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# Dispersive Estimates for 1D Discrete Schrödinger and Klein-Gordon Equations

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**Abstract**

We derive the long-time asymptotics for solutions of the discrete 1D Schrödinger and Klein-Gordon equations.

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## 1 Introduction

In this paper, we establish the long-time behavior of the solutions to the discrete Schrödinger and Klein-Gordon equations in one space dimension. We extend a general strategy introduced by Vainberg [12], Jensen-Kato [6], and Murata [8], which concerns the wave, Klein-Gordon, and Schrödinger equations, to the discrete case. Namely, we establish the Puiseux expansion for a resolvent of a stationary problem. Then the long-time asymptotics can be obtained by means of the (inverse) Fourier-Laplace transform.

We adopt the general scheme of [8] and make all constructions for the concrete case in detail. We restrict ourselves to a "nonsingular case", in the sense of [8], where the truncated resolvent is bounded at the ends of the continuous spectrum; this holds for a generic potential. It is just this case which allows us to get the desired time decay of order $\sim t^{-3/2}$, as is desirable for applications to scattering problems.

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First we consider the 1D discrete version of the Schrödinger equation

$$\left\{ \begin{array}{l} i\dot{\psi}(x,t) = H\psi(x,t) := (-\Delta + V(x))\,\psi(x,t) \\ \psi\big|_{t=0} = \psi_0 \end{array} \right| \quad x \in \mathbb{Z}, \quad t \in \mathbb{R}. \qquad (1.1)$$

Here $\Delta$ stands for the difference Laplacian in $\mathbb{Z}$, defined by

$$\Delta\psi(x) = \psi(x+1) - 2\psi(x) + \psi(x-1), \quad x \in \mathbb{Z},$$

for functions $\psi : \mathbb{Z} \to \mathbb{C}$. Denote by $\mathcal{S}$ the set of real functions on the lattice $\mathbb{Z}$ with a finite support. For the potential $V$ we assume that $V \in \mathcal{S}$. If we apply the Fourier-Laplace transform

$$\tilde{\psi}(x,\omega) = \int_0^\infty e^{i\omega t}\psi(x,t)\;dt, \quad \operatorname{Im}\omega > 0,$$

to (1.1), then the stationary equation

$$(H-\omega)\tilde{\psi}(\omega) = -i\psi_0, \quad \operatorname{Im}\omega > 0, \qquad (1.2)$$

is obtained. Here $\tilde{\psi}(\omega) := \tilde{\psi}(\cdot,\omega)$. Note that the integral converges, since $\|\psi(\cdot,t)\|_{l^2} = \text{const}$ by charge conservation. Hence we get as the solution

$$\tilde{\psi}(\omega) = -i\,R(\omega)\psi_0, \qquad (1.3)$$

where $R(\omega) = (H-\omega)^{-1}$ is the resolvent of the Schrödinger operator $H$.

We are going to use the function spaces which are the discrete version of the Agmon spaces [1]. These are the weighted Hilbert spaces $l^2_\sigma = l^2_\sigma(\mathbb{Z})$ with the norm

$$\|u\|_{l^2_\sigma} = \left\|(1+x^2)^{\sigma/2}u\right\|_{l^2}, \quad \sigma \in \mathbb{R}.$$

Let us denote

$$B(\sigma,\,\sigma') = \mathcal{L}(l^2_\sigma,\,l^2_{\sigma'}), \quad \mathbf{B}(\sigma,\,\sigma') = \mathcal{L}(l^2_\sigma \oplus l^2_\sigma,\,l^2_{\sigma'} \oplus l^2_{\sigma'})$$

the space of bounded linear operators from $l^2_\sigma$ to $l^2_{\sigma'}$ and from $l^2_\sigma \oplus l^2_\sigma$ to $l^2_{\sigma'} \oplus l^2_{\sigma'}$, respectively. Concerning further notation, we write $K = \operatorname{Op}(K(x,y))$ for the operator with kernel $K(x,y)$, i.e.,

$$(Ku)(x) = \sum_{y\in\mathbb{Z}} K(x,y)u(y), \quad x \in \mathbb{Z}.$$

We prove below that the continuous spectrum of the operator $H$ coincides with the interval $[0,4]$. Then our main results are as follows. For a generic potential $V \in \mathcal{S}$ (see Definition 5.1) satisfying the condition $\sum\limits_{x\in\mathbb{Z}} V(x) \neq 0$, we derive the Puiseux expansion for the resolvent at the singular spectral points $\mu = 0$ and $\mu = 4$ as

$$R(\mu+\omega) = R_0^\mu + R_1^\mu \omega^{1/2} + R_2^\mu \omega + R_3^\mu \omega^{3/2} + ... + \mathcal{O}(|\omega|^{N/2}), \ \omega \to 0. \quad (1.4)$$

This expansion is valid in the norm $B(\sigma, -\sigma)$ with a $\sigma$ depending on $N$. Then taking the inverse Fourier-Laplace transform of (1.3), it follows that for $\sigma > 7/2$

$$\Big\| e^{-itH} - \sum_{j=1}^{n} e^{-it\omega_j} P_j \Big\|_{B(\sigma,-\sigma)} = \mathcal{O}(t^{-3/2}), \quad t \to \infty. \quad (1.5)$$

Here $P_j$ are the orthogonal projections in $l^2$ onto the eigenspaces of $H$, corresponding to the discrete eigenvalues $\omega_j \in \mathbb{R}$.

For the proof, we first calculate an explicit formula for the resolvent of the free equation in the case where $V = 0$. This formula allows us to construct the expansion of the type (1.4) for the free resolvent. Then we prove (1.4) for $V \neq 0$, developing the Fredholm alternative arguments similar to [6], [8]. Finally, Lemma 10.2 of Jensen-Kato [6] plays a crucial role in verifying the decay (1.5).

We also obtain similar results for the discrete Klein-Gordon equation

$$\left\{ \begin{array}{l} \ddot\psi(x,t) = (\Delta - m^2 - V(x))\,\psi(x,t) \\ \psi\big|_{t=0} = \psi_0, \ \dot\psi\big|_{t=0} = \pi_0 \end{array} \right| \quad x \in \mathbb{Z}, \quad t \in \mathbb{R}. \quad (1.6)$$

Set $\mathbf{\Psi}(t) \equiv \big(\psi(\cdot,t), \dot\psi(\cdot,t)\big)$, $\mathbf{\Psi}_0 \equiv \big(\psi_0, \pi_0\big)$. Then (1.6) takes the form

$$i\dot{\mathbf{\Psi}}(t) = \mathbf{H}\mathbf{\Psi}(t), \quad t \in \mathbb{R}; \quad \mathbf{\Psi}(0) = \mathbf{\Psi}_0, \quad (1.7)$$

where

$$\mathbf{H} = \begin{pmatrix} 0 & i \\ i(\Delta - m^2 - V) & 0 \end{pmatrix}$$

The resolvent $\mathbf{R}(\omega) = (\mathbf{H} - \omega)^{-1}$ of the operator $\mathbf{H}$ can be expressed in terms of the resolvent $R(\omega)$, and this expression yields the corresponding properties of $\mathbf{R}(\omega)$. In particular, we derive the asymptotic expansion of the type (1.4) for $\mathbf{R}(\omega)$, and also the long-time asymptotics of the type (1.5) for the solution.

Let us comment on previous results in this direction. Eskina [3] and Shaban–Vainberg [10] considered the difference Schrödinger equation in dimensions $n \geq 1$. They proved the limiting absorption principle and applied it to the Sommerfeld radiation condition. However, [3, 10] do not concern the asymptotic expansion of $R(\omega)$ and the long-time asymptotics of the type (1.5).

The asymptotic expansion of the resolvent and the asymptotics (1.5) for continuous hyperbolic equations were obtained in [7], [11], [12], [13], and for Schrödinger equation in [4], [5], [6], [8]; also see [9] for an up-to-date review and many references concerning dispersive properties of solutions to the continuous Schrödinger equation in various norms. For the discrete Schrödinger and Klein-Gordon equations, the asymptotic expansion (1.4) and long-time asymptotics (1.5) seem to be obtained for the first time in the present paper.

The paper is organized as follows. In Section 2 we obtain an explicit formula for the free resolvent. In Section 3 we derive the asymptotic expansion of the free resolvent. The limiting absorption principle for the perturbed resolvent is proved in Section 4. In Sections 5 and 6 we get the Puiseux expansion of the perturbed resolvent. In Section 7 we prove the long-time asymptotics (1.5). In Section 8 we extend the results to the discrete Klein-Gordon equation. Finally, in an appendix we illustrate the presence of a discrete spectrum for potentials which are supported at one or two points.

# 2 The free resolvent

We start with an investigation of the unperturbed problem for equation (1.1) with $V = 0$.The discrete Fourier transform of $u : \mathbb{Z} \to \mathbb{C}$ is defined by the formula

$$\hat{u}(\theta) = \sum_{x\in\mathbb{Z}} u(x)e^{i\theta x}, \quad \theta \in T := \mathbb{R}/2\pi\mathbb{Z}.$$

After taking the Fourier transform, the operator $H_0 = -\Delta$ becomes the operator of multiplication by $\phi(\theta) = 2 - 2\cos\theta$:

$$-\widehat{\Delta u}(\theta) = \phi(\theta)\widehat{u}(\theta).$$

Thus, the operator $H_0$ is selfadjoint and its spectrum is absolutely continuous. It coincides with the range of the function $\phi$, that is $\operatorname{Spec} H_0 = [0, 4]$. Denote by $R_0(\omega) = (H_0 - \omega)^{-1}$ the resolvent of the difference Laplacian. Then the kernel of the resolvent $R_0(\omega) = (H_0 - \omega)^{-1}$ reads as

$$R_0(\omega, x, y) = \frac{1}{2\pi}\int\limits_T \frac{e^{-i\theta(x-y)}}{\phi(\theta) - \omega}\, d\theta, \quad \omega \in \mathbb{C} \setminus [0, 4]. \tag{2.1}$$

Let us calculate an explicit formula for $R_0(\omega, x, y)$ using the Cauchy residue theorem.

**Lemma 2.1.** *For $\omega \in \mathbb{C} \setminus [0,4]$ the resolvent is given by*

$$R_0(\omega,x,y) = -i\,\frac{e^{-i\theta(\omega)|x-y|}}{2\sin\theta(\omega)}, \quad x,y \in \mathbb{Z}, \tag{2.2}$$

*where $\theta(\omega)$ is the unique solution of the equation*

$$2 - 2\cos\theta = \omega \tag{2.3}$$

*in the domain $D := \{-\pi \le \operatorname{Re}\theta \le \pi,\ \operatorname{Im}\theta < 0\}$.*

*Proof.* First let us assume that $x - y \ge 0$. Denote by $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ the path indicated in Fig. 1, where

$$\begin{aligned}
\Gamma_1 &: \quad \operatorname{Re}\theta = -\pi,\ \operatorname{Im}\theta \in [-\infty, 0],\\
\Gamma_2 &: \quad \operatorname{Im}\theta = 0,\ \operatorname{Re}\theta \in [-\pi, 0],\\
\Gamma_3 &: \quad \operatorname{Im}\theta = 0,\ \operatorname{Re}\theta \in [0, \pi],\\
\Gamma_4 &: \quad \operatorname{Re}\theta = \pi,\ \operatorname{Im}\theta \in [0, -\infty].
\end{aligned}$$

The map $\theta \longmapsto \phi(\theta) = 2 - 2\cos\theta$ transforms the paths $\Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4$ to the (oriented) intervals of the real axis $(\infty, 4],\ [4,0],\ [0,4],\ [4,\infty)$ respectively. Note, that the path $\Gamma_c : \operatorname{Re}\theta = 0,\ -\infty < \operatorname{Im}\theta \le 0$ is mapped onto the interval $(-\infty, 0)$ and the region $D$ is transformed to the complex plane with the cut $[0,4]$. Hence, there exists a unique solution $\theta(\omega)$ of the equation $\phi(\theta) = \omega,\ \omega \notin [0,4]$, in the domain $D$.

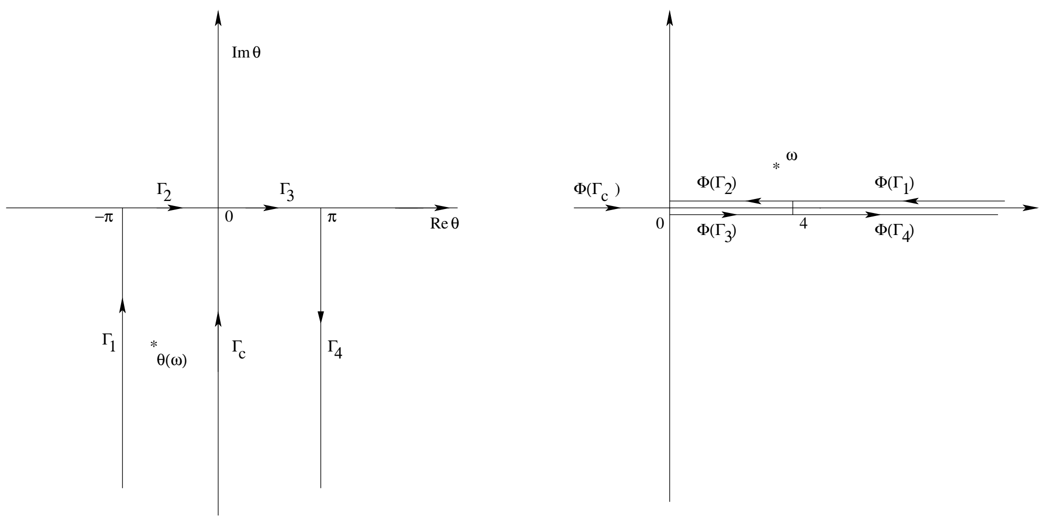


Figure 1: Conformal mapping $\phi(\theta)$

Therefore the integrand in (2.1) has one simple pole at the point $\theta(\omega)$, and from the Cauchy residue theorem it follows that

$$R_0(\omega,x,y)=\frac{1}{2\pi}\int\limits_{\Gamma}\frac{e^{-i\theta(x-y)}}{\phi(\theta)-\omega}\,d\theta=-i\,\mathrm{res}_{\theta(\omega)}\left(\frac{e^{-i\theta(x-y)}}{\phi(\theta)-\omega}\right).$$

This implies (2.2) for $x-y\geq 0$. If $x-y\leq 0$, we choose a similar path in the upper half-plane $\operatorname{Im}\theta>0$ and get the same formula (2.2). □

# 3 Puiseux expansion of the free resolvent

The free resolvent $R_0(\omega)$ is an analytic function with values in $B(0,0)$ for $\omega\in\mathbb{C}\setminus[0,4]$. This follows directly from the formula (2.2) since $\operatorname{Im}\theta(\omega)<0$, and the kernel (2.2) decays exponentially. For $\omega\in(0,4)$, the decay fails due to $\operatorname{Im}\theta(\omega)=0$, whereas for $\omega=0$ and $\omega=4$ the kernel does not exist since then $\sin\theta(\omega)=0$. Nevertheless, for the free resolvent the following limiting absorption principle holds.

**Lemma 3.1.** *For $\sigma>1/2$ the following limit exists as $\varepsilon\to 0+$:*

$$R_0(\omega\pm i\varepsilon)\ \xrightarrow{\mathcal{B}(\sigma,-\sigma)}\ R_0(\omega\pm i0),\quad \omega\in(0,4). \tag{3.1}$$

*Proof.* $R_0(\omega)$ is the operator with the kernel $R_0(\omega,x,y)$. If $\sigma>1/2$ and $\omega\notin\{0,4\}$, then the formula (2.2) implies that this is a Hilbert-Schmidt operator in the space $B(\sigma,-\sigma)$. For $\omega\in(0,4)$ and $x,y\in\mathbb{Z}$, there exists the pointwise limit

$$R_0(\omega\pm i\varepsilon,x,y)\to\ R_0(\omega\pm i0,x,y),\quad \varepsilon\to 0+\,.$$

Moreover, $|R_0(\omega\pm i\varepsilon,x,y)|\leq C(\omega)$. Therefore,

$$\sum_{x,y\in\mathbb{Z}}(1+x^2)^{-\sigma}|R_0(\omega\pm i\varepsilon,x,y)-R_0(\omega\pm i0,x,y)|^2(1+y^2)^{-\sigma}\to 0$$

as $\varepsilon\to 0+$ by the Lebesgue dominated convergence theorem. Hence the Hilbert-Schmidt norm of the difference $R_0(\omega\pm i\varepsilon)-R_0(\omega\pm i0)$ converges to zero, and (3.1) is proved. □

**Remark 3.1.** *Note that*

$$R_0(\omega-i0,x,y)=\overline{R_0(\omega+i0,x,y)},\quad \omega\in(0,4). \tag{3.2}$$

*This is a consequence of the relation $\overline{\theta(\omega)}=-\theta(\overline{\omega})$ for $\omega\in\mathbb{C}\setminus[0,4]$.*

Further, we need more information on the behavior of $R_0(\omega)$ near $\omega = 0$ and $\omega = 4$. Without loss of generality we consider only the case $\omega = 0$. By means of Taylor expansion we obtain from (2.3) that

$$\frac{1}{\sin\theta(\omega)} = \Big(\omega - \frac{\omega^2}{4}\Big)^{-1/2} = -\frac{1}{\sqrt{\omega}}(1 + \frac{\omega}{8} + \frac{3\omega^2}{128} + \dots),\ \omega \to 0,$$

where $\operatorname{Im}\sqrt{\omega} > 0$. This choice of the branch provides $\operatorname{Im}\theta(\omega) < 0$ that corresponds to the exponentially decay of the kernel (2.2). Similarly,

$$e^{-i\theta(\omega)} = \cos\theta(\omega) - i\sin\theta(\omega) = 1 - \frac{\omega}{2} + i\sqrt{\omega}(1 - \frac{\omega}{8} - \frac{\omega^2}{128} - \dots),\ \omega \to 0.$$

Therefore, we get the formal expansion

$$R_0(\omega, x, y) \sim \sum_{j=-1}^{\infty} \omega^{j/2} R_0^j(x, y), \quad \omega \to 0, \tag{3.3}$$

where $R_0^{-1}(x, y) = \frac{i}{2}$, $R_0^0(x, y) = -\frac{1}{2}|x - y|$, and $R_0^j(x, y) = \sum\limits_{k=0}^{j+1} c_{kj}\,|x - y|^k$ for $j \in \mathbb{N}$, with suitable coefficients $c_{kj} \in \mathbb{C}$.

For the next result, cf. [6, Lemma 2.3].

**Lemma 3.2.** *i) If* $\sigma > 1/2 + j + 1$, *then* $R_0^j = \operatorname{Op}\big(R_0^j(x, y)\big) \in B(\sigma, -\sigma)$.
*ii) The asymptotics* (3.3) *hold in the operator sense:*

$$R_0(\omega) = \sum_{j=-1}^{N} \omega^{j/2} R_0^j + r_N(\omega), \quad \omega \to 0, \tag{3.4}$$

*where* $\|r_N(\omega)\|_{B(\sigma,-\sigma)} = \mathcal{O}(|\omega|^{(N+1)/2})$ *with* $\sigma > 1/2 + N + 2$.
*iii) In the same sense,* (3.4) *can be differentiated* $N + 2$ *times in* $\omega$*:*

$$(d/d\omega)^r R_0(\omega) = \sum_{j=-1}^{N} (d/d\omega)^r \omega^{j/2} R_0^j + \tilde{r}_N(\omega), \quad \omega \to 0,$$

*where* $\|\tilde{r}_N(\omega)\|_{B(\sigma,-\sigma)} = \mathcal{O}(|\omega|^{(N+1)/2-r})$ *with the same* $\sigma > 1/2 + N + 2$.

*Proof.* By Taylor expansion with remainders, it is possible to check that

$$r_N(\omega, x, y) = \Big(\sum_{k=0}^{N+2} b_k(\omega)|x - y|^k\Big)\,\omega^{(N+1)/2},$$

where all $b_k(\omega) = \mathcal{O}(1)$. It remains to note that for $k = 0, \dots, N + 2$ the kernels $|x - y|^k$ define Hilbert-Schmidt operators in the spaces $B(\sigma, -\sigma)$, provided that $\sigma > 1/2 + N + 2$; this is due to the fact that $|x - y|^{2k} \le C((1 + x^2)^k + (1 + y^2)^k)$. $\square$

# 4 The limiting absorbtion principle

Let $M < \infty$ be the number of points in the support of $V$. Then the rank of the operator of multiplication by $V$ equals $M$. Therefore we have the following result.

**Lemma 4.1.** *i)* $\mathrm{Spec}_{\mathrm{ess}} H = [0,4]$.
*ii) The spectrum of $H$, outside the interval $[0,4]$, consists of real eigenvalues $\omega_j$, $j = 1, ..., n$, where $n \le M$.*

Unfortunately we do not know an example of a potential $V$ for which the discrete spectrum is empty. In the appendix we provide some illustration by showing that the discrete spectrum is nonempty, if the support of $V$ consists of one or two points.

In the next lemma we develop the results of [3], [10] for the 1D case and prove the limiting absorption principle in the sense of the operator convergence. It will be needed for the proof of the long-time asymptotics (1.5).

**Lemma 4.2.** *Let $V \in \mathcal{S}$ and $\sigma > 1/2$. Then the following limits exist as $\varepsilon \to 0+$*

$$R(\omega \pm i\varepsilon) \quad \stackrel{B(\sigma,-\sigma)}{\longrightarrow} \quad R(\omega \pm i0), \quad \omega \in (0,4). \tag{4.1}$$

*Proof. Step i)* First we verify that for $\omega \in (0,4)$ the operator $1 + VR_0(\omega \pm i0)$ has only a trivial kernel; for instance, we consider the "+"-case. Let $h$ be a solution of

$$h + VR_0(\omega + i0)h = 0. \tag{4.2}$$

Note that $V(x) = 0$ for some $x \in \mathbb{Z}$ also yields $h(x) = 0$, i.e., $h \in \mathcal{S}$. Now for $x \in \operatorname{supp} V$, (4.2) implies

$$\sum_{y\in\mathbb{Z}} R_0(\omega + i0, x, y)h(y) = -\frac{h(x)}{V(x)}. \tag{4.3}$$

Multiplying (4.3) by $\overline{h}(x)$ and taking the sum over $x \in \operatorname{supp} V$, we get from (2.2) and Lemma 3.1,

$$\mathrm{Im}\Big[\sum_{x,y\in\mathbb{Z}} i\,\frac{e^{-i\theta_+|x-y|}}{2\sin\theta_+}\,h(y)\overline{h}(x)\Big] = 0, \tag{4.4}$$

where $\theta_+ = \theta(\omega + i0) \in (-\pi, 0)$. Since $\theta_+$ is real, also $\sin\theta_+$ is a real number. Thus (4.4) implies

$$\sum_{x,y\in\mathbb{Z}} \cos\big(\theta_+(x-y)\big)h(y)\overline{h}(x) = 0,$$

and therefore

$$\left|\sum_{x\in\mathbb{Z}}\cos(\theta_+x)h(x)\right|^2+\left|\sum_{x\in\mathbb{Z}}\sin(\theta_+x)h(x)\right|^2=0.$$

In summary, if $\omega\in(0,4)$ and $h$ is such that (4.2) holds, then $\widehat{h}(\theta_+)=0$ for $\theta_+=\theta(\omega+i0)$. Moreover, equality $\theta_-=\theta(\omega-i0)=-\theta_+$ implies that $\widehat{h}(\theta_-)=0$. Hence the function $\widehat{\psi}(\theta)=\dfrac{\widehat{h}(\theta)}{\phi(\theta)-\omega}$ is an entire function of $\theta\in\mathbb{C}$. It is easy to check that the trigonometric polynomial $\phi(\theta)-\omega$ has simple roots for $\omega\in(0,4)$, and therefore $\widehat{\psi}(\theta)$ is also a trigonometric polynomial. This implies that $\psi(x)$ has a finite support; see [10, Thm. 9] for a similar argument. Moreover, $\psi$ is the unique solution of the equation

$$(-\Delta-\omega)\psi=h. \tag{4.5}$$

Next we prove that also $\varphi=R_0(\omega+i0)h$ is a solution to (4.5). Indeed, the function $R_0(\eta)h$ satisfies (4.5) with $\omega=\eta\notin(0,4)$, and from Lemma 3.1 it follows that one can pass to the limit in the equation as $\eta\to\omega+i0$. Thus the uniqueness for (4.5) yields that $\psi=\varphi=R_0(\omega+i0)h$. Consequently,

$$(-\Delta-\omega+V)\psi=0, \tag{4.6}$$

since $(-\Delta-\omega+V)\psi=h+V\psi=h+VR_0(\omega+i0)h=0$ by (4.2). But the only solution of (4.6) with a finite support is $\psi\equiv 0$, which implies $h\equiv 0$.

*Step ii)* Fix $\omega\in(0,4)$ and $\sigma>1/2$. Then Lemma 3.1 yields

$$1+VR_0(\omega\pm i\varepsilon)\ \xrightarrow{\mathcal{B}(\sigma,\sigma)}\ 1+VR_0(\omega\pm i0),\quad \varepsilon\to 0+\,;$$

For this, recall that the potential $V$ is assumed to be compactly supported in $\mathbb{Z}$. Therefore the convergence $R_0(\omega\pm i\varepsilon)\to R_0(\omega\pm i0)$ in $B(\sigma,-\sigma)$ is improved to convergence in $B(\sigma,\sigma)$ through multiplication by $V$. By Step i), the operator $1+VR_0(\omega\pm i0)$ has only a trivial kernel. Hence, being Fredholm if index zero, $1+VR_0(\omega\pm i0)$ is invertible, and moreover

$$\big(1+VR_0(\omega\pm i\varepsilon)\big)^{-1}\ \xrightarrow{\mathcal{B}(\sigma,\sigma)}\ \big(1+VR_0(\omega\pm i0)\big)^{-1},\quad \varepsilon\to 0+\,.$$

Then the representation $R=R_0(1+VR_0)^{-1}$ implies (4.1). □

**Remark 4.1.** *Equation* (3.2) *implies*

$$R(\omega-i0,x,y)=\overline{R(\omega+i0,x,y)},\quad \omega\in(0,4).$$

# 5 Fredholm alternative argument

In this section we are going to obtain an asymptotic expansion for the perturbed resolvent $R(\omega)$. In particular, we will show that no term of order $\omega^{-1/2}$ appears in the series for $R(\omega)$ in the case of a generic potential $V \in \mathcal{S}$, regardless of the singularity of $R_0(\omega)$.

**Definition 5.1.** *i) A set $\mathcal{V} \subset \mathcal{S}$ is called generic, if for each $V \in \mathcal{S}$ we have $\alpha V \in \mathcal{V}$, with the possible exception of a discrete set of $\alpha \in \mathbb{C}$.*
*ii) We say that a property holds for a "generic" $V$, if it holds for all $V$ from a generic subset of $\mathcal{S}$.*

We consider the asymptotic behavior of $R(\omega)$ at the singular points $\omega = 0$ and $\omega = 4$. For instance, we focus on $\omega = 0$ and construct the resolvent $R(\omega)$ for small $|\omega|$ in the case of a generic potential $V$. This will be achieved by means of the relation

$$R(\omega) = (1 + R_0(\omega)V)^{-1}R_0(\omega).$$

According to Section 3, it remains to construct $(1 + R_0(\omega)V)^{-1}$. First we note that

$$T(\omega) = 1 + R_0(\omega)V = \text{Op}[\delta(x-y) + R_0(\omega, x, y)V(y)]. \tag{5.1}$$

Taking into account (3.3) we decompose (5.1) as

$$T(\omega) = T_r(\omega) + T_s(\omega), \tag{5.2}$$

with

$$T_r(\omega) = \text{Op}\Big[\delta(x-y) + \Big(R_0(\omega, x, y) - \frac{i}{2}\,\omega^{-1/2}\Big)V(y)\Big] \tag{5.3}$$

and

$$T_s(\omega) = \text{Op}\,\big[\frac{i}{2}\,\omega^{-1/2}V(y)\big] \tag{5.4}$$

which isolates the singular term in the expansion of $T(\omega)$. This operator acts as

$$(T_s(\omega)u)(x) = \frac{i}{2}\,\omega^{-1/2}\langle V, u\rangle := \frac{i}{2}\,\omega^{-1/2}\sum_{y\in\mathbb{Z}} V(y)u(y), \tag{5.5}$$

and hence its range is the one-dimensional subspace of constant functions. To determine

$$u(\omega) := R(\omega)\psi = (1 + R_0(\omega)V)^{-1}R_0(\omega)\psi$$

for a given function $\psi$, put $f(\omega) = R_0(\omega)\psi$. Thus we are looking for solutions $u(\omega) \in l^2_{-\sigma}$, $\sigma > 3/2$ of the equation $T(\omega)u(\omega) = f(\omega)$. Accordingly, we decompose the space $l^2_{-\sigma}$ as the sum of orthogonal subspaces as $l^2_{-\sigma} = V^\perp + V^\parallel$, where the orthogonality refers to the $l^2$ inner product $\langle\cdot,\cdot\rangle$, and $V^\parallel$ is the one-dimensional subspace spanned by $V$. Therefore we can write

$$u(\omega) = u^\perp(\omega) + c(\omega)v, \quad v := V/\|V\|, \tag{5.6}$$

with suitable $u^\perp(\omega) \in V^\perp$ and $c(\omega) \in \mathbb{C}$; here $\|V\| = \|V\|_{l^2}$. By (5.5) we have $V^\perp \subset \ker T_s(\omega)$. Thus $T_s(\omega)u^\perp(\omega) = 0$, and consequently $T(\omega)u(\omega) = f(\omega)$ is equivalent to

$$T_r(\omega)u^\perp(\omega) + c(\omega)T(\omega)v = f(\omega). \tag{5.7}$$

**Lemma 5.1.** *Let $\sigma > 3/2$. Then for a generic potential $V \in \mathcal{S}$ the operator $T_r(\omega) : l^2_{-\sigma} \to l^2_{-\sigma}$ is invertible, provided that $|\omega|$ is sufficiently small.*

*Proof.* First we show that for a generic potential $V \in \mathcal{S}$ the operator $T_r(0) : l^2_{-\sigma} \to l^2_{-\sigma}$ is invertible. Since

$$T_r(0) = \mathrm{Op}\Big[\delta(x-y) - \frac{1}{2}|x-y|V(y)\Big],$$

it suffices to prove that the operator

$$\mathrm{Op}\Big[(1+x^2)^{-\sigma/2}\Big(\delta(x-y) - \frac{1}{2}\,|x-y|\,V(y)\Big)(1+y^2)^{\sigma/2}\Big]$$

is an invertible operator in $l^2$. And this holds generically. Indeed, for a given potential $V \in \mathcal{S}$ we introduce

$$\begin{aligned}\mathcal{A}(\alpha) &= \mathrm{Op}\Big[(1+x^2)^{-\sigma/2}\Big(\delta(x-y) - \frac{\alpha}{2}\,|x-y|\,V(y)\Big)(1+y^2)^{\sigma/2}\Big] \\ &= 1 + \alpha\mathcal{K}, \quad \alpha \in \mathbb{C}.\end{aligned}$$

Due to $\sigma > 3/2$, the function

$$K(x,y) = -\frac{1}{2}\,(1+x^2)^{-\sigma/2}|x-y|\,V(y)(1+y^2)^{\sigma/2} \in l^2(\mathbb{Z}\times\mathbb{Z}).$$

Hence $K(x,y)$ is a Hilbert-Schmidt kernel, and accordingly the operator $\mathcal{K} = \mathrm{Op}(K(x,y))$: $l^2 \to l^2$ is compact. Further, $\mathcal{A}(\alpha)$ is analytic in $\alpha \in \mathbb{C}$ and $\mathcal{A}(0)$ is invertible. It follows that $\mathcal{A}(\alpha)$ is invertible for all $\alpha \in \mathbb{C}$ outside a discrete set; see [2]. Thus we could replace the original potential $V$ by $\alpha V$ with $\alpha$ arbitrarily close to 1, if necessary, to have $T_r(0)$ invertible. Since $T_r(\omega) - T_r(0) \to 0$ as $\omega \to 0$, also $T_r(\omega)$ is invertible for sufficiently small $|\omega|$. □

Put

$$w(\omega) = (T_r^{-1}(\omega))^* v,$$

where $T_r^{-1}(\omega)$ exists by Lemma 5.1. Since $v \in l^2_\sigma$ for any $\sigma \in \mathbb{R}$, we also get

$$w(\omega) \in \bigcap_{\sigma > 3/2} l^2_\sigma.$$

Furthermore, for $v^\perp \in V^\perp$ one obtains

$$\langle w(\omega), T_r(z)v^\perp \rangle = \Big\langle (T_r^{-1}(\omega))^* v,\, T_r(\omega)v^\perp \Big\rangle = \langle v, v^\perp \rangle = 0,$$

so that

$$w(\omega) \;\perp\; T_r(\omega)V^\perp.$$

Now, taking the inner product of (5.7) with $w(\omega)$ we find

$$c(\omega) = \frac{\langle f(\omega), w(\omega)\rangle}{\langle T(\omega)v, w(\omega)\rangle}\,, \tag{5.8}$$

provided that

$$\langle T(\omega)v, w(\omega)\rangle \neq 0.$$

**Lemma 5.2.** *For a generic potential $V \in \mathcal{S}$ with $\sum\limits_{x\in\mathbb{Z}} V(x) \neq 0$, the relation $\langle T(\omega)v, w(\omega)\rangle \neq 0$ holds for sufficiently small $|\omega| \neq 0$.*

*Proof.* Denote

$$T_r(0,\alpha) = \mathrm{Op}\Big[\delta(x-y) - \frac{\alpha}{2}\,|x-y|\,V(y)\Big], \quad \alpha \in \mathbb{C}.$$

Then $T_r(0,1) = T_r(0)$, $T_r(0,0) = \mathrm{Op}\,[\delta(x-y)]$, and $\langle T_r(0,0)^{-1}1, V\rangle = \langle 1, V\rangle \neq 0$. Hence, the meromorphic function $\alpha \mapsto \langle T_r(0,\alpha)^{-1}1, V\rangle$ does not vanish identically, and thus we have $\langle T_r(0,\alpha)^{-1}1, V\rangle \neq 0$ for all $\alpha \in \mathbb{C}$ outside a discrete set. Therefore we could replace the original potential $V$ by $\alpha V$ with $\alpha$ arbitrarily close to 1, if necessary, to ensure that

$$\langle T_r^{-1}(0)1, V\rangle \neq 0 \tag{5.9}$$

Then for a generic potential $V \in \mathcal{S}$ with $\langle 1, V\rangle = \sum\limits_{x\in\mathbb{Z}} V(x) \neq 0$, we have

$$\begin{aligned}
\langle T(\omega)v, w(\omega)\rangle &= \langle T_r(\omega)v, w(\omega)\rangle + \langle T_s(\omega)v, w(\omega)\rangle \\
&= \Big\langle T_r(\omega)v, (T_r^{-1}(\omega))^* v \Big\rangle + \frac{i}{2}\omega^{-1/2}\langle V, v\rangle\langle 1, w(\omega)\rangle \\
&= 1 + \frac{i}{2}\omega^{-1/2}\|V\|\langle T_r^{-1}(\omega)1, v\rangle \qquad (5.10) \\
&= \frac{i}{2}\omega^{-1/2}\langle T_r^{-1}(0)1, V\rangle + o(\omega^{-1/2}) \neq 0
\end{aligned}$$

for sufficiently small $|\omega| \neq 0$. □

By Lemma 5.1, (5.7) yields

$$u^{\perp}(\omega) = T_r^{-1}(\omega)\Big(f(\omega) - c(\omega)\,T(\omega)v\Big).$$

Thus (5.6) implies that

$$u(\omega) = T_r^{-1}(\omega)\Big(f(\omega) - c(\omega)\,T(\omega)v\Big) + c(\omega)v.$$

Hence we can summarize the foregoing arguments as follows:

**Theorem 5.1.** *Let $\sigma > 3/2$. Then for a generic potential $V \in \mathcal{S}$ with $\sum\limits_{x\in\mathbb{Z}} V(x) \neq 0$, the resolvent $R(\omega) = (H-\omega)^{-1}$ can be expressed as*

$$R(\omega)\psi = T_r^{-1}(\omega)\Big(f(\omega) - c(\omega)\,T(\omega)v\Big) + c(\omega)v, \tag{5.11}$$

*where $T_r(\omega)$ is from (5.3) and invertible by Lemma 5.1, $f(\omega) = R_0(\omega)\psi$, $c(\omega)$ is given by* (5.8)*, and $T(\omega) = 1 + R_0(\omega)V$.*

# 6 Puiseux expansion

**Theorem 6.1.** *Let $\sigma > 7/2$. Then for a generic potential $V \in \mathcal{S}$ with $\sum\limits_{x\in\mathbb{Z}} V(x) \neq 0$, the resolvent $R(\omega)$ has the expansion*

$$R(\omega) = R^0 + \mathcal{O}(|\omega|^{1/2}), \quad \omega \to 0, \tag{6.1}$$

*where the asymptotics hold in the norm of $B(\sigma, -\sigma)$. See (6.6) below for the explicit form of $R^0$.*

*Proof. Step i).* Fix $\sigma > 7/2$. Equations (3.3) and (5.3) imply that for small $|\omega|$,

$$T_r(\omega) = T_0 + \omega^{1/2}\,T_1 + \mathcal{O}(|\omega|)$$

in $B(-\sigma, -\sigma)$, where

$$\begin{aligned} T_0 &= T_r(0) = \mathrm{Op}\Big[\delta(x-y) - \frac{1}{2}\,|x-y|\,V(y)\Big], \\ T_1 &= \mathrm{Op}\Big[\sum_{k=0}^{2} c_{k1}\,|x-y|^k\,V(y)\Big] = \frac{i}{4}\,\mathrm{Op}\Big[\Big(\frac{1}{4} - |x-y|^2\Big)V(y)\Big]. \end{aligned}$$

Note that again the compact support of $V$ is used here. Next we write down the Neumann series for $T_r^{-1}(\omega)$ about the invertible $T_0 = T_r(0)$ to obtain

$$T_r^{-1}(\omega) = S_0 + \omega^{1/2} S_1 + \mathcal{O}(|\omega|), \quad \omega \to 0, \tag{6.2}$$

in $B(-\sigma, -\sigma)$, where

$$S_0 = T_0^{-1} = T_r(0)^{-1}, \quad S_1 = -T_0^{-1} T_1 T_0^{-1}.$$

*Step ii).* Now let us calculate $c(\omega)$. From (6.2) we deduce

$$(T_r^{-1}(\omega))^* = S_0^* + \omega^{1/2} S_1^* + \mathcal{O}(|\omega|)$$

in $B(\sigma, \sigma)$ for $\sigma > 7/2$. Thus

$$w(\omega) = (T_r^{-1}(\omega))^* v = w_0 + \omega^{1/2} w_1 + \mathcal{O}(|\omega|) \tag{6.3}$$

in $l_\sigma^2$ for $\sigma > 7/2$, where

$$w_0 = S_0^* v, \quad w_1 = \mathcal{S}_1^* v.$$

By (3.3),

$$R_0(\omega) = \frac{i}{2}\,\omega^{-1/2} \mathrm{Op}(1) + R_0^0 + \omega^{1/2} R_0^1 + \mathcal{O}(|\omega|) \tag{6.4}$$

in $\mathcal{B}(\sigma, -\sigma)$ for $\sigma > 7/2$. Hence the numerator of (5.8) admits the asymptotic expansion

$$\begin{aligned}
\langle f(\omega), w(\omega)\rangle &= \langle R_0(\omega)\psi, w(\omega)\rangle \\
&= \Big\langle \frac{i}{2}\,\omega^{-1/2}\mathrm{Op}(1)\psi + R_0^0\psi + \omega^{1/2} R_0^1\psi + \mathcal{O}(|\omega|), \\
&\qquad w_0 + \omega^{1/2} w_1 + \mathcal{O}(|\omega|)\Big\rangle \\
&= \frac{i}{2}\,\omega^{-1/2}\langle 1, \psi\rangle\langle 1, w_0\rangle + \frac{i}{2}\,\langle 1, \psi\rangle\langle 1, w_1\rangle + \langle R_0^0\psi, w_0\rangle \\
&\quad + \mathcal{O}(|\omega|^{1/2}).
\end{aligned}$$

Next we have to expand the denominator of (5.8). By (5.10) and (6.3),

$$\begin{aligned}
\langle T(\omega)v, w(\omega)\rangle &= 1 + \frac{i}{2}\,\omega^{-1/2}\|V\|\langle 1, (T_r^{-1}(\omega))^* v\rangle \\
&= 1 + \frac{i}{2}\,\omega^{-1/2}\|V\|\langle 1, w_0 + \omega^{1/2} w_1 + \mathcal{O}(|\omega|)\rangle \\
&= \frac{i}{2}\,\omega^{-1/2}\|V\|\langle 1, w_0\rangle + 1 + \frac{i}{2}\|V\|\langle 1, w_1\rangle + \mathcal{O}(|\omega|^{1/2}).
\end{aligned}$$

We already noticed that for a generic potential

$$\langle 1, w_0\rangle = \langle 1, S_0^* v\rangle = \langle 1, (T_r^{-1}(0))^* v\rangle = \langle T_r^{-1}(0)1, v\rangle \neq 0,$$

recall (5.9). Hence (5.8) implies

$$\begin{aligned} c(\omega) &= \frac{\langle f(\omega), w(\omega)\rangle}{\langle T(\omega)v, w(\omega)\rangle} \\ &= \frac{\frac{i}{2}\,\omega^{-1/2}\langle 1,\psi\rangle\langle 1, w_0\rangle + \frac{i}{2}\langle 1,\psi\rangle\langle 1, w_1\rangle + \langle R_0^0\psi, w_0\rangle + \mathcal{O}(|\omega|^{1/2})}{\frac{i}{2}\,\omega^{-1/2}\|V\|\langle 1, w_0\rangle + 1 + \frac{i}{2}\|V\|\langle 1, w_1\rangle + \mathcal{O}(|\omega|^{1/2})} \\ &= c_0 + \omega^{1/2} c_1 + \mathcal{O}(|\omega|), \end{aligned} \tag{6.5}$$

where $c_0 = \|V\|^{-1}\langle 1,\psi\rangle$ and $c_1 \in \mathbb{C}$ is appropriate.
*Step iii).* Substituting (5.2), (5.4), (6.2), (6.4), and (6.5) into (5.11), and noting the key relation

$$\frac{i}{2}\,\omega^{-1/2}\,\mathrm{Op}(1)\psi - c_0\mathrm{Op}\left[\frac{i}{2}\,\omega^{-1/2}\,V(y)\right]v = \frac{i}{2}\,\omega^{-1/2}\Big(\langle 1,\psi\rangle - c_0\langle V, v\rangle\Big) = 0,$$

we obtain the following asymptotic expansion for $R(\omega)\psi$.

$$\begin{aligned} R(\omega)\psi &= T_r^{-1}(\omega)\Big(R_0(\omega)\psi - c(\omega)\,[T_r(\omega) + T_s(\omega)]v\Big) + c(\omega)v \\ &= T_r^{-1}(\omega)\Big(\frac{i}{2}\,\omega^{-1/2}\,\mathrm{Op}(1)\psi + R_0^0\psi + \mathcal{O}(|\omega|^{1/2}) \\ &\qquad - (c_0 + \omega^{1/2}c_1 + \mathcal{O}(|\omega|))\,\mathrm{Op}\left[\frac{i}{2}\,\omega^{-1/2}\,V(y)\right]v\Big) \\ &= T_r^{-1}(\omega)\Big(R_0^0\psi + \mathcal{O}(|\omega|^{1/2}) - \frac{i}{2}\,(c_1 + \mathcal{O}(|\omega|^{1/2}))\|V\|\Big) \\ &= \Big(S_0 + \mathcal{O}(|\omega|^{1/2})\Big)\Big(R_0^0\psi - \frac{i}{2}\,c_1\|V\| + \mathcal{O}(|\omega|^{1/2})\Big) \\ &= S_0\Big(R_0^0\psi - \frac{i}{2}\,c_1\|V\|\Big) + \mathcal{O}(|\omega|^{1/2}). \end{aligned}$$

This expansion does not contain singular terms in $\omega^{-1/2}$, since they have cancelled. Therefore defining $R^0\psi = S_0(R_0^0\psi - \frac{i}{2}\,c_1\|V\|)$, the proof of Theorem 6.1 is complete; the explicit form of the operator $R^0$ can be obtained by calculating $c_1 = c_1(\psi) \in \mathbb{C}$ from (6.5). More precisely, it is found that

$$c_1 = \frac{\|V\|\langle R_0^0\psi, w_0\rangle - \langle 1,\psi\rangle}{\frac{i}{2}\|V\|^2\langle 1, w_0\rangle},$$

so that

$$R^0\psi = \left( S_0 R_0^0 \psi - \frac{\langle S_0 R_0^0 \psi, V\rangle}{\langle S_0(1), V\rangle} S_0(1) \right) + \frac{\langle \psi, 1\rangle}{\langle S_0(1), V\rangle} S_0(1) \tag{6.6}$$

is obtained. Here the first operator makes the projection of $S_0 R_0^0 \psi$ onto the space $V^\perp$ along the vector $S_0(1)$ and the second operator is of range 1. □

**Corollary 6.1.** *Let $\sigma > 7/2$. Then for a generic potential $V \in \mathcal{S}$ with $\sum_{x\in\mathbb{Z}} V(x) \neq 0$, the resolvent expansion of $R(\omega)$ from (6.1) may be differentiated in $\omega$ three times, and for $r = 1, 2, 3$,*

$$(d/d\omega)^r R(\omega) = \mathcal{O}(|\omega|^{1/2-r}), \quad \omega \to 0, \tag{6.7}$$

*in $B(\sigma, -\sigma)$.*

*Proof.* Note that

$$R(\omega) = (1 + R_0(\omega)V)^{-1} R_0(\omega),$$

and $R_0(\omega)$ has a differentiable asymptotic series by Lemma 3.2. Hence it suffices to argue that the asymptotic series for $(1+R_0(\omega)V)^{-1}$ is differentiable. For the latter, it may be shown that

$$(d/d\omega)(1 + R_0 V)^{-1} = -(1 + R_0 V)^{-1} R_0' V (1 + R_0 V)^{-1},$$

and after some lengthy but straightforward calculation also (6.7) is found. □

**Remark 6.1.** *A similar expansion of $R(\omega)$ is valid as $\omega \to 4$.*

# 7 Long-time asymptotics

**Theorem 7.1.** *Let $\sigma > 7/2$. Then for a generic potential $V \in \mathcal{S}$ with $\sum_{x\in\mathbb{Z}} V(x) \neq 0$, the asymptotics* (1.5) *hold, i.e.,*

$$\left\| e^{-itH} - \sum_{j=1}^{n} e^{-it\omega_j} P_j \right\|_{B(\sigma,-\sigma)} = \mathcal{O}(t^{-3/2}), \quad t \to \infty.$$

*Here $P_j$ denote the projections on the eigenspaces corresponding to the eigenvalues $\omega_j \in \mathbb{R} \setminus [0, 4]$, $j = 1, \ldots, n$.*

*Proof.* The estimate for $e^{-itH}$ is based on the formula

$$e^{-itH} = -\frac{1}{2\pi i} \oint\limits_{|\omega|=C} e^{-it\omega} R(\omega) d\omega, \ C > \max\{4; |\omega_j|, \ j = 1, ..., n\}. \tag{7.1}$$

Encircling the spectrum $[0,4] \cup \{\omega_j : j = 1, \ldots, n\}$ of $H$ by smaller and smaller pathes, it follows from

$$P_j = -\frac{1}{2\pi i} \oint\limits_{|\omega-\omega_j|=\varepsilon} R(\omega)\, d\omega$$

for $\varepsilon > 0$ sufficiently small and Remark 4.1 that

$$\begin{aligned} e^{-itH} - \sum_{j=1}^{n} e^{-it\omega_j} P_j &= \frac{1}{2\pi i} \int\limits_{[0,4]} e^{-it\omega} (R(\omega + i0) - R(\omega - i0))\, d\omega \\ &= \frac{1}{\pi} \int\limits_{[0,4]} e^{-it\omega} \operatorname{Im} R(\omega + i0)\, d\omega = \int\limits_{[0,4]} e^{-it\omega} P(\omega) d\omega, \end{aligned}$$

where $P(\omega) = \dfrac{1}{\pi} \operatorname{Im} R(\omega + i0)$. The asymptotic expansion for $P(\omega)$ at the singular points $\mu = 0$ and $\mu = 4$ can be deduced from (6.1). Thus, restricting to $\omega \in \mathbb{R}$, we have

$$P(\mu + \omega) = \mathcal{O}(|\omega|^{1/2}), \ \omega \to 0. \tag{7.2}$$

To prove the desired decay for large $t$, it is convenient to represent the function $P(\omega)$ for $\omega \in [0,4]$ as

$$P(\omega) = \phi_1(\omega) P(\omega) + \phi_2(\omega) P(\omega), \tag{7.3}$$

where $\phi_j(\omega) \in C_0^\infty(\mathbb{R})$ for $j = 1, 2$, $\phi_1(\omega) + \phi_2(\omega) = 1$ for $\omega \in [0,4]$, $\operatorname{supp} \phi_1 \subset (-1, 3)$, and $\operatorname{supp} \phi_2 \subset (1, 5)$. Due to (7.2) and Corollary 6.1, we can apply Lemma 7.1 below with $F = \phi_1 P$, $a = 3$, $\mathbf{B} = B(\sigma, -\sigma)$ where $\sigma > 7/2$, and $\theta = 1/2$ to get

$$\int\limits_{[0,4]} e^{-it\omega} \phi_1(\omega) P(\omega) d\omega = \mathcal{O}(t^{-3/2}), \quad t \to \infty,$$

in $B(\sigma, -\sigma)$. Since the same argument can be used for $F = \phi_2 P$, (7.3) shows that the proof is complete. □

The following result is a special case of [6, Lemma 10.2].

**Lemma 7.1.** *Assume $\mathcal{B}$ is a Banach space, $a > 0$, and $F \in C(0,a;\mathbf{B})$ satisfies $F(0) = F(a) = 0$, $F' \in L^1(0,a;\mathbf{B})$, as well as $F''(\omega) = \mathcal{O}(\omega^{\theta-2})$ as $\omega \searrow 0$ for some $\theta \in (0,1)$. Then*

$$\int_0^a e^{-it\omega}F(\omega)d\omega = \mathcal{O}(t^{-1-\theta}), \quad t \to \infty.$$

# 8 The Klein-Gordon equation

Now we extend the results of Sections 5-7 to the case of the Klein-Gordon equation (1.6)-(1.7). The operator $\mathbf{H}$ is not selfadjoint in $l^2 \oplus l^2$. First we prove the existence and uniqueness of the global solution $\mathbf{\Psi} := e^{-it\mathbf{H}}\mathbf{\Psi}_0$.

**Lemma 8.1.** *For any initial data $\mathbf{\Psi}_0(x) \in l^2 \oplus l^2$ there exists a unique solution $\mathbf{\Psi}(x,t) \in C(\mathbb{R}, l^2 \oplus l^2)$ of (1.7).*

*Proof.* The existence of a local solution for sufficiently small $|t|$ is shown by the contraction mapping method. That this local solution can be extended to a global solution follows from the energy a priori estimate. In fact, multiplying (1.6) by $\dot{\psi}(x,t)$ and taking the sum over $x \in \mathbb{Z}$, we have

$$\frac{d}{dt}\left(\left\|\dot{\psi}(t)\right\|_{l^2}^2 + \|\nabla\psi(t)\|_{l^2}^2 + m^2\|\psi(t)\|_{l^2}^2\right) + 2\sum_{x\in\mathbb{Z}} V(x)\psi(x,t)\dot{\psi}(x,t) = 0,$$

where $(\nabla\psi)(x) = \psi(x+1) - \psi(x)$ for $x \in \mathbb{Z}$. Put $\alpha = -\min_{x\in\mathbb{Z}} V(x) \geq 0$. Since $\|\nabla\psi\|_{l^2} \leq 2\|\psi\|_{l^2}$, we get

$$\left\|\dot{\psi}(t)\right\|_{l^2}^2 + \|\nabla\psi(t)\|_{l^2}^2 + m^2\|\psi(t)\|_{l^2}^2 \leq (4+m^2)\|\mathbf{\Psi}_0\|_{l^2\oplus l^2}^2 + \alpha\int_0^t \|\mathbf{\Psi}(s)\|_{l^2\oplus l^2}^2\, ds$$

and therefore

$$\|\mathbf{\Psi}(t)\|_{l^2\oplus l^2}^2 \leq C\|\mathbf{\Psi}_0\|_{l^2\oplus l^2}^2 + \alpha_1\int_0^t \|\mathbf{\Psi}(s)\|_{l^2\oplus l^2}^2\, ds.$$

for suitable constants $C > 0$ and $\alpha_1 > 0$. The Gronwall inequality implies that

$$\|\mathbf{\Psi}(t)\|_{l^2\oplus l^2}^2 \leq Ce^{\alpha_1 t}\|\mathbf{\Psi}_0\|_{l^2\oplus l^2}^2, \quad t > 0.$$

which gives the desired bound. □

Now we can apply the Fourier-Laplace transform

$$\tilde{\mathbf{\Psi}}(x,\omega)=\int\limits_0^\infty e^{i\omega t}\mathbf{\Psi}(x,t)\;dt,\quad \operatorname{Im}\omega>\alpha_1>0,$$

and get the stationary equation

$$(\mathbf{H}-\omega)\tilde{\mathbf{\Psi}}(\omega)=-i\mathbf{\Psi}_0,\quad \operatorname{Im}\omega>\alpha_1.$$

Let us first consider the resolvent $\mathbf{R}(\omega)=(\mathbf{H}-\omega)^{-1}$ of the operator $\mathbf{H}$.

**Lemma 8.2.** *If $\omega^2-m^2\in\mathbb{C}\setminus[0,4]$, then the resolvent $\mathbf{R}(\omega)$ can be expressed in terms of the resolvent $R(\omega)$ from* (1.3) *as*

$$\mathbf{R}(\omega)=\left(\begin{array}{cc}\omega R(\omega^2-m^2) & iR(\omega^2-m^2)\\ -i(1+\omega^2R(\omega^2-m^2)) & \omega R(\omega^2-m^2)\end{array}\right). \tag{8.1}$$

*Proof.* The expression for the resolvent $\mathbf{R}_0(\omega)=(\mathbf{H}_0-\omega)^{-1}$ of the free equation with $V=0$ in the case where $\omega^2-m^2\in\mathbb{C}\setminus[0,4]$ can be obtained by inverse Fourier transform $F^{-1}_{\theta\to x-y}$ of the matrix

$$\frac{1}{\phi(\theta)-(\omega^2-m^2)}\left(\begin{array}{cc}\omega & i\\ -i(\phi(\theta)+m^2) & \omega\end{array}\right).$$

Using that by (2.1)

$$F^{-1}_{\theta\to x-y}\left(\frac{1}{\phi(\theta)-(\omega^2-m^2)}\right)=R_0(\omega^2-m^2,x,y),$$

we get

$$\mathbf{R}_0(\omega)=\left(\begin{array}{cc}\omega R_0(\omega^2-m^2) & iR_0(\omega^2-m^2)\\ -i(1+\omega^2R_0(\omega^2-m^2)) & \omega R_0(\omega^2-m^2)\end{array}\right).$$

Put

$$\mathbf{V}=\left(\begin{array}{cc}0 & 0\\ V & 0\end{array}\right).$$

Then the formula

$$\mathbf{R}(\omega)=(\mathbf{I}-i\mathbf{R}_0(\omega)\mathbf{V})^{-1}\mathbf{R}_0(\omega)$$

for the full resolvent yields (8.1). □

The representation (8.1) implies the following properties of the operator $\mathbf{H}$.

1) By Lemma 4.1 we have that

$$\mathrm{Spec}_{\mathrm{ess}}\mathbf{H} = [-\sqrt{m^2+4}, -m] \cup [m, \sqrt{m^2+4}].$$

The discrete spectrum of $\mathbf{H}$ is $\tilde\omega_j^\pm = \pm\sqrt{m^2+\omega_j}$, where $\omega_j$ are the eigenvalues of the operator $H$. Note that either $\tilde\omega_j^\pm \in \mathbb{R}$ or $\tilde\omega_j^\pm \in i\mathbb{R}$.

2) Let $\sigma > 1/2$. By Lemma 4.2, the following limits exist as $\varepsilon \to 0+$.

$$\mathbf{R}(\omega \pm i\varepsilon) \quad \xrightarrow{\mathbf{B}(\sigma,-\sigma)} \quad \mathbf{R}(\omega \pm i0),$$

and moreover

$$\mathbf{R}(\omega - i0, x, y) = \overline{\mathbf{R}(\omega + i0, x, y)}.$$

Both relations hold for $\omega \in (-\sqrt{m^2+4}, -m) \cup (m, \sqrt{m^2+4})$.

3) Let $\sigma > 7/2$. By Theorem 6.1, we have for a generic potential $V \in \mathcal{S}$ with $\sum\limits_{x\in\mathbb{Z}} V(x) \neq 0$ the following asymptotic expansion of the resolvent $\mathbf{R}$ in $\mathbf{B}(\sigma, -\sigma)$:

$$\mathbf{R}(\mu + \omega) = \mathbf{R}_0^\mu + \mathcal{O}(|\omega|^{1/2}), \quad \omega \to 0,$$

where $\mu = \pm m$ or $\mu = \pm\sqrt{m^2+4}$.

4) Let $\sigma > 7/2$. By Theorem 7.1, for a generic potential $V \in \mathcal{S}$ with $\sum\limits_{x\in\mathbb{Z}} V(x) \neq 0$, the following asymptotics hold:

$$\left\| e^{-it\mathbf{H}} - \sum_{\pm}\sum_{j=1}^{n} e^{-it\tilde\omega_j^\pm}\mathbf{P}_j^\pm \right\|_{\mathbf{B}(\sigma,-\sigma)} = \mathcal{O}(t^{-3/2}), \quad t \to \infty.$$

Here $\mathbf{P}_j^\pm$ are the projections onto the eigenspaces corresponding to the eigenvalues $\tilde\omega_j^\pm$, $j = 1, \ldots, n$.

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# A Appendix

Let the number of the points in the support of the potential $V$ equal 1 or 2. We will show that for such a potential the operator $H = -\Delta + V$ always has a real eigenvalue outside the interval $[0, 4]$.

**Example I**. Let $V(x) = V_1\delta(x - x_1)$. We seek the solution of the equation

$$(-\Delta - \omega + V)\psi = 0 \tag{A.1}$$

in the form

$$\psi = (-\Delta - \omega)^{-1}h.$$

Then (A.1) becomes

$$h(x) + V(x)((-\Delta - \omega)^{-1}h)(x) = 0. \tag{A.2}$$

Substituting the explicit formula (2.2) for the resolvent in (A.2) we obtain

$$h(x) + V_1\delta(x - x_1)\left[-i\sum_{y\in\mathbb{Z}} \frac{e^{-i\theta(\omega)|x-y|}}{2\sin\theta(\omega)}\, h(y)\right] = 0. \tag{A.3}$$

Thus $h(x) = 0$ for $x \neq x_1$, and (A.3) simplifies to

$$h(x_1)\left(1 - \frac{iV_1}{2\sin\theta(\omega)}\right) = 0. \tag{A.4}$$

Hence one has to solve the following equation for the eigenvalue $\omega$ of the operator $H$.

$$2\sin\theta(\omega) = iV_1. \tag{A.5}$$

First we consider the case where $V_1 < 0$ and seek the solution to (A.5) in the form $\theta(\omega) = is$ for $s \in \mathbb{R}$. Then (A.5) implies $s = \operatorname{arcsinh}(V_1/2) < 0$. Therefore $\theta(\omega) = is \in \Gamma_c$, and consequently $\omega \in (-\infty, 0)$ is a real eigenvalue of the operator $H$. Similarly, if $V_1 > 0$, then we get a real eigenvalue $\omega \in (4, \infty)$. It is easy to check that the corresponding eigenfunctions belong to $l^2$.

**Example II**. Let $V(x) = V_1\delta(x - x_1) + V_2\delta(x - x_2)$. Similarly to (A.4), we now get the system

$$\begin{cases} h(x_1)\left(\dfrac{iV_1}{2\sin\theta(\omega)} - 1\right) + h(x_2)\dfrac{iV_1}{2\sin\theta(\omega)}\, e^{-i\theta(\omega)|x_2-x_1|} = 0 \\ \\ h(x_1)\dfrac{iV_2}{2\sin\theta(\omega)}\, e^{-i\theta(\omega)|x_2-x_1|} + h(x_2)\left(\dfrac{iV_2}{2\sin\theta(\omega)} - 1\right) = 0 \end{cases}.$$

The determinant of this system equals

$$D(\omega) = (iV_1 - 2\sin\theta(\omega))\,(iV_2 - 2\sin\theta(\omega)) + V_1V_2e^{-2i\theta(\omega)|x_2-x_1|}.$$

We want to determine a real $\omega$ which is a solution to the equation $D(\omega) = 0$. Denoting $z = e^{-i\theta(\omega)}$, this reads as

$$\big(V_1 + \frac{1}{z} - z\big)\big(V_2 + \frac{1}{z} - z\big) = V_1V_2\, z^{2|x_2-x_1|}. \tag{A.6}$$

Put $N = |x_2 - x_1| \geq 1$, $a = 1/V_1$, and $b = 1/V_2$. Then (A.6) becomes

$$(az^2 - z - a)(bz^2 - z - b) = z^{2N+2}. \tag{A.7}$$

Denote by $L(z)$ and $R(z)$ the left hand side and the right hand side of (A.7), respectively. It is easy to check that the graphs $y = L(z)$ and $y = R(z)$ intersect each other at the points $z = \pm 1$. Moreover, $R(0) = 0$ and $R(z) > 0$ for $z \neq 0$.

First we consider the case where $a, b > 0$. Then the polynomial $L(z)$ has two roots in the interval $(-1, 0)$, and $L(0) = ab > 0$. Therefore these graphs also have an intersection at a point $z = z_0$, with $-1 < z_0 < 0$. It is straightforward to prove that this point corresponds to a value $\omega \in (4, \infty)$.

The case where $a, b < 0$ is handled similarly, and in this case we get a solution $\omega \in (-\infty, 0)$ of the equation $D(\omega) = 0$.

Finally, if $a$ and $b$ have opposite signs, then $L(0) < 0$. Calculating the first derivatives of $L(z)$ and $R(z)$ at $z = \pm 1$, we obtain

$$L'(-1) = -2a - 2b - 2, \quad L'(1) = -2a - 2b + 2,$$
$$R'(-1) = -2N - 2, \quad R'(1) = 2N + 2.$$

If $N > a + b$, then $R'(-1) < L'(-1)$ and $R(z) < L(z)$ for $z > -1$ and $z + 1$ small enough. On the other hand, $L(0) < R(0)$. Thus the graphs of $L(z)$ and $R(z)$ have an intersection in $(-1, 0)$. Similarly, if $N > -a - b$, then these graphs have an intersection in $(0, 1)$. Therefore we have at least one root of (A.7) in $(-1, 1) \setminus \{0\}$.

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