega\in\mathbb{C}\setminus\overline{\Gamma}, \tag{3.8}$$

$$\mathcal{R}''(\omega)=\mathcal{O}((\omega-m)^{-2}\log^{-2}(\omega-m)),\quad \omega\to m,\quad \omega\in\mathbb{C}\setminus\overline{\Gamma}. \tag{3.9}$$

Integrating (3.8), we get

$$\mathcal{R}(\omega)=\mathcal{A}^++\mathcal{O}(\log^{-1}(\omega-m)),\quad \omega\to m,\quad \omega\in\mathbb{C}\setminus\overline{\Gamma} \tag{3.10}$$

in $\mathcal{L}(L^2_\sigma,L^2_{-\sigma})$ with $\sigma>5/2$. Now we can refine asymptotics (3.5) and (3.6). Namely, asymptotics (3.3) and (3.10) imply

$$\left.\begin{array}{l}(1+\mathcal{R}_0(\omega)\mathcal{V})^{-1}\mathcal{A}_0^+=\mathcal{D}_1^+\log^{-1}(\omega-m)+\mathcal{O}(\log^{-2}(\omega-m))\\[2ex] \mathcal{A}_0^+(1+\mathcal{V}\mathcal{R}_0(\omega))^{-1}=\mathcal{D}_2^+\log^{-1}(\omega-m)+\mathcal{O}(\log^{-2}(\omega-m))\end{array}\right|\quad \omega\to m,\quad \omega\in\mathbb{C}\setminus\overline{\Gamma} \tag{3.11}$$

in $\mathcal{L}(H^{-1}_\sigma,H^1_{-\sigma})$ with $\sigma>5/2$. Note, that $\mathcal{D}_2^+$ is one-dimensional operator. Substituting (3.11) and (2.12) into the first formula of (3.7), we obtain asymptotics (3.2) for the first derivative. Finally, integrating these asymptotics we obtain asymptotics (3.2) for $R(\zeta)$. Proposition 3.3 is proved. □

Finally, for $\mathcal{R}(\omega)$ the limiting absorption principle holds:

Lemma 3.4. *Let condition (1.4) holds. Then for $\omega\in\Gamma$ the convergence holds*

$$\mathcal{R}(\omega\pm i\varepsilon)\to\mathcal{R}(\omega\pm i0),\quad \varepsilon\to 0+ \tag{3.12}$$

in $\mathcal{L}(L^2_\sigma,L^2_{-\sigma})$ with $\sigma>1/2$.

The proof of the lemma in the case of 3D Dirac equation one can find in [17]. For the 2D equation the proof is similar. Note that under conditions (1.4) and (3.1) convergence (3.12) for ω close enough to $\pm m$ and $\sigma>1$ immediately follows from lemma 3.2 and the Lebesgue dominated convergence theorem. Below we use the limiting absorption principle just for such ω and σ.

3.2 Time decay

Our main result is the following.

main **Theorem 3.5.** *Let conditions (1.4) and (3.1) hold. Then for any $\sigma > 5/2$ one has*

$$\|e^{-it\mathcal{H}}P_c\|_{\mathcal{L}(L^2_\sigma,L^2_{-\sigma})} = \mathcal{O}(|t|^{-1}\log^{-2}|t|), \quad t\to\pm\infty. \tag{3.13}$$ maind

We estimate propagation near thresholds and propagation far from thresholds separately. Choose arbitrary $\chi \in C_0^\infty(\mathbb{R}^2,\mathbb{C}^2)$ with the support in a sufficiently small neighborhood of $[-m,m]$. Then

$$\|e^{-it\mathcal{H}}P_c\|_{\mathcal{L}(L^2_\sigma,L^2_{-\sigma})} \le \|e^{-it\mathcal{H}}P_c\chi(\mathcal{H})\|_{\mathcal{L}(L^2_\sigma,L^2_{-\sigma})} + \|e^{-it\mathcal{H}}P_c(1-\chi(\mathcal{H}))\|_{\mathcal{L}(L^2_\sigma,L^2_{-\sigma})}. \tag{3.14}$$ split

Propagation near thresholds.

p1 **Proposition 3.6.** *Let conditions (1.4) and (3.1) hold. Then for any $\sigma > 5/2$ one has*

$$\|e^{-it\mathcal{H}}P_c\chi(\mathcal{H})\|_{\mathcal{L}(L^2_\sigma,L^2_{-\sigma})} = \mathcal{O}(|t|^{-1}\log^{-2}|t|), \quad t\to\pm\infty. \tag{3.15}$$ split1

Proof. Lemmas 3.2 and 3.4 imply the representation (cf.(2.15)):

$$e^{-it\mathcal{H}}P_c\chi(\mathcal{H}) = \frac{1}{2\pi i}\int\limits_\Gamma e^{-i\omega t}\Big[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\Big]\chi(\omega)\,d\omega \tag{3.16}$$ Gin

in $\mathcal{L}(L^2_\sigma,L^2_{-\sigma})$ with $\sigma>1$. Then we can apply "two-dimensional" modification of Lemma 10.2 from [6]:

JK79 **Lemma 3.7.** *(see [11, Lemma 3.12]) Assume $\mathcal{B}$ be a Banach space, and $F\in C(a,b;\mathbf{B})$ satisfies $F(a)=0$ and $F(\omega)=0$ for $\omega>b>a$, $F'\in L^1(a+\delta,b;\mathbf{B})$ for any $\delta>0$. Moreover, $F'(\omega)=\mathcal{O}((\omega-a)^{-1}\ln^{-3}(\omega-a))$ as well as $F''(\omega)=\mathcal{O}((\omega-a)^{-2}\log^{-2}(\omega-a))$ as $\omega\to+a$. Then*

$$\int\limits_a^\infty e^{-it\omega}F(\omega)d\omega = \mathcal{O}(|t|^{-1}\ln^{-2}|t|), \quad t\to\pm\infty$$

.

Due to (3.2), we can apply this lemma with $F=\chi(\omega)\big(\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\big)$, $\mathcal{B}=\mathcal{L}(L^2_\sigma,L^2_{-\sigma})$ with $\sigma>5/2$ and appropriate values a and b to obtain (3.15). □

Propagation far from thresholds.

p3 **Proposition 3.8.** *Let condition (1.4) holds. Then for any $\sigma>0$ one has*

$$\|e^{-it\mathcal{H}}(1-\chi(\mathcal{H}))\|_{\mathcal{L}(L^2_\sigma,L^2_{-\sigma})} = \mathcal{O}(|t|^{-\sigma}), \quad t\to\pm\infty. \tag{3.17}$$ split2

Following [3] we deduce asymptotics (3.15) from the *minimal escape velocity estimates* obtained in [3, Theorem 2.1] for Dirac equation. Note that in [3] the 3D equation was considered, however in the 2D case the proof is similar. Denote

$$A = -i\frac{1}{2}\{\mathcal{H}_0^{-1}\nabla\cdot x + x\cdot\nabla\mathcal{H}_0^{-1}\},$$

and let $\mathbf{I}_M$ be the characteristic function of a set M.

Lemma 3.9. *(Minimal escape velocity estimate) Let condition (1.4) holds. Then there exists $\theta > 0$ such that for any $\gamma > 0$, $v \in (0, \theta)$ and $a \in \mathbb{R}$ one has*

$$\|\mathbf{I}_{A \le a+v|t|} e^{-itH}(1-\chi(H))\mathbf{I}_{A\ge a}\|_{\mathcal{L}(L^2,L^2)} \le C\langle t\rangle^{-\gamma}, \tag{3.18}$$

where C does not depends on a and t.

Proof of Proposition 3.8. We consider only integer $\sigma = 1, 2, ...$ since it is sufficient for our purposes. For any $c \ge 0$ one has

$$\langle A\rangle^{-\sigma} = \langle A\rangle^{-\sigma}\mathbf{I}_{\pm A\le c|t|} + \mathcal{O}(|t|^{-\sigma}), \quad |t| \ge 1$$

in $\mathcal{L}(L^2, L^2)$. Hence,

$$\langle A\rangle^{-\sigma}e^{-it\mathcal{H}}(1-\chi(\mathcal{H}))\langle A\rangle^{-\sigma} = \langle A\rangle^{-\sigma}\mathbf{I}_{A\le(\theta-\varepsilon)|t|/2}e^{-it\mathcal{H}}(1-\chi(\mathcal{H}))\mathbf{I}_{A\ge -\theta|t|/2}\langle A\rangle^{-\sigma} + \mathcal{O}(|t|^{-\sigma}).$$

Choosing $a = -\frac{\theta|t|}{2}$ and $v = \theta - \frac{\varepsilon}{2}$ in Lemma 3.9, we obtain

$$\|\langle A\rangle^{-\sigma}e^{-it\mathcal{H}}(1-\chi(\mathcal{H}))\langle A\rangle^{-\sigma}\|_{\mathcal{L}(L^2,L^2)} = \mathcal{O}(|t|^{-\sigma}), \quad t \pm \infty.$$

Now (3.15) follows since for integer σ the operators $\langle A\rangle^{-\sigma}\langle x\rangle^{\sigma}$ and $\langle x\rangle^{\sigma}\langle A\rangle^{-\sigma}$ are bounded in $\mathcal{L}(L^2, L^2)$ by the standard technique of PDOs (see. [2, 14]).

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