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On Convergence to Equilibrium Distribution for Schrödinger Equation

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Abstract

Consider the Schrödinger equation with constant or variable coefficients in $\mathbb{R}^3$. We study a distribution μ_t of a *random solution* at a time $t \in \mathbb{R}$. An initial measure μ_0 has translation-invariant correlation matrices, zero mean and finite mean density of energy. It also satisfies a Rosenblatt- or Ibragimov-Linnik-type mixing condition. The main result is the convergence of μ_t to a Gaussian measure as $t \to \pm\infty$ which gives the Central Limit Theorem for the Schrödinger equation. The proof for the case of constant coefficients is based on a spectral cutoff and an analysis of long time asymptotics of the solution in the Fourier representation and Bernstein's 'room-corridor' argument. The case of variable coefficients is reduced to that of constant ones by a version of a scattering theory for infinite energy solutions.

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1 Introduction

This paper studies a long-time behavior of random solutions of the Schrödinger equations. We prove the convergence to statistical equilibrium at $t = \pm\infty$. This is an extension of the results [4]-[7], where similar convergence is proved for wave and Klein-Gordon equations and for harmonic crystals. The case of the nonrelativistic Schrödinger equation presents new difficulties which are absent for the relativistic wave and Klein-Gordon equations studied in [4, 5]. Main novelties are connected with an infinite speed of propagation for the nonrelativistic Schrödinger equation. Even the existence of a random solution, with a finite local charge, is a nontrivial fact.

It is important to identify a natural property of the initial distribution guaranteeing the convergence to statistical equilibrium. We follow an idea of Dobrushin and Suhov [3] and use a 'space'-mixing condition of Rosenblatt- or Ibragimov-Linnik-type (see (1.5) below). Such a condition is natural from physical point of view. It replaces a 'quasiergodic hypothesis' and allows us to avoid introducing a 'thermostat' with a prescribed time-behaviour. Similar conditions have been used in [1, 2, 16, 17] and [4]-[7]. In this paper, mixing is defined and applied in the context of the Schrödinger equations.

We consider the linear Schrödinger equation in $\mathbb{R}^3$:

$$\left\{ \begin{array}{l} i\dot{\psi}(x,t) = L\psi(x,t) := -\Delta\psi(x,t) + V(x)\,\psi(x,t) \\ \psi\big|_{t=0} = \psi_0(x) \end{array} \right. \Bigg| \quad x \in \mathbb{R}^3, \qquad (1.1)$$

where $V(x)$ is a potential of an external electrostatic field. Denote

$$Y(t) = (Y^0(t), Y^1(t)) := (\mathrm{Re}\,\psi, \mathrm{Im}\,\psi),\ Y_0 = (Y_0^0, Y_0^1) := (\mathrm{Re}\,\psi_0, \mathrm{Im}\,\psi_0).$$

Then (1.1) becomes

$$\dot{Y}(t) = \mathcal{A}Y(t), \quad t \in \mathbb{R}; \quad Y(0) = Y_0. \qquad (1.2)$$

Here we denote

$$\mathcal{A} = \begin{pmatrix} 0 & -\Delta + V \\ \Delta - V & 0 \end{pmatrix}.$$

We suppose that the initial state Y_0 is a random element of a functional phase spase $\mathcal{H}_\alpha$, see Definition 2.3 below. The distribution of Y_0 is a probability measure μ_0 of mean zero satisfying some additional assumptions, see Conditions **S2**–**S4** below.

We assume that the initial correlation functions are translation-invariant, i.e. for $i, j = 0, 1$,

$$Q_0^{ij}(x,y) := E\Big(Y_0^i(x) Y_0^j(y)\Big) = q_0^{ij}(x-y), \ \ x, y \in \mathbb{R}^3. \qquad (1.3)$$

Next, we assume that the initial 'mean energy density' is finite:

$$e_0 := E\,|Y_0(x)|^2 = q_0^{00}(0) + q_0^{11}(0) < \infty. \tag{1.4}$$

Finally, we assume that μ_0 satisfies a mixing condition. Roughly speaking, it means that

$$Y_0(x) \quad \text{and} \quad Y_0(y) \text{ are asymptotically independent as } |x-y| \to \infty. \tag{1.5}$$

Denote by μ_t, $t \in \mathbb{R}$, the measure on $\mathcal{H}_\alpha$ giving the distribution of the random solution $Y(t)$ to problem (1.2). Our main result is the (weak) convergence

$$\mu_t \rightharpoondown \mu_\infty, \quad t \to \infty \tag{1.6}$$

to a limiting measure μ_∞ which is a stationary Gaussian probability measure on $\mathcal{H}_\alpha$.

We prove the convergence (1.6) using the following strategy. At first, we prove (1.6) for the free equations (i.e. $V(x) \equiv 0$) in three steps.

I. The family of measures μ_t, $t \geq 0$, is compact in an appropriate space.

II. The correlation functions converge to a limit: for $i,j = 0,1$

$$Q_t^{ij}(x,y) := \int Y^i(x) Y^j(y) \mu_t(dY) \to Q_\infty^{ij}(x,y), \quad t \to \infty. \tag{1.7}$$

III. The characteristic functionals converge to the Gaussian:

$$\hat{\mu}_t(\Psi) = \int \exp\{i\,\langle Y, \Psi\rangle\}\,\mu_t(dY) \to \exp\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi,\Psi)\}, \quad t \to \infty, \tag{1.8}$$

where Ψ is an arbitrary element of the dual space and $\mathcal{Q}_\infty$ is the quadratic form with the integral kernel $\big(Q_\infty^{ij}(x,y)\big)_{i,j=0,1}$.

Property I follows from the Prokhorov Compactness Theorem by methods of [20]. First, one proves a uniform bound for the mean local energy in measure μ_t with the help of the Fourier transform. The conditions of Prokhorov's Theorem then follow from Sobolev's Embedding Theorem. Property II is deduced from an analysis of oscillatory integrals arising in the Fourier transform. An important role is attributed to Proposition 4.1 below, reflecting properties of the correlation functions in the Fourier transform, which are deduced from the mixing condition.

The proof of the convergence (1.8) is based on Bernstein's 'room-corridor' method. We modify the approach developed in [4] for the Klein-Gordon equation and [5] for the wave equation. We construct a representation of the solution as a sum of weakly dependent random variables. Then (1.8) follows from the Ibragimov-Linnik Central Limit Theorem [12]. All three steps I-III of the argument rely on the mixing condition.

Finally, we prove the convergence in (1.6) for the problem (1.1) with a potential $V(x)$. In this case explicit formulas for the solution are unavailable. To prove (1.6) in this case, we construct a version of a scattering theory for infinite energy solution. This allows us to reduce the proof to the case of constant coefficients.

Our approach is a modification of methods [4, 5]. In particular, we have two new difficulties caused by an infinite speed of propagation for the non-relativistic Schrödinger equation:

i) First, this hinders the proof of the existence of a dynamics in the space $L^2_{loc}(\mathbb{R}^3)$. Namely, a translation-invariant random initial function is not in $L^2(\mathbb{R}^3)$ with probability one. Therefore, we have to construct a dynamics for initial functions which do not belong to $L^2(\mathbb{R}^3)$. For $V(x) \equiv 0$ it is easy to construct a solution in the space of tempered distribution $C(\mathbb{R}, S'(\mathbb{R}^3))$ using the Fourier transform. However, we have to prove that the solution is in $L^2_{loc}(\mathbb{R}^3)$ since physical solutions have a finite local charge. This fact is non-trivial even for the case $V(x) \equiv 0$. We prove that the solution belongs to $L^2_{loc}(\mathbb{R}^3)$ with probability one using the translation-invariance of the random process and finite mean energy density. (However, we cannot prove this for any fixed initial function which is not in $L^2(\mathbb{R}^3)$). Our construction relies on the Wiener isometry idea which is standard in the theory of stochastic integrals. The solution is a random process defined with probability one.

ii) The second novelty arises in the 'room-corridor' Bernstein's method. Namely, the number of the rooms is a priori infinite for every $t > 0$, since the speed of the propagation is infinite. To avoid this difficulty, we first choose a test function $\Psi(x)$ with a compact support in the Fourier space:

$$\hat{\Psi}(k) = 0, \ |k| > R. \tag{1.9}$$

Then an analogue of the Huyghen's principle holds for solution $\Phi(t, x)$ of the free Schrödinger equation with the initial function $\Psi(x)$: for any $p \geq 0$

$$|\Phi(t,x)| \leq C(p)(1 + |t| + |x|)^{-p}, \ |x| > 2Rt,$$

since $2R$ is the maximal value of the group velocity $|\nabla k^2|$ for $|k| \leq R$. Then we get rid of the spectral restriction (1.9).

The paper is organized as follows. In Section 2 we state our main result. Sections 3-7 deal with the case of constant coefficients: the main results are stated in Section 3, the compactness (Property **I**) and the convergence (1.7) are proved in Section 4, and the convergence (1.8) in Sections 5,6. In Section 7 we check the Lindeberg condition. In Section 9 we construct the scattering theory, and in Section 10 we establish convergence (1.6) for variable coefficients. In Appendix A we have collected the Fourier calculations. In Appendix B we construct the solution as a random process using the Wiener isometry method.

2 Main results

2.1 Notations

Definition 2.1. *Denote by D the space of real functions $C_0^\infty(\mathbb{R}^3)$. We set $\mathcal{D} = D \oplus D$.*

Definition 2.2. *For any real α, s denote by $H^s_\alpha(\mathbb{R}^3)$ the space of real valued functions on $\mathbb{R}^n$ defined by*

$$H^s_\alpha(\mathbb{R}^3) = \left\{ u(x) : (1+|x|^2)^{\alpha/2} u(x) \in H^s(\mathbb{R}^3) \right\}. \tag{2.1}$$

Definition 2.3. *$\mathcal{H}^s_\alpha$ is the space of pairs $Y(x) \equiv (Y^0(x), Y^1(x)) \in H^s_\alpha(\mathbb{R}^3) \oplus H^s_\alpha(\mathbb{R}^3)$, $\mathcal{H}_\alpha = \mathcal{H}^0_\alpha$.*

We set $\langle Y, \Psi\rangle = \langle Y^0, \Psi^0\rangle + \langle Y^1, \Psi^1\rangle$ for $Y = (Y^0, Y^1) \in \mathcal{H}_\alpha$, and $\Psi = (\Psi^0, \Psi^1) \in \mathcal{D}$. We assume for the potential $V(x)$ the following conditions:

E1. $V(x)$ is a real-valued measurable function and

$$|V(x)| \le C(1+|x|)^{-\beta},\ \beta > 4.$$

Further, we assume the absence of a discrete negative spectrum of the Schrödinger operator $H = -\Delta + V$:

E2. A discrete negative spectrum of H does not exist.

The decay imposed on $V(x)$ in **E1** eliminates the existence of positive eigenvalues by Kato Theorem and implies the following Puiseux expansion for the resolvent $R(\zeta) = (H-\zeta)^{-1}$ (see [10]):

$$R(\zeta) = -\zeta^{-1} P_0 + \zeta^{-1/2} C_{-1} + C_0 + \zeta^{1/2} C_1 + O(\zeta),\ \zeta \to 0. \tag{2.2}$$

The asymptotics hold in appropriate Agmon's norms . Here $P_0 = 0$ if the point $\zeta = 0$ is not an eigenvalue of H, and $C_{-1} = 0$ if the equation $H\Psi = 0$ does not have resonance (i.e. a non-zero solution in a space slightly larger than $L^2(\mathbb{R}^3)$). It is well known that, for a generic potential V, we have $P_0 = 0$ and $C_{-1} = 0$, which is called, by definition ([10]), that $\zeta = 0$ is a regular point of the resolvent. We will assume:

E3. $\zeta = 0$ is a regular point of the resolvent $R(\zeta)$.

The conditions **E2** and **E3** provide a good decay of the nonstationary finite energy solutions, as $t^{-3/2}$ (see [10]). The decay follows by the Fourier transform $F_{\zeta\to t}$ of (2.2), with $P_0 = C_{-1} = 0$, since then the resolvent is sufficiently smooth at the singular point $\zeta = 0$. The decay will play a key role in our construction of the scattering theory for the case of the perturbed Schrödinger equation with $V(x) \not\equiv 0$.
Example The conditions **E1**– **E3** are satisfied if $V(x) \in C_0^\infty$ is a nonnegative function. It follows from [19, Thms 13,14].

2.2 Random solution. Convergence to equilibrium

Let (Ω, Σ, P) be a probability space with the expectation E, and $\mathcal{B}(\mathcal{H}_\alpha)$ denote the Borel σ-algebra in $\mathcal{H}_\alpha$ with $-\beta \le \alpha < -3/2$. We assume that $Y_0 = Y_0(\omega, \cdot)$ in (1.2) is a measurable random function with values in $\big(\mathcal{H}_\alpha,\, \mathcal{B}(\mathcal{H}_\alpha)\big)$. In other words, $(\omega, x) \mapsto Y_0(\omega, x)$ is a measurable map $\Omega \times \mathbb{R}^n \to \mathbb{R}^2$ with respect to the (completed) σ-algebras $\Sigma \times \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^2)$. Denote $\mathcal{L}^2_\alpha = L^2(\Omega, \Sigma, P; \mathcal{H}_\alpha)$, $\mathcal{C}_\alpha = C(\mathbb{R}, \mathcal{L}^2_\alpha)$.

Proposition 2.1. *Let **E1** holds and $-\beta \le \alpha < -3/2$. Then*
i) There exists a unique random process $Y(t) \in \mathcal{C}_\alpha$ which is a solution to the Cauchy problem (1.2) *in the following sense: for any $\Phi \in \mathcal{D}$*

$$\langle Y(t), \Phi\rangle = \langle Y_0, \Phi\rangle + \int_0^t \langle Y(s), \mathcal{A}'\Phi\rangle\, ds, \quad t \in \mathbb{R}, \tag{2.3}$$

where the identity (2.3) *holds in $L^2(\Omega, \Sigma, P)$ and*

$$\mathcal{A}' = \begin{pmatrix} 0 & \Delta - V \\ -\Delta + V & 0 \end{pmatrix}. \tag{2.4}$$

ii)The operator $\mathcal{U}(t) : Y_0 \to Y(t)$ is a continuous linear map in $\mathcal{L}^2_\alpha$.

Remark 2.1. *The existence of dynamics does not relies on the mixing condition* (1.5) *(see **S4**).*

We prove the Proposition 2.1 in Appendix B. Denote by $\mu_0(dY_0)$ the Borel probability measure in $\mathcal{H}_\alpha$ which is the distribution of the Y_0. Without loss of generality, we may assume $(\Omega, \Sigma, P) = (\mathcal{H}_\alpha,\, \mathcal{B}(\mathcal{H}_\alpha),\, \mu_0)$ and $Y_0(\omega, x) = \omega(x)$ for $\mu_0(d\omega)\times dx$-almost all $(\omega, x) \in \mathcal{H}_\alpha \times \mathbb{R}^n$.

Definition 2.4. *μ_t is the Borel probability measure in $\mathcal{H}_\alpha$ which is the distribution of $Y(t)$:*

$$\mu_t(B) = P\big(Y(t) \in B\big), \quad \forall B \in \mathcal{B}(\mathcal{H}_\alpha), \quad t \in \mathbb{R}.$$

Our main goal is to derive the weak convergence of the measures μ_t in the Hilbert spaces $\mathcal{H}_\alpha^{-\varepsilon}$ with $-\beta \le \alpha < -5/2$, $\varepsilon > 0$:

$$\mu_t \overset{\mathcal{H}_\alpha^{-\varepsilon}}{\longrightarrow} \mu_\infty, \quad t \to \infty, \tag{2.5}$$

where μ_∞ is a limiting measure on the space $\mathcal{H}_\alpha^{-\varepsilon}$. This means the convergence

$$\int f(Y)\, \mu_t(dY) \to \int f(Y)\, \mu_\infty(dY) \quad \text{as} \quad t \to \infty$$

for any bounded continuous functional f on the space $\mathcal{H}_\alpha^{-\varepsilon}$.

Definition 2.5. *The correlation functions of a measure μ_t are defined by*

$$Q_t^{ij}(x, y) := E\big(Y^i(x,t) Y^j(y,t)\big), \; i, j = 0, 1, \tag{2.6}$$

where the equality is understood in the sense of distributions, i.e.

$$\langle Q_t^{ij}(x,y), \Psi(x)\Psi(y)\rangle = E\langle Y^i, \Psi\rangle\langle Y^j, \Psi\rangle, \; \forall \Psi \in D.$$

For a probability measure μ on $\mathcal{H}_\alpha$, denote by $\hat{\mu}$ the characteristic functional (Fourier transform)

$$\hat{\mu}(\Psi) := \int \exp\{i\, \langle Y, \Psi\rangle\}\, \mu(dY), \quad \Psi \in \mathcal{D}.$$

A probability measure μ is called Gaussian probability measure (of mean zero) if its characteristic functional has the form

$$\hat{\mu}(\Psi) = \exp\Big\{-\frac{1}{2}\mathcal{Q}(\Psi, \Psi)\Big\}, \quad \Psi \in \mathcal{D},$$

where $\mathcal{Q}$ is a real nonnegative quadratic form in $\mathcal{D}$. A measure μ is called translation-invariant if

$$\mu(T_h B) = \mu(B), \quad \forall B \in \mathcal{B}(\mathcal{H}_\alpha), \quad h \in \mathbb{R}^n,$$

where $T_h Y(x) = Y(x - h)$, $x \in \mathbb{R}^n$.

2.3 Mixing condition

Let $O(r)$ denote the set of all pairs of open bounded subsets $\mathbf{A}$, $\mathbf{B} \subset \mathbb{R}^3$ at a distance $\mathrm{dist}(\mathbf{A}, \mathbf{B}) \geq r$ and let $\sigma(\mathbf{A})$ be the σ-algebra of the subsets in $\mathcal{H}_\alpha$ generated by all linear functionals $Y \mapsto \langle Y, \Psi\rangle$, where $\Psi \in \mathcal{D}$ with $\mathrm{supp}\,\Psi \subset \mathbf{A}$. We define the Ibragimov-Linnik mixing coefficient of a probability measure μ_0 on $\mathcal{H}_\alpha$ by (cf [12, Def 17.2.2])

$$\varphi(r) \equiv \sup_{(\mathbf{A},\mathbf{B})\in O(r)} \sup_{\substack{A\in\sigma(\mathbf{A}),\, B\in\sigma(\mathbf{B}),\\ \mu_0(B)>0}} \frac{|\mu_0(A\cap B) - \mu_0(A)\mu_0(B)|}{\mu_0(B)}.$$

Definition 2.6. *A measure μ_0 satisfies the strong uniform Ibragimov-Linnik mixing condition if*

$$\varphi(r) \to 0, \quad r \to \infty.$$

Below, we specify the rate of the decay of φ (see Condition **S4**).

2.4 Main theorem

We assume that the measure μ_0 satisfies the following conditions:

S1. μ_0 has zero expectation value,

$$EY_0(x) \equiv 0, \quad a.a.\ x \in \mathbb{R}^3.$$

S2. μ_0 has translation invariant correlation functions, i.e. (1.3) holds for almost all $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$.

S3. μ_0 has a finite mean energy density, i.e. (1.4) holds for almost all $x \in \mathbb{R}^3$.

S4. μ_0 satisfies the strong uniform Ibragimov-Linnik mixing condition, with

$$\bar{\varphi} \equiv \int_0^\infty r^2 \varphi^{1/2}(r)\, dr < \infty. \tag{2.7}$$

Define, for almost all $x, y \in \mathbb{R}^3$, the matrix-valued function

$$Q_\infty(x, y) = \Big(Q_\infty^{ij}(x, y)\Big)_{i,j=0,1} = \Big(q_\infty^{ij}(x - y)\Big)_{i,j=0,1},$$

where

$$\Big(q_\infty^{ij}\Big)_{i,j=0,1} = \frac{1}{2}\begin{pmatrix} q_0^{00} + q_0^{11} & q_0^{01} - q_0^{10} \\ q_0^{10} - q_0^{01} & q_0^{00} + q_0^{11} \end{pmatrix}. \tag{2.8}$$

Denote by $\mathcal{Q}_\infty$ the real quadratic form in $\mathcal{H}_0$ defined by

$$\mathcal{Q}_\infty(\Psi,\Psi)=\sum_{i,j=0,1}\int\limits_{\mathbb{R}^n\times\mathbb{R}^n} Q_\infty^{ij}(x,y)\Psi^i(x)\Psi^j(y)\,dx\,dy. \tag{2.9}$$

The form $\mathcal{Q}$ is continuous in $\mathcal{H}_0$ as the functions $Q_\infty^{ij}(x,y)$ are bounded by **S3**. Our main result is the following theorem.

Theorem A. *Let **E1**– **E3**, **S1**–**S4** hold and $-\beta\le\alpha<-5/2$. Then*
i) The convergence (2.5) *holds for any $\varepsilon>0$.*
ii) The limiting measure μ_∞ is a Gaussian measure on $\mathcal{H}_\alpha$.
iii) The limiting characteristic functional has the form

$$\hat\mu_\infty(\Psi)=\exp\Big\{-\frac{1}{2}\mathcal{Q}_\infty(W\Psi,W\Psi)\Big\},\quad \Psi\in\mathcal{D},$$

where $W:\mathcal{D}\to\mathcal{H}_\alpha$ is a linear continuous operator.
iv) $W=I$ in the case $V(x)\equiv 0$.

3 Free Schrödinger equations

In Sections 3–8 we consider the Cauchy problem (1.1) with $V(x)\equiv 0$, i.e.

$$\begin{cases} i\dot\psi(x,t)=-\Delta\psi(x,t), & x\in\mathbb{R}^3,\\ \psi\big|_{t=0}=\psi_0(x). \end{cases} \tag{3.1}$$

Rewrite (3.1) in the form similar to (1.2):

$$\dot Y(t)=\mathcal{A}_0Y(t),\quad t\in\mathbb{R};\quad Y(0)=Y_0, \tag{3.2}$$

where

$$\mathcal{A}_0=\begin{pmatrix}0 & -\Delta\\ \Delta & 0\end{pmatrix}. \tag{3.3}$$

Denote by $U_0(t)$, $t\in\mathbb{R}$, the dynamical group for the problem (3.2):

$$Y(x,t)=U_0(t)Y_0(x)=F_{k\to x}\hat{\mathcal{G}}_t(k)\hat Y_0(k).$$

Then $Y(t)=U_0(t)Y_0$ a.s. by Remark B.1 i). Therefore, Definition 2.4 implies that in this case

$$\mu_t(B)=\mu_0\big(U_0(-t)B\big),\quad B\in\mathcal{B}(\mathcal{H}_\alpha),\quad t\in\mathbb{R}.$$

The main result for the problem (3.2) is the following

Theorem B. *Let conditions* ***S1–S4*** *hold and* $\alpha < -3/2$. *Then the conclusions of Theorem A hold with* $W = I$, *and the limiting measure* μ_∞ *is translation-invariant.*

Theorem B can be deduced from Propositions 3.1 and 3.2 below, by the same arguments as in [20, Thm XII.5.2].

Proposition 3.1. *The family of measures* $\{\mu_t, t \ge 0\}$ *is weakly compact in* $\mathcal{H}_\alpha^{-\varepsilon}$ *with any* $\alpha < -3/2, \varepsilon > 0$, *and the bounds hold:*

$$\sup_{t\ge 0} E\,\|U_0(t)Y_0\|^2_{\mathcal{H}_\alpha} < \infty. \tag{3.4}$$

Let $\mathcal{S} := S(\mathbb{R}^3) \oplus S(\mathbb{R}^3)$ denote the Schwartz space of smooth test functions with rapid decay at infinity.

Proposition 3.2. *For any* $\Psi \in \mathcal{S}$,

$$\hat\mu_t(\Psi) \equiv \int \exp\Big\{i\,\langle Y, \Psi\rangle\Big\}\,\mu_t(dY) \to \exp\Big\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi,\Psi)\Big\}, \quad t\to\infty. \tag{3.5}$$

Propositions 3.1 and 3.2 are proved in Sections 4 and 5–8, respectively.

4 Compactness of the measures family

Here we prove Proposition 3.1 with the help of Fourier transform.

4.1 Mixing in terms of spectral density

The next proposition gives the mixing condition in terms of the Fourier transform $\hat q_0^{ij} := F q_0^{ij}$ of the initial correlation functions q_0^{ij}. Let us note that $q_0^{ij}(z)$ is a measurable bounded function by **S3**.

Proposition 4.1. *Let the assumptions of Theorem B hold. Then* $\hat q_0^{ij}(k) \in L^p(\mathbb{R}^3)$ *with any* $p \in [1,\infty]$.

Proof. Conditions **S1**, **S3** and **S4** imply by [12, Lemma 17.2.3] (see also Lemma 6.2-i) below) that the functions q_0^{ij} are bounded by mixing coefficient:

$$\big|q_0^{ij}(z)\big| \le C e_0 \varphi^{1/2}\big(|z|\big), \quad z\in\mathbb{R}^3. \tag{4.1}$$

The bounds (4.1) and (2.7) imply that $q_0^{ij}(x) \in L^1(\mathbb{R}^3)$, hence $\hat q_0^{ij}(k) \in C(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. Now consider the case $i = j$. The function $\hat q_0^{ii}$ is nonnegative by the Bohner theorem. Hence,

$$\int \big|\hat q_0^{ii}(k)\big|\,dk = \int \hat q_0^{ii}(k)\,dk = q_0^{ii}(0) < \infty,$$

due to **S3**. Hence, $\hat{q}_0^{ii}(k) \in L^p(\mathbb{R}^3)$ with any $p \in [1,\infty]$ by interpolation. Finally, for the case $i \neq j$ we use again the Bohner theorem. Namely, the matrix $\big(\hat{q}_0^{ij}(k)\big)_{i,j=0,1}$ is symmetric and nonnegative, hence its determinant is nonnegative. Therefore, $\left|\hat{q}_0^{ij}(k)\right|^2 \le \hat{q}_0^{00}(k)\hat{q}_0^{11}(k)$, $k \in \mathbb{R}^3$. □

Corollary 4.1. *Formula* (2.8) *implies that the functions* $\hat{q}_\infty^{ij}(k)$ *belong to* $L^p(\mathbb{R}^3)$ *with any* $p \in [1,\infty]$, $i,j = 0,1$.

4.2 Compactness

We now prove the bound (3.4). Proposition 3.1 then can be deduced with the help of the Prokhorov Theorem [20, Lmm II.3.1] as in [20, Thm XII.5.2]. The formulas (A.2), (A.3), and Proposition 4.1 imply that for almost all $x, y \in \mathbb{R}^3 \times \mathbb{R}^3$

$$E\Big(Y^0(x,t)Y^0(y,t)\Big) := q_t^{00}(x-y) = \frac{1}{(2\pi)^3}\int e^{-ik(x-y)}\Big[\frac{1+\cos 2\,|k|^2\,t}{2}\hat{q}_0^{00}(k) + \frac{\sin 2\,|k|^2\,t}{2}\big(\hat{q}_0^{01}(k)+\hat{q}_0^{10}(k)\big) + \frac{1-\cos 2\,|k|^2\,t}{2}\hat{q}_0^{11}(k)\Big]\,dk, \quad (4.2)$$

where the integral converges and defines a continuous function of $x - y$. Similar representations hold for q_t^{ij} with all $i,j = 0,1$. Therefore, we have as in (1.4),

$$e_t := q_t^{00}(0) + q_t^{11}(0) = \frac{1}{(2\pi)^3}\int\Big(\hat{q}_t^{00}(k) + \hat{q}_t^{11}(k)\Big)\,dk. \quad (4.3)$$

It remains to estimate the last integral. (4.2) implies the following representation for $\hat{q}_t^{00}$:

$$\hat{q}_t^{00}(k) = \frac{1+\cos 2\,|k|^2 t}{2}\hat{q}_0^{00}(k) + \frac{\sin 2\,|k|^2 t}{2}\big(\hat{q}_0^{01}(k)+\hat{q}_0^{10}(k)\big) + \frac{1-\cos 2\,|k|^2 t}{2}\hat{q}_0^{11}(k) \quad (4.4)$$

Similarly, formulas (A.2), (A.3) imply that

$$\hat{q}_t^{11}(k) = \frac{1-\cos 2\,|k|^2 t}{2}\hat{q}_0^{00}(k) - \frac{\sin 2\,|k|^2 t}{2}\big(\hat{q}_0^{01}(k)+\hat{q}_0^{10}(k)\big) + \frac{1-\cos 2\,|k|^2 t}{2}\hat{q}_0^{11}(k) \quad (4.5)$$

Therefore, (4.3) implies that $e_t \le C_1(\varphi)\,e_0$. Hence, according to (B.6), for any $\alpha < -3/2$

$$E\,\|U_0(t)Y_0\|^2_{\mathcal{H}_\alpha} = e_t\int\big(1+|x|^2\big)^\alpha\,dx = C_\alpha e_t \le C_1(\varphi)\,e_0. \quad (4.6)$$

Corollary 4.2. *By the Fubini theorem, the bound* (4.6) *implies the convergence of the integrals in* (2.6) *for a.a.* $x, y \in \mathbb{R}^3 \times \mathbb{R}^3$.

4.3 Convergence of covariance functions

This convergence is used in Section 6.

Lemma 4.1. *The following convergence holds for $i, j = 0, 1$:*

$$q_t^{ij}(z) \to q_\infty^{ij}(z), \quad t \to \infty, \quad z \in \mathbb{R}^3. \tag{4.7}$$

Proof. (4.4) and (4.5) imply the convergence for $i = j$: the oscillatory terms there converge to zero by the Lebesgue-Riemann theorem, since they are absolutely continuous and summable by Proposition 4.1. For $i \neq j$ the proof is similar. □

We will use the convergence (4.7) we will use in Section 6.

5 Bernstein's argument

In this and subsequent section we develop a version of Bernstein's 'room-corridor' method. We use the standard integral representation for solutions, divide the domain of integration into 'rooms' and 'corridors' and evaluate their contribution. As a result, $\langle U_0(t)Y_0, \Psi\rangle$ is represented as the sum of weakly dependent random variables. We evaluate the variances of these random variables which will be important in next section. At first, we will prove Proposition 3.2 under an additional condition on the function $\Psi \in \mathcal{S}$:

$$\text{SPECTRAL CONDITION}: \qquad \hat{\Psi}(k) = 0, \qquad |k| > R_0. \tag{5.1}$$

We will get rid of this condition in Section 8.

5.1 Dual dynamics

First, we evaluate $\langle Y(t), \Psi\rangle$ in (3.5) by using a duality arguments. For $t \in \mathbb{R}$, introduce a 'formal adjoint' operator $U_0'(t)$ in the space $\mathcal{S}$:

$$\langle Y, U_0'(t)\Psi\rangle = \langle U_0(t)Y, \Psi\rangle\,, \quad \Psi \in \mathcal{S}, \quad Y \in \mathcal{S}', \quad t \in \mathbb{R}. \tag{5.2}$$

The adjoint groups admit a convenient description. Lemma 5.1 below displays that the action of the group $U_0'(t)$ coincides with the action of $U_0(t)$ up to the order of the components. In particular, $U_0'(t)$ is a continuous operator in $\mathcal{S}$.

Lemma 5.1. *For $\Psi = (\Psi^0, \Psi^1) \in \mathcal{S}$*

$$U_0'(t)\Psi = \big(\operatorname{Im}\phi(\cdot, t), \operatorname{Re}\phi(\cdot, t)\big), \tag{5.3}$$

where $\phi(x, t)$ is the solution of Eq. (3.1) with the initial state $\psi_0 = \Psi^1 + i\Psi^0$.

Proof. Differentiating (5.2) with $Y, \Psi \in \mathcal{S}$, we obtain

$$\left\langle Y, \dot{U}_0'(t)\Psi \right\rangle = \left\langle \dot{U}_0(t)Y, \Psi \right\rangle.$$

The group $U_0(t)$ has the generator (3.3). Hence, the generator of $U_0'(t)$ is the 'transposed' operator

$$\mathcal{A}_0' = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}. \tag{5.4}$$

Therefore, (5.3) holds with $i\dot{\phi} = -\Delta\phi$. □

Denote $\Phi(\cdot, t) = U_0'(t)\Psi$. Then (5.2) means that

$$\langle Y(t), \Psi \rangle = \langle Y_0, \Phi(\cdot, t) \rangle, \quad t \in \mathbb{R}. \tag{5.5}$$

Further, (5.4) and (A.1) imply that in the Fourier representation, $\dot{\hat{\Phi}}(k,t) = \hat{\mathcal{A}}_0'(k)\hat{\Phi}(k,t)$ and $\hat{\Phi}(k,t) = \hat{\mathcal{G}}_t'(k)\hat{\Psi}(k)$, where $\hat{\mathcal{G}}_t'(k)$ is the transposed matrix to $\hat{\mathcal{G}}_t(k)$. Therefore,

$$\Phi(x,t) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}^3} e^{-ikx} \hat{\mathcal{G}}_t'(k)\hat{\Psi}(k)\, dk. \tag{5.6}$$

Now we can obtain the long-time asymptotics of the function $\Phi(x,t)$.

Lemma 5.2. *Let* (5.1) *hold. Then for any* $p > 0$

$$|\Phi(x,t)| \le C(\Psi)(1+t)^{-3/2}, \qquad |x| < 2R_0 t, \tag{5.7}$$

$$|\Phi(x,t)| \le C(\Psi,p)(1+|x|+t)^{-p}, \quad |x| > 2R_0 t. \tag{5.8}$$

Proof. Denote $\phi(x,t) := \Phi^1(x,t) + i\Phi^0(x,t)$, $\psi_0(x) := \Psi^1(x) + i\Psi^0(x)$. Then $\phi(x,t)$ is the solution of Eq.(3.1) and ϕ can be written as the integral

$$\phi(x,t) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}^3} \exp\bigl\{-i(kx + |k|^2 t)\bigr\}\, \hat{\psi}_0(k)\, dk, \tag{5.9}$$

Let us prove the asymptotics (5.7) along each ray $x = vt$ with any $v \in \mathbb{R}^3$. We get from (5.9)

$$\phi(vt,t) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}^3} \exp\bigl\{-it(kv + |k|^2)\bigr\}\, \hat{\psi}_0(k)\, dk. \tag{5.10}$$

This is an oscillatory integral with the phase function $\omega(k) = kv + |k|^2$ and $\nabla\omega_(k) = v + 2k$. If $k = -v/2 \in \operatorname{supp}\hat{\Psi}$, the phase function has one stationary point and the Hessian is nondegenerate everywhere. Hence, $\phi(x,t) = \mathcal{O}(t^{-3/2})$ according to the standard stationary phase method (see [14]). If $|v| > 2R_0$, the phase function does not have stationary points. Therefore, $|\phi(x,t)| \le C(\Psi,p)(1+|x|+t)^{-p}$ for any $p > 0$. □

5.2 Rooms and corridors

We are going to partition the ball $B_{2R_0t} = \{x \in \mathbb{R}^3 : |x| \le 2R_0t\}$ into N_t 'rooms' R_t^j, separated by 'corridors' C_t^j. Given $t > 1$, choose $d \equiv d_t \ge 1$ and $\rho \equiv \rho_t > 0$. Asymptotical relations between t, d_t and ρ_t are specified below. Define $h = d + \rho$ and

$$a^j = -2R_0t + (j-1)h, \quad b^j = a^j + d, \quad 1 \le j \le N_t, \quad N_t \sim \frac{t}{h}. \tag{5.11}$$

Let us set $R_t^j = \{x \in B_{2R_0t} : a^j \le x^3 \le b^j\}$ and $C_t^j = \{x \in B_{2R_0t} : b^j \le x^3 \le a^{j+1}\}$. Here $x = (x^1, \dots, x^3)$, d is the width of the room, and ρ of the corridor. Now we have

$$B_{2R_0t} = \bigcup_{j=1}^{N_t} \big(R_t^j \cup C_t^j\big).$$

Denote by χ_t^j, ξ_t^j, η_t the indicators of the sets R_t^j, C_t^j and $\mathbb{R}^3 \setminus B_{2R_0t}$, respectively. Let us define random variables r_t^j, c_t^j, l_t by formulas

$$r_t^j = \Big\langle Y_0, \chi_t^j \Phi(\cdot, t)\Big\rangle, \; c_t^j = \Big\langle Y_0, \xi_t^j \Phi(\cdot, t)\Big\rangle, \; l_t = \Big\langle Y_0, \eta_t \Phi(\cdot, t)\Big\rangle, \;\; 1 \le j \le N_t. \tag{5.12}$$

Then (5.5) implies that

$$\langle U_0(t) Y_0, \Psi \rangle = \sum_1^{N_t} (r_t^j + c_t^j) + l_t. \tag{5.13}$$

Lemma 5.3. *Let **S1**–**S4** hold. Then the following bounds hold for* $t > 1$ *and* $1 \le j \le N_t$

$$E\big|r_t^j\big|^2 \le C(\Psi)\, d_t/t, \tag{5.14}$$

$$E\big|c_t^j\big|^2 \le C(\Psi)\, \rho_t/t, \tag{5.15}$$

$$E\big|l_t\big|^2 \le C_{p_1}(\Psi)\, t^{-p_1}, \quad \forall p_1 > 0. \tag{5.16}$$

Proof. (5.16) follows from (5.8). We discuss the bound (5.14) only, the bound (5.15) is done in a similar way. Rewrite the LHS of (5.14) as the integral of correlation matrices. Definition (5.12) and Corollary 4.2 imply by Fubini Theorem that

$$E\big|r_t^j\big|^2 = \left\langle \chi_t^j(x)\chi_t^j(y) q_0(x-y), \Phi(x,t) \otimes \Phi(y,t) \right\rangle. \tag{5.17}$$

According to (5.7), the equation (5.17) implies that

$$E\big|r_t^j\big|^2 \le C\,t^{-3}\int\limits_{R_t^j\times R_t^j} \|q_0(x-y)\|\;dx\,dy \le C\,t^{-3}\int\limits_{R_t^j} dx\int\limits_{\mathbb{R}^3}\|q_0(z)\|\;dz, \tag{5.18}$$

where $\|q_0(z)\|$ stands for the norm of a matrix $\big(q_0^{ij}(z)\big)$. Therefore, (5.14) follows, since $\|q_0(z)\| \in L^1(\mathbb{R}^3)$ by (4.1) and (2.7). □

6 Convergence of characteristic functionals

In this section we complete the proof of Proposition 3.2 for the functions Ψ with the additional condition (5.1): we will remove it in Section 8. We use a version of the Central Limit Theorem developed by Ibragimov and Linnik. This gives the convergence to an equilibrium Gaussian measure. If $\mathcal{Q}_\infty(\Psi,\Psi)=0$, Proposition 3.2 follows from the Chebyshev inequality. Thus, we may assume that

$$\mathcal{Q}_\infty(\Psi,\Psi)\neq 0. \tag{6.1}$$

Choose $0<\sigma<1$ and

$$\rho_t\sim t^{1-\sigma},\quad d_t\sim\frac{t}{\ln t},\quad t\to\infty. \tag{6.2}$$

Lemma 6.1. *The following limit holds true:*

$$N_t\left(\varphi(\rho_t)+\Big(\frac{\rho_t}{t}\Big)^{1/2}\right)+N_t^2\left(\varphi^{1/2}(\rho_t)+\frac{\rho_t}{t}\right)\to 0,\quad t\to\infty. \tag{6.3}$$

Proof. The function $\varphi(r)$ is nonincreasing, hence by (2.7)

$$r^3\varphi^{1/2}(r)=3\int_0^r s^2\varphi^{1/2}(r)\,ds\le 3\int_0^r s^2\varphi^{1/2}(s)\,ds\le C\,\overline{\varphi}<\infty. \tag{6.4}$$

Then convergence (6.3) follows, since (6.2) and (5.11) imply that $N_t\sim\ln t$. □

By the triangle inequality,

$$\begin{aligned}
|\hat\mu_t(\Psi)-\hat\mu_\infty(\Psi)| &\le \Big|E\exp\Big\{i\,\langle U_0(t)Y_0,\Psi\rangle\Big\}-E\exp\Big\{i\sum_t r_t^j\Big\}\Big|\\
&+\Big|\exp\Big\{-\frac12\sum_t E\big|r_t^j\big|^2\Big\}-\exp\Big\{-\frac12\mathcal{Q}_\infty(\Psi,\Psi)\Big\}\Big|\\
&+\Big|E\exp\Big\{i\sum_t r_t^j\Big\}-\exp\Big\{-\frac12\sum_t E\big|r_t^j\big|^2\Big\}\Big|\\
&\equiv I_1+I_2+I_3,
\end{aligned}$$

where the sum $\sum\limits_t$ stands for $\sum\limits_{j=1}^{N_t}$. We are going to show that all the summands I_1, I_2, I_3 tend to zero as $t \to \infty$.
Step i) Eq. (5.13) implies

$$\begin{aligned} I_1 = \Big| E \exp\Big\{ i \sum_t r_t^j \Big\} \Big(\exp\Big\{ i \sum_t c_t^j + i l_t \Big\} - 1 \Big) \Big| &\le \sum_t E\big|c_t^j\big| + E\big|l_t\big| \\ &\le \sum_t \Big(E\big|c_t^j\big|^2 \Big)^{1/2} + \Big(E\big|l_t\big|^2 \Big)^{1/2}. \end{aligned} \tag{6.5}$$

From (6.5), (5.15), (5.16) and (6.3) we obtain that

$$I_1 \le C_1 N_t \left(\frac{\rho_t}{t} \right)^{1/2} + C_2 t^{-p_1} \to 0, \quad t \to \infty.$$

Step ii) By the triangle inequality,

$$\begin{aligned} I_2 \le & \frac{1}{2} \Big| \sum_t E\big|r_t^j\big|^2 - \mathcal{Q}_\infty(\Psi, \Psi) \Big| \le \frac{1}{2} \Big| \mathcal{Q}_t(\Psi, \Psi) - \mathcal{Q}_\infty(\Psi, \Psi) \Big| \\ & + \frac{1}{2} \Big| E\Big(\sum_t r_t^j \Big)^2 - \sum_t E\big|r_t^j\big|^2 \Big| + \frac{1}{2} \Big| E\Big(\sum_t r_t^j \Big)^2 - \mathcal{Q}_t(\Psi, \Psi) \Big| \\ & \equiv I_{21} + I_{22} + I_{23}, \end{aligned} \tag{6.6}$$

where $\mathcal{Q}_t$ is the quadratic form with the integral kernel $\Big(Q_t^{ij}(x, y) \Big)$. Eq.(4.7) implies that $I_{21} \to 0$. As to I_{22}, we first have that

$$I_{22} \le \sum_{j<l} E\big|r_t^j r_t^l\big|. \tag{6.7}$$

The next lemma is a corollary of [12, Lemma 17.2.3].

Lemma 6.2. *Let* $\mathbf{A}, \mathbf{B} \subset \mathbb{R}^3$, ξ *resp.* η *be random values measurable with respect to the* σ*-algebra* $\sigma(\mathbf{A})$ *resp.* $\sigma(\mathbf{B})$ *and* $\operatorname{dist}(\mathbf{A}, \mathbf{B}) \ge r > 0$.
i) Let $\left(E\,|\xi|^2 \right)^{1/2} \le a$, $\left(E\,|\eta|^2 \right)^{1/2} \le b$. *Then*

$$|E\xi\eta - E\xi E\eta| \le Cab\, \varphi^{1/2}(r).$$

ii) Let $|\xi| \le a$, $|\eta| \le b$ *almost sure. Then*

$$|E\xi\eta - E\xi E\eta| \le Cab\, \varphi(r).$$

We apply Lemma 6.2 to deduce that $I_{22} \to 0$ as $t \to \infty$. Note that $r_t^j = \langle Y_0, \chi_t^j \Phi(\cdot, t) \rangle$ is measurable with respect to the σ-algebra $\sigma(R_t^j)$. The distance between different rooms R_t^j is greater or equal to ρ_t according to (5.11). Then (6.7) and **S2**, **S4** imply, together with Lemma 6.2-i), that

$$I_{22} \le C N_t^2 \varphi^{1/2}(\rho_t), \tag{6.8}$$

which tends to 0 as $t \to \infty$ because of (6.3). Finally, it remains to check that $I_{23} \to 0$, $t \to \infty$. By the Cauchy - Schwartz inequality,

$$\begin{aligned} I_{23} \le \Big| E\Big(\sum_t r_t^j\Big)^2 - E\Big(\sum_t r_t^j + \sum_t c_t^j + l_t\Big)^2 \Big| \le C N_t \sum_t E\big|c_t^j\big|^2 \\ + C\bigg(E\Big(\sum_t r_t^j\Big)^2\bigg)^{1/2} \Big(N_t \sum_t E\big|c_t^j\big|^2 + E\big|l_t\big|^2\Big)^{1/2} + CE\big|l_t\big|^2. \end{aligned} \tag{6.9}$$

Then (5.14), (6.7) and (6.8) imply

$$E\Big(\sum_t r_t^j\Big)^2 \le \sum_t E\big|r_t^j\big|^2 + 2\sum_{j<l} E\big|r_t^j r_t^l\big| \le C N_t \frac{d_t}{t} + C_1 N_t^2 \varphi^{1/2}(\rho_t) \le C_2 < \infty.$$

Now (5.14), (6.9) and (6.3) yield

$$I_{23} \le C_1 N_t^2 \frac{\rho_t}{t} + C_2 N_t \left(\frac{\rho_t}{t}\right)^{1/2} + C_3 t^{-p_1} \to 0, \quad t \to \infty.$$

So, all the terms I_{21}, I_{22}, I_{23} in (6.6) tend to zero. Then (6.6) implies that

$$I_2 \le \frac{1}{2} \Big| \sum_t E\big|r_t^j\big|^2 - \mathcal{Q}_\infty(\Psi, \Psi) \Big| \to 0, \quad t \to \infty. \tag{6.10}$$

Step iii) It remains to verify that

$$I_3 = \Big| E \exp\Big\{ i \sum_t r_t^j \Big\} - \exp\Big\{ -\frac{1}{2} E \sum_t \big|r_t^j\big|^2 \Big\} \Big| \to 0, \quad t \to \infty.$$

Using Lemma 6.2-ii) we obtain:

$$\begin{aligned}
&\Big|E\exp\Big\{i\sum_t r_t^j\Big\}-\prod_1^{N_t}E\exp\{ir_t^j\}\Big| \\
&\quad\le\Big|E\exp\{ir_t^1\}\exp\Big\{i\sum_2^{N_t}r_t^j\Big\}-E\exp\{ir_t^1\}E\exp\Big\{i\sum_2^{N_t}r_t^j\Big\}\Big| \\
&\qquad+\Big|E\exp\{ir_t^1\}E\exp\Big\{i\sum_2^{N_t}r_t^j\Big\}-\prod_1^{N_t}E\exp\{ir_t^j\}\Big| \\
&\qquad\qquad\le C\varphi(\rho_t)+\Big|E\exp\Big\{i\sum_2^{N_t}r_t^j\Big\}-\prod_2^{N_t}E\exp\{ir_t^j\}\Big|.
\end{aligned}$$

We then apply Lemma 6.2-ii) recursively and get, according to Lemma 6.1,

$$\Big|E\exp\Big\{i\sum_t r_t^j\Big\}-\prod_1^{N_t}E\exp\{ir_t^j\}\Big|\le CN_t\varphi(\rho_t)\to 0,\quad t\to\infty.$$

It remains to check that

$$\Big|\prod_1^{N_t}E\exp\Big\{ir_t^j\Big\}-\exp\Big\{-\frac{1}{2}\sum_t E\big|r_t^j\big|^2\Big\}\Big|\to 0,\quad t\to\infty.$$

According to the standard statement of the Central Limit Theorem (see, e.g. [13, Thm 4.7]), it suffices to verify the Lindeberg condition: $\forall\varepsilon>0$

$$\frac{1}{\sigma_t}\sum_t E_{\varepsilon\sqrt{\sigma_t}}\big|r_t^j\big|^2\to 0,\quad t\to\infty.$$

Here $\sigma_t\equiv\sum_t E\big|r_t^j\big|^2$ and $E_\delta f\equiv E(X_\delta f)$, where X_δ is the indicator of the event $|f|>\delta^2$. At last, (6.10) and (6.1) imply that

$$\sigma_t\to\mathcal{Q}_\infty(\Psi,\Psi)\neq 0,\quad t\to\infty.$$

Hence, it remains to verify that $\forall\varepsilon>0$

$$\sum_t E_\varepsilon\big|r_t^j\big|^2\to 0,\quad t\to\infty. \tag{6.11}$$

We check (6.11) in Section 7. This will complete the proof of Proposition 3.2.

7 Lindeberg condition

The proof of (6.11) can be reduced to the case when we have almost sure that

$$|u_0(x)| + |v_0(x)| \le \Lambda < \infty, \quad x \in \mathbb{R}^3 \tag{7.1}$$

for some $\Lambda \ge 0$. Then the proof of (6.11) is reduced to the convergence

$$\sum_t E\big|r_t^j\big|^4 \to 0, \quad t \to \infty \tag{7.2}$$

by Chebyshev's inequality. The general case can be covered by standard cut-off arguments taking into account that the bound (5.14) for $E\big|r_t^j\big|^2$ depends only on e_0 and φ. We deduce (7.2) from

Theorem 7.1. *Let the conditions of Theorem B hold and assume that* (7.1) *is fulfilled. Then for any $\Psi \in \mathcal{D}$ there exists a constant $C(\Psi)$ such that*

$$E\big|r_t^j\big|^4 \le C(\Psi)\Lambda^4 d_t^2/t^2, \quad t > 1. \tag{7.3}$$

Proof. *Step i)* Given four points $x_1, x_2, x_3, x_4 \in \mathbb{R}^3$, set:

$$M_0^{(4)}(x_1,\dots,x_4) = E\big(Y_0(x_1) \otimes \cdots \otimes Y_0(x_4)\big).$$

Then the equations (7.1) and (5.12) imply by the Fubini Theorem that similarly to (5.17)

$$E\big|r_t^j\big|^4 = \Big\langle \chi_t^j(x_1)\dots\chi_t^j(x_4) M_0^{(4)}(x_1,\dots,x_4),\ \Phi(x_1,t) \otimes \cdots \otimes \Phi(x_4,t) \Big\rangle. \tag{7.4}$$

Let us analyze the domain of the integration $(R_t^j)^4$ in the right hand side of (7.4). We partition $(R_t^j)^4$ into three parts W_2, W_3 and W_4:

$$(R_t^j)^4 = \bigcup_{i=2}^4 W_i,\ W_i = \big\{\bar{x} = (x_1,\dots,x_4) \in (R_t^j)^4 : |x_1 - x_i| = \max_{p=2,3,4} |x_1 - x_p|\big\}$$

Furthermore, given $\bar{x} = (x_1, x_2, x_3, x_4) \in W_i$, divide R_t^j in three parts S_j, $j = 1, 2, 3$: $R_t^j = S_1 \cup S_2 \cup S_3$, by two hyperplanes orthogonal to the segment $[x_1, x_i]$ and partitioning it in three equal segments, where $x_1 \in S_1$ and $x_i \in S_3$. Denote by x_p, x_q the two remaining points with $p, q \ne 1, i$. Set: $\mathbf{A}_i = \{\bar{x} \in W_i : x_p \in S_1, x_q \in S_3\}$, $\mathbf{B}_i = \{\bar{x} \in W_i : x_p, x_q \notin S_1\}$ and $\mathbf{C}_i = \{\bar{x} \in W_i : x_p, x_q \notin S_3\}$, $i = 2, 3, 4$. Then $W_i = \mathbf{A}_i \cup \mathbf{B}_i \cup \mathbf{C}_i$. Define a function $\mathrm{m}_0^{(4)}(\bar{x})$, $\bar{x} \in (R_t^j)^4$, in the following way:

$$\mathrm{m}_0^{(4)}(\bar{x})\Big|_{W_i} = \begin{cases} M_0^{(4)}(\bar{x}) - q_0(x_1 - x_p) \otimes q_0(x_i - x_q), & \bar{x} \in \mathbf{A}_i, \\ M_0^{(4)}(\bar{x}), & \bar{x} \in \mathbf{B}_i \cup \mathbf{C}_i. \end{cases}$$

This determines $\mathrm{m}_0^{(4)}(\bar{x})$ correctly for almost all quadruples $\bar{x}$. Note that

$$\begin{aligned}\big\langle \chi_t^j(x_1)\dots\chi_t^j(x_4) q_0(x_1-x_p)\otimes q_0(x_i-x_q),\ \Phi(x_1,t)\otimes\dots\otimes\Phi(x_4,t)\big\rangle \\ = \big\langle \chi_t^j(x_1)\chi_t^j(x_p) q_0(x_1-x_p),\ \Phi(x_1,t)\otimes\Phi(x_p,t)\big\rangle \\ \big\langle \chi_t^j(x_i)\chi_t^j(x_q) q_0(x_i-x_q),\ \Phi(x_i,t)\otimes\Phi(x_q,t)\big\rangle.\end{aligned}$$

Each factor here is bounded by $C(\Psi)\,d_t/t$. Similarly to (5.14), this can be deduced from an expression of type (5.17) for the factors. Therefore, the proof of (7.3) is reduced to the proof of the bound

$$\begin{aligned}J_t(j) := \Big|\Big\langle \chi_t^j(x_1)\dots\chi_t^j(x_4)\,\mathrm{m}_0^{(4)}(x_1,\dots,x_4),\ \Phi(x_1,t)\otimes\dots\otimes\Phi(x_4,t)\Big\rangle\Big| \\ \le C(\Psi)\Lambda^4 d_t^2/t^2, \qquad t>1.\end{aligned}$$

Step ii) Similarly to (5.18), the asymptotics (5.7) with $N=2$ implies,

$$J_t(j)\ \le\ C(\Psi)\,t^{-6}\int\limits_{(R_t^j)^4}\big|\mathrm{m}_0^{(4)}(x_1,\dots,x_4)\big|\,dx_1\,dx_2\,dx_3\,dx_4.$$

Let us estimate $\mathrm{m}_0^{(4)}$ using Lemma 6.2-ii).

Lemma 7.1. *For each $i=2,3,4$ and almost all $\overline{x}\in W_i$ the following bound holds*

$$\big|\mathrm{m}_0^{(4)}(x_1,\dots,x_4)\big|\le C\Lambda^4\varphi\big(|x_1-x_i|/3\big).$$

Proof. For $\bar{x}\in\mathbf{A}_i$ we apply Lemma 6.2-ii) to $\mathbb{R}^2\otimes\mathbb{R}^2$-valued random variables $\xi=Y_0(x_1)\otimes Y_0(x_p)$ and $\eta=Y_0(x_i)\otimes Y_0(x_q)$. Then (7.1) implies the bound for almost all $\bar{x}\in\mathbf{A}_i$

$$\big|\mathrm{m}_0^{(4)}(\bar{x})\big|\le C\Lambda^4\varphi\big(|x_1-x_i|/3\big).$$

For $\bar{x}\in\mathbf{B}_i$, we apply Lemma 6.2-ii) to $\xi=Y_0(x_1)$ and $\eta=Y_0(x_p)\otimes Y_0(x_q)\otimes Y_0(x_i)$. Then **S1** implies a similar bound for almost all $\bar{x}\in\mathbf{B}_i$,

$$\begin{aligned}\big|\mathrm{m}_0^{(4)}(\bar{x})\big| = \Big|M_0^{(4)}(\bar{x}) - EY_0(x_1)\otimes E\Big(Y_0(x_p)\otimes Y_0(x_q)\otimes Y_0(x_i)\Big)\Big| \\ \le C\Lambda^4\varphi\big(|x_1-x_i|/3\big),\end{aligned}$$

and the same for almost all $\bar{x}\in\mathbf{C}_i$. □

Step iii) It remains to prove the following bounds for each $i=2,3,4$:

$$V_i(t)\ :=\int\limits_{(R_t^j)^4} X_i(\overline{x})\,\varphi\big(|x_1-x_i|/3\big)\,dx_1\,dx_2\,dx_3\,dx_4\ \le\ Cd_t^2t^4, \tag{7.5}$$

where X_i is an indicator of the set W_i. In fact, this integral does not depend on i, hence set $i = 2$ in the integrand:

$$V_2(t) \le C \int\limits_{(R_t^j)^2} \varphi\big(|x_1 - x_2|/3\big) \left[\int\limits_{R_t^j} \Big(\int\limits_{R_t^j} X_2(\overline{x})\, dx_4 \Big)\, dx_3 \right] dx_1\, dx_2.$$

Now a key observation is that the inner integral in dx_4 is $\mathcal{O}\big(|x_1 - x_2|^3\big)$, since $X_2(\overline{x}) = 0$ for $|x_4 - x_1| > |x_1 - x_2|$. This implies

$$V_2(t) \le C_1 \int\limits_{R_t^j} \Big(\int\limits_{R_t^j} \varphi\big(|x_1 - x_2|/3\big)\, |x_1 - x_2|^3\, dx_2 \Big)\, dx_1 \int\limits_{R_t^j} dx_3. \qquad (7.6)$$

The inner integral in dx_2 is bounded as

$$\int\limits_{R_t^j} \varphi\big(|x_1 - x_2|/3\big)\, |x_1 - x_2|^3\, dx_2 \le C \int_0^{4R_0 t} r^5 \varphi(r/3)\, dr$$

$$\le C_1 \sup_{r \in [0; 4R_0 t]} r^3 \varphi^{1/2}(r/3) \int_0^{4R_0 t} r^2 \varphi^{1/2}(r/3)\, dr.$$

The 'sup' and the last integral are bounded by (6.4) and (2.7), respectively. Therefore, (7.5) follows from (7.6). This completes the proof of Theorem 7.1. □

Proof of convergence (7.2). The estimate (7.3) implies

$$\sum_t E\big|r_t^j\big|^4 \le \frac{C\Lambda^4 d_t^2}{t^2} N_t \le \frac{C_1 \Lambda^4}{N_t} \to 0, \quad t \to \infty$$ □

since $d_t \le h \sim t/N_t$.

8 Removing spectral condition

Now we remove the spectral condition (5.1). We must prove Proposition 3.2 for any $\Psi \in \mathcal{S}$. Let us split Ψ in the sum of two functions

$$\Psi = \Psi_R + \Theta_R, \qquad (8.1)$$

where

$$\hat{\Psi}_R(k) = \chi_R(k) \hat{\Psi}(k), \quad \hat{\Theta}_R(k) = \big(1 - \chi_R(k)\big) \hat{\Psi}(k)$$

and

$$\chi_R(k) \in C_0^\infty(\mathbb{R}^3), \qquad \chi_R(k) = \begin{cases} 1, & |k| \le R \\ 0, & |k| \ge 2R \end{cases}$$

Then by the triangle inequality

$$\begin{aligned}
&\Big|E\exp\Big\{i\,\langle U(t)Y_0,\Psi\rangle\Big\} - \exp\Big\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi,\Psi)\Big\}\Big| \\
=&\Big|E\exp\Big\{i\,\langle U(t)Y_0,\Psi_R\rangle\Big\}\exp\Big\{i\,\langle U(t)Y_0,\Theta_R\rangle\Big\} - \exp\Big\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi,\Psi)\Big\}\Big| \\
\le&\Big|E\exp\Big\{i\,\langle U(t)Y_0,\Psi_R\rangle\Big\}\exp\Big\{i\,\langle U(t)Y_0,\Theta_R\rangle\Big\} - E\exp\Big\{i\,\langle U(t)Y_0,\Psi_R\rangle\Big\}\Big| \\
+&\Big|E\exp\Big\{i\,\langle U(t)Y_0,\Psi_R\rangle\Big\} - \exp\Big\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi_R,\Psi_R)\Big\}\Big| \\
+&\Big|\exp\Big\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi_R,\Psi_R)\Big\} - \exp\Big\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi,\Psi)\Big\}\Big| = I_1 + I_2 + I_3.
\end{aligned} \tag{8.2}$$

We will estimate each term in the RHS.
Step i) Applying the Cauchy-Schwartz inequality, we get with the summation in the repeating indices

$$\begin{aligned}
I_1 &= \Big|E\exp\big\{i\,\langle U(t)Y_0,\Psi_R\rangle\big\}\Big(\exp\big\{i\,\langle U(t)Y_0,\Theta_R\rangle\big\} - 1\Big)\Big| \\
&\le E\,\Big|\exp\Big\{i\,\langle U(t)Y_0,\Theta_R\rangle\Big\} - 1\Big| \le E\,|\langle U(t)Y_0,\Theta_R\rangle| \le \Big(E\,\langle U(t)Y_0,\Theta_R\rangle^2\Big)^{1/2} \\
&\le \Big\langle Q_t^{ij}(x,y), \Theta_R^i(x)\Theta_R^j(y)\Big\rangle^{1/2}.
\end{aligned} \tag{8.3}$$

The RHS of (8.3), in Fourier space, can be written as

$$\Big\langle \hat q_t^{ij}(k), \hat\Theta_R^i(k)\hat\Theta_R^j(k)\Big\rangle^{1/2} \le \mu(R), \quad t > 0.$$

Here $\mu(R) \to 0$, $R \to \infty$, uniformly in $t \ge 0$ since the functions $\hat q_t^{ij}(k)$ admit the summable (in k) majorant independent of t by Proposition 4.1 and formulas of type (4.4) and (4.5) for $\hat q_t^{ij}(k)$.
Step ii) The second term I_2 in the RHS of (8.2) converges to zero as $t \to \infty$ according the results of section 6 since $\hat\Psi_R(k) = 0$ for $|k| \le R$.
Step iii) It remains to verify that the third summand $I_3 \to 0, R \to \infty$ that is the difference $\mathcal{Q}_\infty(\Psi,\Psi) - \mathcal{Q}_\infty(\Psi_R,\Psi_R) \to 0, R \to \infty$. According to (8.1),

$$\begin{aligned}
\mathcal{Q}_\infty(\Psi,\Psi) - \mathcal{Q}_\infty(\Psi_R,\Psi_R) &= \mathcal{Q}_\infty(\Psi,\Psi-\Psi_R) + \mathcal{Q}_\infty(\Psi-\Psi_R,\Psi_R) \\
&= \mathcal{Q}_\infty(\Psi,\Theta_R) + \mathcal{Q}_\infty(\Theta_R,\Psi_R).
\end{aligned}$$

Using Fourier transform, we obtain by (2.9)

$$\mathcal{Q}_\infty(\Psi,\Theta_R)=\sum_{i,j=0,1}\int\limits_{|k|\ge R}\hat q^{ij}_\infty(k)\hat\Psi^i(k)\hat\Theta^j_R(k)\,dk\to 0,\ R\to\infty,$$

since $\hat\Psi^i$ are bounded and $\hat q^{ij}_\infty\in L^1(\mathbb{R}^3)$ by Corollary 4.1. The second summand in RHS of (8.1) converges to zero as $R\to\infty$ by the same argument.

Therefore, $I_1+I_2+I_3$ converges to zero as $t\to\infty$.

9 Scattering theory

In this section we develop a version of a scattering theory to deduce Theorem A from Theorem B. Let us recall that $\alpha<-5/2$ by the conditions of Theorem A. Denote by $U(t):Y_0\to Y(t)$ the dynamical group of the problem (1.2) for finite energy solutions with the values in the space $\mathcal{H}_0=L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$. The results of [10] imply the following lemma:

Lemma 9.1. *Let conditions* ***E1****–****E3*** *hold and* $\alpha_1>5/2$. *Then* $U(t)\in\mathcal{B}(\mathcal{H}_{\alpha_1},\mathcal{H}_\alpha)$ *and*

$$\|U(t)Y\|_{\mathcal{H}_\alpha}\le Ct^{-3/2}\|Y\|_{\mathcal{H}_{\alpha_1}}\tag{9.1}$$

For $t\in\mathbb{R}$ introduce an adjoint operator $U'(t)$ on the space $\mathcal{H}_0$:

$$\langle Y,U'(t)\Psi\rangle=\langle U(t)Y,\Psi\rangle\,,\ Y,\Psi\in\mathcal{H}_0.$$

Consider the family of finite seminorms in $\mathcal{H}_0$,

$$\|\Psi\|^2_{(R)}=\int\limits_{|x|\le R}(|\Psi_0(x)|^2+|\Psi_1(x)|^2)\,dx,\qquad R>0.$$

Denote by $\mathcal{H}_{(R)}$ the subspace of functions of $\mathcal{H}_0$ with a support in the ball B_R.

Definition 9.1. $\mathcal{H}_{comp}$ *denotes the space* $\cup_{R>0}\mathcal{H}_{(R)}$ *endowed with the following convergence: a sequence* Ψ_n *converges to* Ψ *in* $\mathcal{H}_{comp}$ *iff* $\exists R>0$ *such that all* $\Psi_n\in\mathcal{H}_{(R)}$*, and* Ψ_n *converge to* Ψ *in the norm* $\|\cdot\|_{(R)}$.

Below, we speak of continuity of maps in $\mathcal{H}_{comp}$ in the sense of this sequential continuity. The main result of this section is Theorem 9.1.

Theorem 9.1. *Let conditions* ***E1*****–*****E3*** *and* ***S1*****–*****S4*** *hold. Then there exist linear continuous operators* $W, r(t) : \mathcal{H}_{comp} \to \mathcal{H}_0$ *such that for* $\Psi \in \mathcal{H}_c$

$$U'(t)\Psi = U_0'(t)W\Psi + r(t)\Psi, \quad t \geq 0, \tag{9.2}$$

and the following bounds hold $\forall R > 0$ *and* $\Psi \in \mathcal{H}_{(R)}$*:*

$$\|r(t)\Psi\|_{\mathcal{H}_0} \leq C(R)t^{-1/2} \|\Psi\|_{(R)}, \ t \geq 0, \tag{9.3}$$

$$E \langle Y_0, r(t)\Psi \rangle^2 \leq C(R)t^{-1} \|\Psi\|^2_{(R)}, \ t \geq 0. \tag{9.4}$$

Proof. We apply the standard Cook method: see, e.g., [14, Thm XI.4]. Fix $\Psi \in \mathcal{H}_{(R)}$ and define $W\Psi$, formally, as

$$W\Psi = \lim_{t\to\infty} U_0'(-t)U'(t)\Psi = \Psi + \int\limits_0^\infty \frac{d}{dt}U_0'(-t)U'(t)\Psi\, dt.$$

We have to prove the convergence of the integral in the norm of the space $\mathcal{H}_0$. First, observe that

$$\frac{d}{dt}U_0'(t)\Psi = \mathcal{A}_0'U_0'(t)\Psi, \qquad \frac{d}{dt}U'(t)\Psi = \mathcal{A}'U'(t)\Psi,$$

where $\mathcal{A}_0'$ and $\mathcal{A}'$ are the generators of the groups $U_0'(t)$, $U'(t)$, respectively. Similarly to (5.4), we have

$$\mathcal{A}' = \begin{pmatrix} 0 & \Delta - V \\ -\Delta + V & 0 \end{pmatrix}. \tag{9.5}$$

Therefore,

$$\frac{d}{dt}U_0'(-t)U_1'(t)\Psi = U_0'(-t)\mathcal{B}'U'(t)\Psi, \tag{9.6}$$

where

$$\mathcal{B}' = \mathcal{A}' - \mathcal{A}_0' = \begin{pmatrix} 0 & -V \\ V & 0 \end{pmatrix}.$$

Note, that the bound (9.1) holds for the operators $U'(t)$ too, then

$$\begin{aligned}\|U_0'(-t)\mathcal{B}'U'(t)\Psi\|_{\mathcal{H}_0} = \|\mathcal{B}'U'(t)\Psi\|_{\mathcal{H}_0} &\leq C\left\|(1+|x|)^{-\beta}U'(t)\Psi\right\|_{\mathcal{H}_0} \\ &\leq Ct^{-3/2}\|\Psi\|_{\mathcal{H}_{s_1}} \leq C(R)t^{-3/2}\|\Psi\|_{(R)}, \ t \geq 0,\end{aligned}$$

where $s_1 > 5/2$ and $-\beta < -4 < -5/2$ by **E1**. Hence (9.6) implies

$$\int\limits_s^\infty \left\| \frac{d}{dt} U_0'(-t) U'(t) \Psi \right\|_{\mathcal{H}_0} dt \le C(R) t^{-1/2} \left\| \Psi \right\|_{(R)}, \quad s \ge 0. \tag{9.7}$$

Therefore, (9.2) and (9.3) follow. It remains to prove (9.4). First, similarly to (5.17),

$$E \left\langle Y_0, r(t)\Psi \right\rangle^2 = \left\langle q_0^{ij}(x-y), \big(r(t)\Psi(x)\big)^i \big(r(t)\Psi(y)\big)^j \right\rangle. \tag{9.8}$$

Therefore, the Young inequality implies

$$E \left\langle Y_0, r(t)\Psi \right\rangle^2 \le \left\| q_0^{ij} \right\|_{L^1} \left\| r(t)^i \Psi \right\|_{L^2} \left\| r(t)^j \Psi \right\|_{L^2}. \tag{9.9}$$

Finally, (9.3) implies for $\Psi \in H_{(R)}$

$$\left\| r(t)^i \Psi \right\|_{L^2} \le C \left\| r(t)\Psi \right\|_{\mathcal{H}_0} \le C(R) t^{-1/2} \left\| \Psi \right\|_{(R)}. \tag{9.10}$$

Therefore, (9.4) follows from (9.9) since $\left\| q_0^{ij} \right\|_{L^1} < \infty$ by (4.1). $\square$

10 Schrödinger equation with a potential

Theorem A follows from the two propositions below:

Proposition 10.1. *The family of the measures $\{\mu_t, t \in \mathbb{R}\}$ is weakly compact in $\mathcal{H}_\alpha^{-\varepsilon}$, if $\alpha < -5/2$ and $\varepsilon > 0$.*

Proposition 10.2. *for any $\Psi \in \mathcal{D}$*

$$\hat{\mu}_t(\Psi) \equiv \int \exp\Big\{ i \left\langle Y, \Psi \right\rangle \Big\} \, \mu_t(dY) \to \exp\Big\{ -\frac{1}{2} \mathcal{Q}_\infty(W\Psi, W\Psi) \Big\}, \quad t \to \infty.$$

Proposition 10.1 provides the existence of the limiting measures of the family μ_t, and Proposition 10.2 provides the uniqueness of the limiting measure, and hence the convergence (1.8). We deduce these propositions from Propositions 3.1 and 3.2, respectively, by means of Theorem 9.1.

Proof of Proposition 10.1. Proposition 10.1 follows from the bounds

$$\sup_{t \ge 0} E \left\| Y(t) \right\|_{\mathcal{H}_\alpha} < \infty, \qquad \alpha < -5/2. \tag{10.1}$$

similarly to Proposition 3.1. For the proof, write the solution to (1.2) in the form (see (B.8)– (B.11))

$$Y(t) = X(t) + Z(t). \tag{10.2}$$

Then (10.2) implies

$$E\left\|Y(t)\right\|_{\mathcal{H}_\alpha} \le E\left\|X(t)\right\|_{\mathcal{H}_\alpha} + E\left\|Z(t)\right\|_{\mathcal{H}_\alpha}. \tag{10.3}$$

By Proposition 3.1 we have

$$\sup_{t\ge 0} E\left\|X(t)\right\|_{\mathcal{H}_\alpha} = \sup_{t\ge 0} E\left\|U_0(t)Y_0\right\|_{\mathcal{H}_\alpha} < \infty. \tag{10.4}$$

It remains to estimate the second term in the RHS of (10.3). Let us choose $\alpha_1 > 5/2$ s.t. $\alpha_1 - \beta < -3/2$: this is possible since $\beta > 4$ by **E1**. The Duhamel representation (B.11), the bound (9.1) imply

$$\begin{aligned} E\left\|Z(t)\right\|_{\mathcal{H}_\alpha} &\le \int_0^t E\left\|U(t-s)\mathcal{B}X(s)\right\|_{\mathcal{H}_\alpha} ds \le C\int_0^t (t-s)^{-3/2} E\left\|\mathcal{B}X(s)\right\|_{\mathcal{H}_{\alpha_1}} ds \\ &\le C\int_0^t (t-s)^{-3/2} E\left\|X(s)\right\|_{\mathcal{H}_{\alpha_2}} ds \le C_1\int_0^t (t-s)^{-3/2} ds \le C_2, \quad t\in\mathbb{R} \end{aligned} \tag{10.5}$$

by condition **E1** and Proposition 3.1 since $\alpha_2 = \alpha_1 - \beta < -3/2$. Then (10.2),(10.4),(10.5) imply (10.1). □

Proof of Proposition 10.2.

Lemma 10.1. *For any $\Psi \in \mathcal{D}$*

$$\langle Y(t), \Psi\rangle = \langle Y_0, U'(t)\Psi\rangle \text{ a.s.}$$

Proof. Formulas (B.8)-(B.11) imply

$$\begin{aligned} \langle Y(t), \Psi\rangle &= \left\langle U_0(t)Y_0 + \int_0^t U(t-s)\mathcal{B}U_0(s)Y_0 ds, \Psi\right\rangle \\ &= \left\langle Y_0, U_0'(t)\Psi + \int_0^t U_0'(s)\mathcal{B}'U'(t-s)\Psi ds\right\rangle = \left\langle Y_0, U'(t)\Psi\right\rangle, \ a.s., \end{aligned}$$

since the same formulas (B.8)-(B.11) hold also for $Y_0 \in \mathcal{H}_0$. □

Lemma 10.1, (9.2) and (9.4) imply by Cauchy-Schwartz,

$$\big|E\exp i\,\langle Y(t),\Psi\rangle - E\exp i\,\langle Y_0,U_0'(t)W\Psi\rangle\big| \le E\big|\langle Y_0,r(t)\Psi\rangle\big|$$

$$\le \big(E\big|\langle Y_0,r(t)\Psi\rangle\big|^2\big)^{1/2}\to 0,\quad t\to\infty.$$

It remains to prove that

$$E\exp i\langle Y_0,U_0'(t)W\Psi\rangle \to \exp\{-\frac{1}{2}\mathcal{Q}_\infty(W\Psi,W\Psi\ )\},\quad t\to\infty. \tag{10.6}$$

This does not follow directly from Proposition 3.2 since in general, $W\Psi\notin\mathcal{D}$. We can approximate $W\Psi\in\mathcal{H}_0$ with functions of $\mathcal{D}$ since $\mathcal{D}$ is dense in $\mathcal{H}_0$: for any $\epsilon>0$ there exists a $\Phi\in\mathcal{D}$ such that

$$\|W\Psi-\Phi\|_{\mathcal{H}_0}\le\epsilon. \tag{10.7}$$

Therefore, we can derive (10.6) by the triangle inequality

$$\begin{aligned}
&\left|E\exp i\langle Y_0,U_0'(t)W\Psi\rangle - \exp\{-\frac{1}{2}\mathcal{Q}_\infty(W\Psi,W\Psi\ )\}\right|\\
&\le |E\exp i\langle Y_0,U_0'(t)W\Psi\rangle - E\exp i\langle Y_0,U_0'(t)\Phi\rangle|\\
&+E\left|\exp i\langle U_0(t)Y_0,\Phi\rangle - \exp\{-\frac{1}{2}\mathcal{Q}_\infty(\Phi,\Phi\ )\}\right|\\
&+\left|\exp\{-\frac{1}{2}\mathcal{Q}_\infty(\Phi,\Phi\ )\} - \exp\{-\frac{1}{2}\mathcal{Q}_\infty(W\Psi,W\Psi\ )\}\right|.
\end{aligned} \tag{10.8}$$

Applying Cauchy-Schwartz, we get, similarly to (9.8)-(9.9), that

$$\begin{aligned}
E\,|\langle Y_0,U_0'(t)(W\Psi-\Phi)\rangle| &\le (E\,|\langle Y_0,U_0'(t)(W\Psi-\Phi)\rangle|^2)^{1/2}\\
&\le C\,\|U_0'(t)(W\Psi-\Phi)\|_{\mathcal{H}_0}.
\end{aligned}$$

Hence, (10.7) implies

$$E\,|\langle Y_0,U_0'(t)(W\Psi-\Phi)\rangle|\le C\epsilon,\ \ t\ge 0. \tag{10.9}$$

Now we can estimate each term in the right hand side of (10.8). The first term is $\mathcal{O}(\epsilon)$ uniformly in $t>0$ by (10.9). The second term converges to zero as $t\to\infty$ by Proposition 3.2 since $\Phi\in\mathcal{D}$. Finally, the third term is $\mathcal{O}(\epsilon)$ due to (10.7) and the continuity of the quadratic form $\mathcal{Q}_\infty(\Psi,\Psi)$ in $L^2(\mathbb{R}^3)$. The continuity follows from the Shur Lemma since the integral kernels $q_\infty^{ij}(z)\in L^1(\mathbb{R}^3)$ by Corollary 4.1. Now the convergence in (10.6) follows, since $\epsilon>0$ is arbitrary. □

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A Appendix A. Fourier calculations

Consider the correlation functions of the solutions to the system (3.2). Let $F : w \mapsto \hat{w}$ denote the Fourier transform of a tempered distribution $w \in S'(\mathbb{R}^3)$ (see, e.g. [8]). We also use this notation for vector- and matrix-valued functions.

A.1 Dynamics in the Fourier space

In the Fourier representation, the system (3.2) becomes
$\dot{\hat{Y}}(k,t) = \hat{\mathcal{A}}_0(k)\hat{Y}(k,t)$, hence

$$\hat{Y}(k,t) = \hat{\mathcal{G}}_t(k)\hat{Y}_0(k), \quad \hat{\mathcal{G}}_t(k) = \exp\big\{\hat{\mathcal{A}}_0(k)t\big\}. \tag{A.1}$$

Here we denote

$$\hat{\mathcal{A}}_0(k) = \begin{pmatrix} 0 & |k|^2 \\ -|k|^2 & 0 \end{pmatrix}, \quad \hat{\mathcal{G}}_t(k) = \begin{pmatrix} \cos|k|^2 t & \sin|k|^2 t \\ -\sin|k|^2 t & \cos|k|^2 t \end{pmatrix}. \tag{A.2}$$

A.2 Covariance matrices in the Fourier space

The translation invariance (1.3) implies that in the sense of distributions

$$E\Big(\hat{Y}_0(k) \otimes_{\mathbb{C}} \hat{Y}_0(k')\Big) = F_{x\to k}F_{y\to k'}q_0(x-y) = (2\pi)^n\delta(k+k')\hat{q}_0(k),$$

where $\otimes_{\mathbb{C}}$ stands for the tensor product of complex vectors. Now (A.1) and (A.2) give in the matrix notation,

$$E\Big(\hat{Y}(k,t) \otimes_{\mathbb{C}} \hat{Y}(k',t)\Big) = (2\pi)^3\delta(k+k')\hat{\mathcal{G}}_t(k)\hat{q}_0(k)\hat{\mathcal{G}}'_t(k).$$

Therefore,

$$q_t(x-y) := E\Big(Y(x,t) \otimes Y(y,t)\Big) = F^{-1}_{k\to x-y}\hat{\mathcal{G}}_t(k)\hat{q}_0(k)\hat{\mathcal{G}}'_t(k). \tag{A.3}$$

B Appendix B. Existence of dynamics

We prove Proposition 2.1.
Step i) Denote

$$Y_n(t) = \mathcal{G}_{t,n} * Y_0,\ t \in \mathbb{R},\ n \in \mathbb{N}, \tag{B.1}$$

where $\hat{\mathcal{G}}_{t,n}(k) = \hat{\mathcal{G}}_t(k)\chi_n(k)$ and

$$\chi_n(k) \in C_0^\infty(\mathbb{R}^3),\ |\chi_n(k)| \le 1, \qquad \chi_n(k) = \begin{cases} 1, & |k| \le n \\ 0, & |k| \ge 2n \end{cases}$$

Then $Y_n(t) \in \mathcal{H}_\alpha$ a.s., since $\mathcal{G}_{t,n}(x) \in \mathcal{S}$. Moreover, $Y_n(t) \in \mathcal{C}$. Let us note that

$$e_{t,n} = \frac{1}{(2\pi)^3} \int \chi_n^2(k) \left(\hat{q}_t^{00}(k) + \hat{q}_t^{11}(k)\right) dk \to \frac{1}{(2\pi)^3} \int \left(\hat{q}_t^{00}(k) + \hat{q}_t^{11}(k)\right) dk = e_t \tag{B.2}$$

as $n \to \infty$. Then by definition (2.6) and the Fubini theorem

$$\begin{aligned} E\,\|Y_n(\cdot,t)\|^2_{\mathcal{H}_\alpha} &= E\,\|\mathcal{G}_{t,n} * Y_0(\cdot)\|^2_{\mathcal{H}_\alpha} \\ &= \int \left(1+|x|^2\right)^\alpha dx \int\int \mathcal{G}^{ki}_{t,n}(x-y) q_0^{ij}(y-y')\mathcal{G}^{kj}_{t,n}(x-y')\ dy\ dy' \\ &= e_{t,n} \int \left(1+|x|^2\right)^\alpha dx = e_{t,n} C_\alpha \to e_t C_\alpha,\ n \to \infty, \end{aligned} \tag{B.3}$$

since $\mathcal{G}_{t,n}(\cdot) \in \mathcal{S}$ and $\alpha < -3/2$. Similarly, by (B.2), (B.3), we obtain

$$\begin{aligned} &E\,\|Y_n(\cdot,t) - Y_m(\cdot,t)\|^2_{\mathcal{H}_\alpha} \\ &= \frac{1}{(2\pi)^3} C_\alpha \int \left(\chi_n(k) - \chi_m(k)\right)^2 \left(\hat{q}_t^{00}(k) + \hat{q}_t^{11}(k)\right) dk \to 0,\ n,m \to \infty \end{aligned} \tag{B.4}$$

uniformly in $t \in \mathbb{R}$ since $\hat{q}_t^{ii}(\cdot) \in L^1(\mathbb{R}^3)$ according to (4.4) and (4.5). Therefore, $Y_n(\cdot,\cdot)$ is a Cauchy sequence in $\mathcal{C}_\alpha$ and

$$Y_n(\omega,x,t) \xrightarrow{\mathcal{C}_\alpha} Y(\omega,x,t), \quad n \to \infty \tag{B.5}$$

since $\mathcal{C}_\alpha$ is a complete metric space. The convergence (B.5) implies

$$E\,\|Y(\cdot,t)\|^2_{\mathcal{H}_\alpha} = \lim_{n\to\infty} E\,\|Y_n(\cdot,t)\|^2_{\mathcal{H}_\alpha} = e_t C_\alpha. \tag{B.6}$$

Note, that $Y_n(t)$ is the solution to the Cauchy problem (1.2) with $V(x) \equiv 0$ and the initial data $Y_{0n} = F^{-1}\chi_n \hat{Y}_0$ in the sense (2.3). Proceeding to a limit in the equality (2.3), we get that $Y(t)$ is the solution to the same problem

with the initial data Y_0. We set $Y(t) = \mathcal{U}_0(t)Y_0$.
Step ii) Let us prove the uniqueness. Convergence (B.4) implies that for a fixed $t \in \mathbb{R}$

$$Y_n(\omega, x, t) \xrightarrow{\mathcal{L}^2_\alpha} Y(\omega, x, t), \quad n \to \infty.$$

Hence, for a fixed t and some subsequence n_k

$$Y_{n_k}(\omega, \cdot, t) \xrightarrow{\mathcal{H}_\alpha} Y(\omega, \cdot, t), \quad n_k \to \infty, \tag{B.7}$$

for almost all ω. Therefore, for almost all ω

$$Y_{n_k}(\omega, \cdot, t) \xrightarrow{\mathcal{S}'} Y(\omega, \cdot, t), \quad n_k \to \infty,$$

On the other hand, $Y_{n_k}(\omega, \cdot, t) \to U_0(t)Y_0(\omega, \cdot)$, $\quad n_k \to \infty$ in $\mathcal{S}'$ for almost all ω, where $U_0(t)Y_0 = \mathcal{G}_t * Y_0$ is a unique solution of problem (3.2) in $\mathcal{S}'$. Hence, $\mathcal{U}_0(t)Y_0 = U_0(t)Y_0$ for almost all ω and a fixed t. This implies the uniqueness.
Step iii) Let us write a solution to (1.2) in the form

$$Y(t) = X(t) + Z(t), \tag{B.8}$$

where $X(t) = U_0(t)Y_0 \in \mathcal{C}_\alpha$ is the solution to (3.2). Then $Z(t)$ is a solution to the inhomogeneous equation with zero initial value

$$\dot{Z}(t) = \mathcal{A}Z(t) + \mathcal{B}X(t), \ t \in \mathbb{R}; \ Z(0) = 0, \tag{B.9}$$

where the equation holds in the sense similar to (2.3) and

$$\mathcal{B} = \begin{pmatrix} 0 & V \\ -V & 0 \end{pmatrix}. \tag{B.10}$$

Denote by $U(t) : Y_0 \to Y(t)$ the dynamical group of the problem (1.2) for finite energy solutions with $Y_0 \in \mathcal{H}_0 = L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. The Duhamel representation for the solution to (B.9) gives

$$Z(t) = \int\limits_0^t U(t-s)\mathcal{B}X(s)ds, \tag{B.11}$$

where the integral exists in $\mathcal{L}^2_0$ since the integrand is a continuos function of s with the values in $\mathcal{L}^2_0$. Namely, according to **E1**, we have $\mathcal{B}X(s) \in \mathcal{C}_0$, since $-\beta \le \alpha$ and the group $U(t)$ is strongly continuous in $\mathcal{H}_0$. Therefore, $Z(\cdot) \in \mathcal{C}_0$ and then $Y(\cdot) \in \mathcal{C}_\alpha$. The uniqueness of the solution $Z(t)$ follows from (B.11).

Remark B.1. *We have proved that for any $t \in \mathbb{R}$ one has $\mathcal{U}_0(t)Y_0 = U_0(t)Y_0$ for almost all ω.*

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