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On Asymptotic Stability of Stationary Solutions to Nonlinear Wave and Klein-Gordon Equations

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Abstract

We consider nonlinear wave and Klein-Gordon equations with general nonlinear terms, localized in space. Conditions are found which provide asymptotic stability of stationary solutions in local energy norms. These conditions are formulated in terms of spectral properties of the Schrödinger operator corresponding to the linearized problem. They are natural extensions to partial differential equations of the known Lyapunov condition. For the nonlinear wave equation in three-dimensional space we find asymptotic expansions, as $t \to \infty$, of the solutions which are close enough to a stationary asymptotically stable solution.

1. Introduction

We consider the asymptotics as $t \to \infty$ of solutions to nonlinear wave and Klein-Gordon equations and systems in the whole space $\mathbf{R}^n$, of the type

$$\ddot{u}(x,t) = \Delta u(x,t) - m^2 u(x,t) + f(x, u(x,t)), \quad (x,t) \in \mathbf{R}^{n+1}. \tag{1}$$

We assume that $m \geqq 0$, $n \geqq 1$, $u(x,t) \in \mathbf{R}^d$ for $(x,t) \in \mathbf{R}^{n+1}$, and $f(x,u)$ is a smooth enough vector-valued function.

Remark 1.1. Equation (1) is a system of d real scalar equations. The case of wave equations with scalar complex solutions corresponds to $d = 2$.

We assume the nonlinear term $f(x,u)$ has a real potential $V(x,u) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^d)$:

$$f(x,u) \equiv -V'_u(x,u) \quad \text{for } (x,u) \in \mathbf{R}^n \times \mathbf{R}^d. \tag{2}$$

Thus (1) formally is a Hamiltonian system with the Hamiltonian functional

$$\mathscr{H}(u, \dot{u}) = \int [\tfrac{1}{2}|\dot{u}|^2 + \tfrac{1}{2}|\nabla u|^2 + \tfrac{1}{2}m^2|u|^2 + V(x,u)]\, dx. \tag{3}$$

Let $S(x)$ be a stationary solution to (1), that is,

$$0 = \Delta S(x) - m^2 S(x) + f(x, S(x)), \quad x \in \mathbf{R}^n. \tag{4}$$

Remark 1.2. There are many results concerning existence of stationary solutions to equations and systems of type (1): see, for instance, [1, 2] and their bibliographies.

The main purpose of this paper is to find conditions which ensure the asymptotic stability for stationary solutions $S(x)$, i.e., conditions which ensure the convergence

$$u(x, t) \to S(x) \quad \text{as } t \to \infty \tag{5}$$

(in an appropriate sense) for each solution $u(x, t)$ of (1) with initial data "close enough" to initial data of the stationary solution $S(x)$. In other words, the stationary point $(S(x), 0)$ in the phase space of (1) attracts some of its neighborhoods. We also derive the asymptotic expansion as $t \to \infty$ for the solutions $u(x, t)$ when $n = 3$ and $m = 0$.

There are known results by SEGAL [25], STRAUSS [29, 30], MORAWETZ & STRAUSS [21], and others, on the decay of local energy in nonlinear scattering problems for equation (1) with a "stable" spatially homogeneous nonlinear term, similar to $|u|^{p-1}u$, $p > 1$. The spatial homogeneity means that the nonlinear term $f(x, u) \equiv f(u)$ does not depend on x. The "stability" of the nonlinear term means (in the case $d = 1$) that $f(u) \leqq 0$ for $u \geqq 0$ and $f(u) \geqq 0$ for $u \leqq 0$. It leads to the positiveness of energy and to the estimate of solutions uniform in t. In those papers it is also assumed that $f(0) = f'(0) = 0$, so $u(x, t) \equiv 0$ is a solution to (1). With these assumptions the authors proved the global existence of solutions (for $0 \leqq t < \infty$) to the Cauchy problem and proved the decay of the local energy of solutions as $t \to \infty$. The decay of solutions means in particular the asymptotic stability of the zero equilibrium position. The results were used by the authors to construct the wave operators and scattering operators. Further extensions of these results were derived by CHADAM [3], J. GINIBRE & VELO [4, 5], REED [24], and others. GLASSEY & STRAUSS [6, 7] derived the local energy decay for the free and coupled (with a scalar field) Yang-Mills equations with "stable" nonlinear terms.

HÖRMANDER [9] and KLAINERMAN [10] studied asymptotic stability of the zero solution to the very general relativistic nonlinear wave and Klein-Gordon equations without "stability" assumption on nonlinear terms. The initial data are assumed to be small in the Sobolev norms W_l^p with $p = 1$ and $p = 2$ for large enough values of l. Then solution converges to zero as $t \to \infty$ in $L^2(\mathbf{R}^n)$ and uniformly in $x \in \mathbf{R}^n$.

There are recent results by PAYNE, SATTINGER, GRILLAKIS, SHATAH and STRAUSS [22, 26, 27, 28, 8] concerning stability and instability of solutions to wave problems, which are either spatially homogeneous or invariant under a certain group of operators. The results concern the stability, but not the asymptotic stability, of solutions. Thus the question of whether the solution that is singled out attracts all the solutions from some neighbourhood was not studied.

The main features which distinguish our work from those cited above are the following. We study the asymptotic stability of a nontrivial stationary solution $S(x)$. We consider the equations without an assumption of invariance, but we require the nonlinear term $f(x, u)$ in (1) to be localized in x, i.e., for some $a > 0$,

$$f(x, u) \equiv 0 \quad \text{for } |x| \geqq a. \tag{6}$$

We do not assume that the differential $f_u'(x, S(x))\delta u$ of the nonlinear term is identically zero (so that the linearized equation may have variable coefficients). We require neither the non-positiveness of $f_u'(x, S(x))$ nor the positiveness of the energy.

The important element in studying asymptotic stability problems is the choice of metric in which convergence (5) holds. The energy norm is generally used when the stability of solutions is studied. However, one cannot expect to have the asymptotic stability of solutions in the energy norm because the energy conservation law is usually in effect for problems under consideration (see Theorem 3.1). Nevertheless, the convergence (5) may hold in the local energy metric (i.e., in the energy norms in any bounded part of the space). The reason for this convergence is the scattering of waves to infinity for wave equations in the whole space. This scattering plays the role of a dissipation of local energy and may provide the asymptotic stability.

Such a dissipation was discovered initially for linear problems by MORAWETZ, LAX and PHILLIPS [15–20] (see also [31–36]). It plays the key role in the paper cited above concerning the local energy decay in nonlinear problems.

Let us note that for one-dimensional equations (1) with assumption (6) asymptotics of type (5) were established in [11–14] for all solutions of finite energy without the assumption of "stability" for the nonlinear term. This convergence is also due to the scattering of waves to infinity and shows that in this case the wave operators do not exist in general.

In order to specify our main results on stabilization (5) we need a couple of notations. We linearize (1) about the solution $u(x, t) \equiv S(x)$:

$$\ddot{\psi}(x, t) = \Delta\psi(x, t) - m^2\psi(x, t) - q(x)\psi(x, t). \tag{7}$$

Here $q(x)$ is the $d \times d$ real symmetric matrix function (a "potential"):

$$q(x) \equiv -f_u'(x, S(x)) = V_{uu}''(x, u). \tag{8}$$

We denote by

$$H \equiv -\Delta + q(x) \tag{9}$$

the Schrödinger operator corresponding to the linearized problem. Let $\gamma_t v \equiv (v(\cdot, t), \dot{v}(\cdot, t))$ denote the Cauchy data of the function $v = v(x, t)$, and $\|\gamma_t v\|_{E,R}$ denote the "local energy norm":

$$\|\gamma_t v\|_{E,R}^2 \equiv \int_{|x|<R} [|\dot{v}(x, t)|^2 + |\nabla v(x, t)^2 + |v(x, t)|^2]\, dx. \tag{10}$$

Then our main results are the following. We find the "stability conditions" in terms of spectral properties of H which guarantee the decay of local energy of the difference $v(x, t) \equiv u(x, t) - S(x)$. This means that for any solution $u = u(x, t)$ of (1) and for any $R > 0$,

$$\|\gamma_t v\|_{E,R} \to 0 \quad \text{as } t \to \infty \tag{11}$$

if initial data $\gamma_0 v$ of the difference $v = u - S$ have compact support and the energy norm $\|\gamma_0 v\|_{E,\infty}$ of $\gamma_0 v$ is small enough. The rate of the decay of local energy depends on n and m as follows:

$$\|\gamma_t v\|_{E,R} \leqq C_{R,b}\mu(t)\|\gamma_0 v\|_{E,b} \quad \text{for } t > 0, \tag{12}$$

where b is the "size" of supp $\gamma_0 v$ while

$$\mu(t) = \begin{cases} \exp(-\alpha t) \text{ with some } \alpha > 0 & \text{if } n \geqq 1 \text{ is odd and } m = 0, \\ 1/(t\ln^2 t + 1) & \text{if } n \geqq 2 \text{ is even and } m = 0, \\ 1/(t+1)^{3/2} & \text{if } n \geqq 1 \text{ is odd and } m \neq 0. \end{cases} \tag{13}$$

In this paper we do not consider the case when n is even and $m \neq 0$.

In order to formulate the main assumptions which guarantee asymptotic stability we consider the truncated resolvent $R_\chi(\lambda) \equiv \chi(H-\lambda)^{-1}\chi$ of the operator H. Here $\chi \in C_0^\infty(\mathbf{R}^n)$ is some cut-off function. The continuous spectrum of the operator H coincides with the semiaxis $\lambda \geqq 0$. Hence the resolvent $R(\lambda) \equiv (H-\lambda)^{-1}$ is a meromorphic function of λ in the complex plane outside of the semiaxis, and has its poles at eigenvalues of the operator H. The truncated resolvent does not "feel" the continuous spectrum, and has a meromorphic continuation with respect to λ through the semiaxis $\lambda > 0$ on the second sheet of the Riemannian surface (see [31–36]). The truncated resolvent has a branch point at $\lambda = 0$ and it may be unbounded in any neighbourhood of the point $\lambda = 0$.

Now our main assumptions which lead to asymptotic stability are the following:

AS-1. Operator H does not have negative eigenvalues.
AS-2. The truncated resolvent $R_\chi(z) \equiv \chi(H-z)^{-1}\chi$ is bounded in a neighborhood of the point $z = 0$ for some cut-off function $\chi \in C_0^\infty(\mathbf{R}^n)$ such that $\chi(x) = 1$ for $|x| \leqq a$.

Moreover, we show that the same stability conditions provide the following expansion as $t \to \infty$ for the solutions of the wave equation (1) with $m = 0$ when dimension $n = 3$. We assume that the initial data $\gamma_0 v$ of the difference $v(x,t) = u(x,t) - S(x)$ fit the conditions mentioned above (the supports are compact and the energy is small enough). Then the solution has the asymptotic form

$$u(x,t) \sim S(x) + \sum_{k=1}^{\infty} \sum_{r=0}^{r_k} a_{k,r}(x) t^r \exp(-i\omega_k t). \tag{14}$$

Here $r_k < \infty$, k, r are integers, $a_{k,r}(x) \in C^\infty(\mathbf{R}^n)$, while $0 > \operatorname{Im}\omega_k \to -\infty$ as $k \to \infty$.

Let us note that the estimates (12), (13) and asymptotic expansion (14) with $S(x) \equiv 0$ were obtained for linear equations of type (7) in the case when $m = 0$ by MORAWETZ, LAX and PHILLIPS [15–17, 19, 20] and by VAINBERG [31, 32, 34–36]. The case $m \neq 0$ was studied by VAINBERG in [33]. In [31–36] it was shown that assumptions AS-1, AS-2 are equivalent to the decay (12), (13) of solutions to the linearized equation (7). These results concerning linear problems play the key role in the present paper.

Remark 1.3. (i) There are simple sufficient conditions which guarantee AS-1, AS-2. We describe them in the next section.

(ii) Scattering frequencies ω_k in (14) run over the set of all finite sums of scattering frequencies ω_j^0, corresponding in a similar way to the linearized equation (7) (the frequencies ω_j^0 may be repeated several times in these sums). The "combinational principle" seems to be natural to the perturbational procedure.

(iii) Conditions AS-1, AS-2 are natural extensions to partial differential equations of the well-known Lyapunov asymptotic stability condition. In the case of ordinary differential equations, the Lyapunov condition can be formulated in terms of the eigenvalues of the corresponding linearized problem. It is equivalent to the decay of solutions to the linearized problem. As we mentioned earlier, conditions AS-1, AS-2 are equivalent to the decay (12), (13) of solutions ψ to the linearized problem (7). (In fact if n is odd and $m = 0$, then these conditions are equivalent to a decay of solutions ψ to the linearized problem. In other cases the solutions ψ may decay with time while the truncated resolvent is unbounded at the origin. However, if $m = 0$, then AS-1, AS-2 are equivalent to the decay of ψ with the rate which guarantees the convergence of the integral $\int^\infty \| \gamma_t v \|_{E,R}\, dt$.) Thus conditions AS-1, AS-2 really are extensions of the Lyapunov condition. Let us stress, however, that the Lyapunov condition is never fulfilled for the ordinary systems of type $\ddot{x} = F(x)$ with real vector field $F(x)$, while it can be fulfilled for the partial differential equation (1). This distinction is related to the existence of the continuous spectrum of the Schrödinger operator H.

The plan of the paper is as follows: In Section 2 we collect all assumptions on the nonlinear term $f(x, u)$ of the equation (1). In Section 3 an existence theorem is stated for solutions to the Cauchy problem for equation (1) with initial data close enough to a stationary state $S(x)$. We assume that the stability conditions AS-1, AS-2 are fulfilled. The proof of existence of solutions in Section 4 follows the well-known contraction mapping approach (see, for example, [24, 23]), and uses essentially the local energy decay of solutions to the linearized problem. Section 5 concerns the energy conservation law.

The exposition in Section 4 is given in such a way that it could be used in Section 6 to prove asymptotic stability of the stationary state $S(x)$. We conclude Section 6 with a short discussion showing that asymptotic stability does not hold in the global energy norm. In Section 7 we derive the asymptotic expansion (14) in the case when $n = 3$, $m = 0$.

Let us note that everywhere in the present paper we could consider the exterior problems for equation (1) with Dirichlet, Neumann, or mixed boundary conditions instead of the Cauchy problem. Then all the results would still be valid if the non-trapping condition is fulfilled.

2. Assumptions on the nonlinear term

We assume that the nonlinear term $f = f(x, u)$ satisfies the following conditions:

A1. Function f is smooth:

$$f(x, u) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^d, \mathbf{R}^d). \tag{15}$$

A2. The nonlinear term $f(x, u)$ is localized in x, that is, relation (6) is fulfilled.

Remark 2.1. Due to (6) we may choose the potential $V(x, u)$ in (2) so that

$$V(x, u) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^d), \quad V(x, u) \equiv 0 \quad \text{for } |x| \geqq a. \tag{16}$$

From now on we assume that (16) holds.

A3. In the case $n \geqq 2$, we assume that

$$|f(x, u)| \leqq C(1 + |u|)^p, \quad |f_u'(x, u)| \leqq C(1 + |u|)^{p-1} \quad \text{for } (x, u) \in \mathbf{R}^n \times \mathbf{R}^d, \tag{17}$$

where

$$p = p(n), \quad p(n) = \frac{n}{n-2} \quad \text{if } n > 2, \quad p(2) \text{ is arbitrary.} \tag{18}$$

Without loss of generality we may assume that $p(2) \geqq 2$.

Remark 2.2. (i) $p(3) = 3$, $p(4) = 2$, and $1 < p(n) < 2$ when $n > 4$.
(ii) By the Sobolev embedding theorem, $H^1(\mathbf{R}^n) \subset L^{2p(n)}_{\text{loc}}(\mathbf{R}^n)$ is a continuous embedding for $n \geqq 2$, i.e.,

$$\| u(x) \|_{L^{2p}(\mathbf{R}^n)} \leqq C \| u(x) \|_{H^1(\mathbf{R}^n)}, \quad n \geqq 2, \ p \leqq p(n). \tag{19}$$

If $n = 1$, then

$$\max_{x \in \mathbf{R}} |u(x)| \leqq C \| u(x) \|_{H^1(\mathbf{R})}. \tag{20}$$

A4. The stationary solution $S(x)$ is smooth:

$$S(x) \in C^\infty(\mathbf{R}^n). \tag{21}$$

(In fact, some finite smoothness in (15) and in (21) is enough.)
A5. The asymptotic stability conditions AS-1 and AS-2 are fulfilled.

Remark 2.3. (i) Condition AS-2 is fulfilled for $n \geqq 2$ if the equation $H\psi(x) = 0$, $x \in \mathbf{R}^n$, has only the trivial solution in the class of functions satisfying the estimate $|\psi(x)| \leqq C|x|^{2-n}$ in some neighborhood of infinity (see Section 3 in Chapter IX of [36]). It is also true for $n = 1$ if the class of bounded solutions ψ is considered.
(ii) Both conditions AS-1 and AS-2 are fulfilled when the matrix "potential" $q(x)$ defined in (7), (8) is nonnegative (and not identical zero if $n \leqq 2$), i.e.,

$$q(x) \geqq 0 \quad \text{for } x \in \mathbf{R}^n \text{ (and } q(x) \not\equiv 0, \text{ if } n \leqq 2). \tag{22}$$

Relation (22) is fulfilled in particular if the potential $V(x, u)$ is a convex function in $u \in \mathbf{R}^n$ for each $x \in \mathbf{R}^n$.
(iii) The assumptions A1, A2, A4 guarantee that the truncated resolvent $R_\chi(\omega^2)$ of operator (9) is analytic in some half-plane $\operatorname{Im}\omega >$ const, and has a meromorphic continuation with respect to ω over the whole complex plane $\mathbf{C}$ if the dimension n is odd, or over the plane with the cut along the negative part of the imaginary axis if the dimension n is even (see [31–36]). The poles of the meromorphic continuation form a sequence $\omega_j^0 \in \mathbf{C}$ such that

$$\operatorname{Im}\omega_j^0 \to -\infty \quad \text{as } j \to \infty. \tag{23}$$

Assumptions AS-1 and AS-2 provide the inequality

$$\operatorname{Im}\omega_j^0 < 0 \quad \text{for all } j \tag{24}$$

and the decay (12), (13) of local energy of solutions to the linearized equation (see [31–36]).

In (13) in the case of linear equations we may take $\alpha = \alpha_1^0 - \delta > 0$, where $\alpha_1^0 \equiv \min_j |\operatorname{Im}\omega_j^0|$, while $\delta > 0$ may be chosen arbitrarily small.

3. The function spaces and existence of solutions

Let us denote by H^s with an integer $s \geqq 0$ the Sobolev space of vector-valued functions $u(x) = (u_1(x), \ldots, u_d(x))$. Here $u_k(x)$ are real-valued measurable functions, and the norm in H^s is defined by

$$\| u(x) \|_s^2 = \sum_{j=1}^{d} \sum_{|\alpha| \leqq s} \int_{\mathbf{R}^n} |\partial_x^\alpha u_j(x)|^2 \, dx < \infty. \tag{25}$$

We also use the spaces H^s_{loc} with finite seminorms $\| u \|_{s,R}$, for each $R > 0$, defined similarly to (25):

$$\| u(x) \|_{s,R}^2 = \sum_{j=1}^{d} \sum_{|\alpha| \leqq s} \int_{|x|<R} |\partial_x^\alpha u_j(x)|^2 \, dx < \infty. \tag{26}$$

Let us consider the Cauchy problem for equation (1):

$$\begin{gathered} \ddot{u}(x,t) = \Delta u(x,t) - m^2 u(x,t) + f(x, u(x,t)), \quad (x,t) \in \mathbf{R}^{n+1}, \\ u|_{t=0} = u^0(x),\ \dot{u}|_{t=0} = u^1(x) \quad \text{for } x \in \mathbf{R}^n. \end{gathered} \tag{27}$$

Let us introduce the phase space E for (1):

Definition 3.1. $E \equiv H^1 \times H^0$ is the space with the norm

$$\| (u^0, u^1) \|_E = \| u^0 \|_1 + \| u^1 \|_0. \tag{28}$$

We introduce similarly the space $E_{\text{loc}} = H^1_{\text{loc}} \times H^0_{\text{loc}}$ with corresponding finite seminorms as in (10): For each $R > 0$,

$$\| (u^0, u^1) \|_{E,R}^2 \equiv \int_{|x|<R} [|u^1(x)|^2 + |\nabla u^0(x)|^2 + |u^0(x)|^2] \, dx. \tag{29}$$

We define the Hamiltonian functional $\mathscr{H}$ on the phase space E for each $(u^0, u^1) \in E$ by an expression similar to (3):

$$\mathscr{H}(u^0, u^1) \equiv \int [\tfrac{1}{2}|u^1(x)|^2 + \tfrac{1}{2}|\nabla u^0(x)|^2 + \tfrac{1}{2}m^2|u^0(x)|^2 + V(x, u^0(x))]\, dx, \quad (30)$$

where V is the potential function with the property (16).

Proposition 3.1. *The Hamiltonian functional* $\mathscr{H}$ *is* (i) *well defined on* E, *and* (ii) *a continuous function on* E, *endowed with the norm* (28).

This proposition is evident for $n = 1$ because of (20), and it follows from Remarks 3.1 and 5.1 below for the case $n \geqq 2$.

We seek a solution of problem (27) in the space $\mathscr{E}$ of functions "of finite energy":

Definition 3.2. $u(x, t) \in \mathscr{E}$ if

$$(u(\cdot, t), \dot{u}(\cdot, t)) \in C([0, \infty); E). \quad (31)$$

We denote by $\mathscr{E}_{\text{loc}}$ the similar space defined by (31) with E_{loc} instead of E.

Let

$$\gamma_t u \equiv (u(\cdot, t), \dot{u}(\cdot, t)) \quad \text{for } t \in \mathbf{R},\ u \in \mathscr{E}_{\text{loc}}. \quad (32)$$

We now formulate a result on existence and uniqueness of the solution to the Cauchy problem (27).

Theorem 3.1. *Let assumptions* A1–A5 *of Section 2 hold. Let* $(u^0, u^1) \in E_{\text{loc}}$, *and*

$$u^0(x) - S(x) = u^1(x) = 0 \quad \textit{for } |x| > b \quad (33)$$

with some $b < \infty$. *Moreover, let the distance* $\|(u^0 - S, u^1)\|_E$ *between* (u^0, u^1) *and* $(S, 0)$ *be small enough. Then*

(i) *The Cauchy problem* (27) *has a solution* $u(x, t) \in \mathscr{E}_{\text{loc}}$, *and the solution is unique.*

(ii) *Moreover, if* $S(x) \in H^1$, *then* $u(x, t) \in \mathscr{E}$, *and the energy conservation law holds*:

$$\mathscr{H}(\gamma_t u) \equiv \text{const} \quad \textit{for } t \in \mathbf{R}. \quad (34)$$

The proof will be given in Sections 4 and 5. It uses the well-known contraction mapping approach [24] (see also [23]) and the decay of local energy of solutions to the linearized equation.

The remainder of this section is devoted to important technical lemmas which we need for the proof of Theorem 3.1.

We rewrite problem (27) for a new unknown function $v(x, t) \equiv u(x, t) - S(x)$:

$$\begin{aligned} \ddot{v}(x, t) &= \Delta v(x, t) - m^2 v(x, t) + f(x, S(x) + v(x, t)) - f(x, S(x)), \quad x \in \mathbf{R}^n,\ t > 0, \\ \gamma_0 v &= (v^0, v^1), \end{aligned} \quad (35)$$

where $v^0(x) \equiv u^0(x) - S(x)$, and $v^1(x) \equiv u^1(x)$.

Let us denote

$$F(x, v) = f(x, S(x) + v) - f(x, S(x)) + q(x)v \quad \text{for } (x, v) \in \mathbf{R}^n \times \mathbf{R}^d \tag{36}$$

where $q(x)$ was defined in (8). Then by (36), (8) and (6) we have

$$F(x, v) \equiv 0 \quad |x| > a. \tag{37}$$

Now (35) takes the form

$$\begin{gathered} \ddot{v}(x, t) = \Delta v(x, t) - m^2 v(x, t) - q(x)v(x, t) + F(x, v(x, t)), \quad x \in \mathbf{R}^n, \ t > 0, \\ \gamma_0 v = (v^0, v^1). \end{gathered} \tag{38}$$

We are interested in solution $v(x, t)$ belonging to the space $\mathscr{E}_{\text{loc}}$.

The following two lemmas are the main tools for the investigation of problem (38).

Lemma 3.1. *The map $v(x) \mapsto F(x, v(x))$ is bounded from H^1_{loc} to H^0, and the following inequalities are fulfilled:*

$$\| F(x, v(x)) \|_0 \leqq \begin{cases} C(\| v(x) \|^p_{1,a} + \| v(x) \|^2_{1,a}) & \text{if } n = 2 \text{ or } 3, \\ C \| v(x) \|^p_{1,a} & \text{if } n \geqq 4, \\ B(\| v(x) \|_{1,a}) \| v \|^2_{1,a} & \text{if } n = 1, \end{cases} \tag{39}$$

where $p = p(n)$ is defined in (18), *a is defined in* (6), *and $B(\| v(x) \|_{1,a})$ is bounded for bounded $\| v(x) \|_{1,a}$.*

Proof. (i) Let us consider first the case when $n = 2$ or 3. Then from (36) we get by assumptions A2 and A3 that

$$|F(x, v)| \leqq C(|v|^p + |v|^2) \quad \text{for } (x, v) \in \mathbf{R}^3 \times \mathbf{R}^d. \tag{40}$$

Therefore, from (37) it follows that

$$\begin{aligned} \| F(x, v(x)) \|_0^2 &\leqq C \int_{|x| \leqq a} [|v(x)|^p + |v(x)|^2]^2 \, dx \\ &\leqq C\left(\int_{|x| \leqq a} |v(x)|^{2p} dx + \int_{|x| \leqq a} |v(x)|^4 dx \right). \end{aligned}$$

Hence from the Sobolev embedding theorem (19) we get

$$\| F(x, v(x)) \|_0^2 \leqq C(\| v \|^{2p}_{1,a} + \| v \|^4_{1,a}),$$

which is equivalent to (39) when $n = 2$ or 3.

(ii) Estimate (39) in the case $n \geqq 4$ may be proved identically in virtue of the inequality

$$|F(x, v)| \leqq C|v|^p \quad \text{if } n \geqq 4.$$

The last inequality follows from A2, A3 and (36).

(iii) The case $n = 1$ may be considered similarly by using (20) instead of (19). Indeed, in this case we have the following estimate instead of (40). For any $B < \infty$, there exists a constant $C(B)$ such that

$$|F(x, v)| \leqq C(B)|v|^2 \quad \text{if } |v| < B.$$

Then

$$\| F(x, v(x)) \|_0 \leqq (2a)^{1/2} C\left(\sup_{|x|<a} |v(x)| \right) \sup_{|x|<a} |v(x)|^2.$$

Together with (20) this estimate leads to the last of inequalities (39). □

Remark 3.1. The same method leads to assertion (i) of Proportion 3.1 for $n \geqq 2$. Indeed, according to (2) we may put

$$V(x, u) \equiv \int_\gamma F(x, v)\, dv, \tag{41}$$

where γ is an arbitrary smooth path in $\mathbf{R}^d$ connecting points 0 and u (and the integral does not depend on the choice of γ).

Then (16) follows from (6) and (15); moreover, by (17),

$$|V(x, u)| \leqq C(1 + |u|)^{p+1} \quad \text{for } (x, u) \in \mathbf{R}^n \times \mathbf{R}^d. \tag{42}$$

Hence the integral of $V(x, u^0(x))$ contained in (30) converges by the Cauchy integral inequality. Indeed $(1 + |u^0(x)|)^p \in L^2(B_a)$ according to (19) and $(1 + |u^0(x)|) \in L^2(B_a)$.

Lemma 3.2. *The map $v(x) \mapsto F(x, v(x))$ is a Lipschitz map from H^1_{loc} to H^0, i.e., for any $v_1, v_2 \in H^1_{\text{loc}}$,*

$$\| F(x, v_2(x)) - F(x, v_1(x)) \|_0 \leqq \kappa \| v_2(x) - v_1(x) \|_{1,a}, \tag{43}$$

where κ is bounded for bounded $\| v_j \|_{1,a}, j = 1, 2$ and $\kappa \to 0$ when $\| v_j \|_{1,a} \to 0, j = 1, 2$.

Proof. Let $v_1, v_2 \in H^1_{\text{loc}}$. Then by the mean value theorem,

$$|\Delta F(x)| \equiv |F(x, v_2(x)) - F(x, v_1(x))| \leqq \max_{0 \leqq \theta \leqq 1} |F'_v(x, v_1(x) + \theta \Delta v(x))| \cdot |\Delta v(x)|,$$

where $\Delta v(x) \equiv v_2(x) - v_1(x)$, and $0 \leqq \theta \leqq 1$.

Let us consider first the case $n \geqq 2$. Then by Hölder's inequality and by (37) we get

$$\| \Delta F(x) \|_0^2 \leqq \left\| \max_{0 \leqq \theta \leqq 1} |F'_v(x, \hat{v}(x))| \right\|_{L^{2q}(B_a)} \| \Delta v(x) \|_{L^{2p}(B_a)}, \tag{44}$$

where $\hat{v}(x) \equiv v_1(x) + \theta \Delta v(x)$, p is defined in (18), and q is defined by $\frac{1}{q} + \frac{1}{p} = 1$. Hence $q = \frac{p}{p-1}$.

Now from A2, A3 and (36) it follows that

$$|F'_v(x, v)| \leqq \begin{cases} C(|v|^{p-1} + |v|) & \text{if } n = 2 \text{ or } 3, \\ C|v|^{p-1} & \text{if } n \geqq 4. \end{cases} \tag{45}$$

Hence taking into account that $|\hat{v}(x)| \leqq |v_1(x)| + |v_2(x)|$, we get

$$\max_{0 \leqq \theta \leqq 1} |F_v'(x, \hat{v}(x))| \leqq \begin{cases} C((|v_1(x)| + |v_2(x)|)^{p-1} + |v_1(x)| + |v_2(x)|) & \text{if } n = 2 \text{ or } 3, \\ C(|v_1(x)| + |v_2(x)|)^{p-1} & \text{if } n \geqq 4. \end{cases} \tag{46}$$

Then we take the $2q$-power of both sides (46), and integrate over the ball $|x| < a$:

$$\int_{B_a} \max_{0 \leqq \theta \leqq 1} |F_v'(x, \hat{v}(x))|^{2q} dx$$
$$\leqq \begin{cases} C \int_{|x|<a} (|v_1(x)|^{2p} + |v_2(x)|^{2p} + |v_1(x)|^{2q} + |v_2(x)|^{2q}) dx & \text{if } n = 2 \text{ or } 3, \\ C \int_{|x|<a} (|v_1(x)|^{2p} + |v_2(x)|^{2p}) dx & \text{if } n \geqq 4. \end{cases} \tag{47}$$

Since $p \geqq 2$ and $q = \frac{p}{p-1}$, it follows that $q \leqq p$. Hence by the Sobolev embedding theorem (19) we get

$$\left\| \max_{0 \leqq \theta \leqq 1} |F_v'(x, \hat{v}(x))| \right\|_{L^{2q}(B_a)}^{2q}$$
$$\leqq \begin{cases} C(\| v_1 \|_{1,a}^{2p} + \| v_2 \|_{1,a}^{2p} + \| v_1 \|_{1,a}^{2q} + \| v_2 \|_{1,a}^{2q}) & \text{if } n = 2 \text{ or } 3, \\ C(\| v_1 \|_{1,a}^{2p} + \| v_2 \|_{1,a}^{2p}) & \text{if } n \geqq 4. \end{cases} \tag{48}$$

By the same Sobolev theorem,

$$\| \Delta v(x) \|_{L^{2p}(B_a)} \leqq C \| \Delta v(x) \|_{1,a}. \tag{49}$$

Hence from (48) and (44) we get (43) in the case $n \geqq 2$.

The case $n = 1$ may be considered similarly by using (20).

4. An integral equation and a contraction mapping

In this section we prove assertion (i) of Theorem 3.1. We have reduced the problem (27) to the investigation of (38). We rewrite (38) as the first-order system

$$\frac{d}{dt} \gamma_t v = A \gamma_t v + J(\gamma_t v), \quad t > 0,$$
$$\gamma_0 v = v_0 \tag{50}$$

for an unknown function $v(x, t) \in \mathscr{E}_{\text{loc}}$. The notation $\gamma_t v$ was introduced in (32), $v_0 \equiv (v^0, v^1)$, and A denotes the matrix operator

$$A = \begin{pmatrix} 0 & 1 \\ -H - m^2 & 0 \end{pmatrix}, \tag{51}$$

where H is the Schrödinger operator (9). By $J(\cdot)$ we denote the vector-valued function

$$J(\gamma_t v) \equiv (0, F(\cdot, v(\cdot, t))). \tag{52}$$

In (50) and everywhere below we treat all vectors $\gamma_t v$, $J(\gamma_t v), \ldots,$ as column vectors.

If $J \equiv 0$, then the system (50) is linear and has a unique solution $v(x, t) \in \mathscr{E}_{\text{loc}}$ for each $v_0 \in E$ [24], and, in fact, $v(x, t) \in \mathscr{E}$. We denote by $U(t)$ the group of linear operators in E, which map $v_0 \mapsto \gamma_t v$, where $v \in \mathscr{E}$ is the solution to the linear problem.

For $u(x, t) \in \mathscr{E}_{\text{loc}}$ the system (50) is equivalent to the integral equation

$$\gamma_t v = \int_0^t U(t-\tau) J(\gamma_\tau v)\, d\tau + U(t) v_0, \quad t > 0. \tag{53}$$

Indeed, if $v \in \mathscr{E}_{\text{loc}}$, then $v(\cdot, t) \in C([0, \infty); H^1_{\text{loc}})$, and hence $J(\gamma_t v) \in C([0, \infty); E)$ according to Lemma 3.2. Then (53) is equivalent to (50) according to the Duhamel principle for solutions to the nonhomogeneous linear systems with right-hand sides continuous with respect to t. Moreover, it follows from (53) that the solutions $u(x, t)$ from $\mathscr{E}_{\text{loc}}$ belong in fact to $\mathscr{E}$.

Now we are going to consider equation (53) only for $|x| < a$. Let us denote by P the operator of restriction which maps any function $f(x)$ defined for $x \in \mathbf{R}^n$ to the same function with domain $|x| < a$. We introduce spaces $H^s_{(a)} \equiv PH^s$, $E_{(a)} \equiv PE$ with norms given by (26), (29) respectively, where $R = a$. Similarly, $\mathscr{E}_{(a)} = P\mathscr{E}_{\text{loc}}$.

For any $w = (w^0, w^1) \in E_{(a)}$ we consider $J(w(x))$ as a function with domain $\mathbf{R}^n$, which is equal to zero outside the ball $|x| < a$:

$$J(w^0, w^1)(x) \equiv \begin{cases} (0, F(0, w^0(x))), & |x| < a, \\ 0, & |x| > a. \end{cases} \tag{54}$$

Hence

$$J(Pv_0)(x) \equiv J(v_0(x)) \quad \text{for each } v_0(x) \in E_{\text{loc}} \tag{55}$$

due to (37) and (52).

Now let us consider equation (53) in the ball $|x| < a$:

$$\gamma_t w = \int_0^t PU(t-\tau) J(\gamma_\tau w)\, d\tau + PU(t) v_0, \quad t > 0, \tag{56}$$

where $w \in \mathscr{E}_{(a)}$.

This equation is equivalent to (53). Indeed, let $v \in \mathscr{E}$ be a solution to (53). Then applying the restriction operator P to both sides of (53) we get that $w \equiv Pv \in \mathscr{E}_{(a)}$ is the solution to (56), because $J(\gamma_\tau w) \equiv J(\gamma_\tau v)$ according to (55).

Conversely, let $w \in \mathscr{E}_{(a)}$ be the solution to (56). Then we define the corresponding solution $v(x, t) \in \mathscr{E}$ to (53) by extension of $w(x, t)$ for $|x| > R$. Namely, we define $\gamma_t v(x, t)$ as the right-hand side of (53) with $\gamma_\tau w$ instead of $\gamma_\tau v$ in the integrand. Such a function $\gamma_\tau v$ is equal to $\gamma_\tau w$ for $|x| < a$ according to (56), and hence $J(\gamma_t v) \equiv J(\gamma_t w)$. Therefore the constructed function $\gamma_t v$ is a solution to (53).

We rewrite (56) as an equation for the vector-valued function $z(t) = (z^0(t), z^1(t))$:

$$z(t) = \int_0^t PU(t-\tau)J(z(\tau))d\tau + PU(t)v_0, \quad t > 0 \tag{57}$$

with $z \in C(0, \infty; E_{(a)})$.

This equation is equivalent to (56), because from (57) it follows that $z^1(t) = \dot{z}^0(t)$, and therefore each solution to (57) may be written in the form $z(t) \equiv \gamma_t w$. Thus all equations (38), (50), (56), (57) are equivalent to one another, and we study (57) instead of (38). We are going to apply the contraction mapping theorem to the integral equation (57).

Let $\mathscr{B}$ be the Banach space $\mathscr{B} = C([0, \infty); E_{(a)})$ with the norm

$$\|z\|_{\mathscr{B}} \equiv \sup_{0<t<\infty} \|z(t)\|_{E_{(a)}}. \tag{58}$$

We denote by $\mathscr{B}_\delta$ the ball in $\mathscr{B}$ of radius δ with center at 0.

Let us rewrite equation (57) in operator form as

$$z = T(z) + r, \tag{59}$$

where $T(z)$ is the integral term in the right-hand side of (57), while $r \equiv PU(t)v_0$.

We show that the contraction mapping theorem can be applied to (59) in the ball $\mathscr{B}_\delta$ with a small enough $\delta > 0$, if $r \in \mathscr{B}_{\delta/2}$. For this purpose we use the results mentioned in the Introduction on the decay of local energy for the group $U(t)$, corresponding to the linearized equation (7). These results can be formulated in the form of the following Proposition 4.1.

Definition 4.1. For any $b > 0$ denote by E_b the subspace of functions $\psi_0(x) \in E$ such that

$$\psi_0(x) \equiv 0 \quad \text{for } |x| \geqq b. \tag{60}$$

Proposition 4.1 ([31–36]). *Let assumptions* AS-1 *and* AS-2 *hold. Then for each* $R > 0$ *and for all* $\psi_0(x) \in E_b$,

$$\|U(t)\psi_0\|_{E,R} \leqq C_{R,b}\mu(t)\|\psi_0\|_{E,b} \quad \textit{for } t > 0, \tag{61}$$

where $\mu(t)$ *is defined in* (13).

As was mentioned in the Introduction, the estimate (61) (decay of local energy) for some classes of linear problems was obtained earlier by Morawetz, Lax, and Phillips.

Lemma 4.1. *For* $\delta > 0$ *small enough and for any* $r \in \mathscr{B}_{\delta/2}$, *the operator* $Q: z \mapsto Tz + r$ *is a contraction map from* $\mathscr{B}_\delta$ *to* $\mathscr{B}_\delta$.

Proof of Lemma 4.1. We first prove that the operator T maps $\mathscr{B}_\delta$ into $\mathscr{B}_{\delta/2}$ for $\delta > 0$ small enough. Let us first check that for $z \in \mathscr{B}$ we have

$$Tz(t) \equiv \int_0^t PU(t-\tau)J(z(\tau))d\tau \in C(0, \infty; E_{(a)}). \tag{62}$$

Indeed, from the inclusion $z^0 \in C([0, \infty); H^1_{(a)})$ and from Lemma 3.2 it follows that

$$J(z(\cdot)) \in C([0, \infty); E_a). \tag{63}$$

Then (62) follows because the norms of the operators $U(t)$ in the space E are bounded for bounded t due to the usual energy estimates for linear equations.

Now let $z \in \mathscr{B}_\delta$. Then from Lemma 3.1 and from (61) with $R = a$ we get

$$\| Tz \|_{\mathscr{B}} \leqq C \sup_{t>0} \int_0^t \mu(t-\tau)(\| z(\tau) \|^p_{E,a} + \| z(\tau) \|^2_{E,a})d\tau \tag{64}$$

(where the quadratic term in the integrand may be omitted if $n \geqq 4$). Hence

$$\begin{aligned}\| Tz \|_{\mathscr{B}} &\leqq C \sup_{t>0} \int_0^t \mu(t-\tau)\, d\tau(\| z \|^p_{\mathscr{B}} + \| z \|^2_{\mathscr{B}}) \\ &\leqq C \sup_{t>0} \int_0^t \mu(s)ds(\delta^p + \delta^2) \leqq C \int_0^\infty \mu(s)ds(\delta^p + \delta^2).\end{aligned} \tag{65}$$

So $\| Tz \|_{\mathscr{B}} \leqq \delta/2$ for $\delta > 0$ small enough, because $\int_0^\infty \mu(s)ds < \infty$ for each function μ in (13) and $p > 1$ for each n.

Hence the operator Q transforms the ball $\mathscr{B}_\delta$ into itself if $r \in \mathscr{B}_{\delta/2}$ and $\delta > 0$ is small enough. To prove Lemma 4.1 it remains to check that the operator Q is a contraction on $\mathscr{B}_\delta$ (for any $r \in \mathscr{B}_{\delta/2}$ and $\delta > 0$ small enough).

If $z_1, z_2 \in \mathscr{B}_\delta$, then

$$Qz_2 - Qz_1 = Tz_2 - Tz_1 = \int_0^t PU(t-\tau)[J(z_2(\tau)) - J(z_1(\tau))]\, d\tau.$$

As in (64), (65) we thus get by Lemma 3.2 that

$$\begin{aligned}\| Qz_2 - Qz_1 \|_{\mathscr{B}} &\leqq C \sup_{t>0} \int_0^t \mu(t-\tau) \| J(z_2(\tau)) - J(z_1(\tau)) \|_{E,a} d\tau \\ &\leqq C\kappa \| z_2 - z_1 \|_{\mathscr{B}}.\end{aligned}$$

Here $\kappa \to 0$ as $\delta \to 0$ by Lemma 3.2, and hence Lemma 4.1 is proved.

Now let us prove (i) of Theorem 3.1. First of all, we choose $\delta > 0$ so that the assertion of Lemma 4.1 holds. According to (33) and (61) with $R = a$ we have

$$\begin{aligned}\| r \|_{\mathscr{B}} &\equiv \| PU(t)v_0 \|_{\mathscr{B}} \equiv \sup_{t>0} \| U(t)v_0 \|_{E,a} \\ &\leqq C \sup_{t>0} \mu(t) \| v_0 \|_{E,b} \leqq C \| v_0 \|_{E,b} \leqq \delta/2\end{aligned} \tag{66}$$

if $\| v_0 \|_{E,b}$ is small enough. Let us consider problem (35) with initial data $v_0 \in E_{\mathrm{loc}}$ satisfying the last inequality in (66). Then the map Q is a contraction in $\mathscr{B}_\delta$. Hence the equation (57) has a solution $z \in \mathscr{B}_\delta$, and this solution is unique in $\mathscr{B}_\delta$.

Equations (57), (56), (53), (50), (38) are equivalent to problem (27), and therefore assertion (i) of Theorem 3.1 is proved.

5. Energy conservation law

Let us prove the assertion (ii) of Theorem 3.1.

From (53) and (52) it follows that the solution $v \in \mathscr{E}_{\mathrm{loc}}$ to (35) constructed above belongs in fact to $\mathscr{E}$. Hence $u \equiv v + S(x)$ also belongs to $\mathscr{E}$, if $S(x) \in H^1$.

It remains to prove the energy conservation law (34). It is enough to establish (see [18]) the standard identity for the derivative of the potential energy, contained in (3):

$$\frac{d}{dt} \int V(x, u(x, t))\, dx = - \int f(x, u(x, t)) \dot{u}(x, t)\, dx, \quad t \geqq 0$$

for $u \in \mathscr{E}$. To prove this let us write

$$\frac{V(x, u(x, t + \Delta t)) - V(x, u(x, t))}{\Delta t} = V'_u(x, u(x, t)) \frac{\Delta u(x, t)}{\Delta t} + \frac{r(x, t)}{\Delta t}.$$

Here $\Delta u(x, t) \equiv u(x, t + \Delta t) - u(x, t)$, and

$$|r(x, t)| \leqq C | V''_{uu}(x, u(x, t) + \theta(x, t) \Delta u(x, t)) | (\Delta u)^2, \quad \text{where } 0 < \theta(x, t) < 1. \tag{67}$$

Since $u \in \mathscr{E}$, we have $V'_u(x, u(x, t)) = -f(x, u(x, t)) \in H^0$ for each $t \geqq 0$ by Lemma 3.1. Moreover $\Delta u / \Delta t \to \dot{u}$ in H^0 as $\Delta t \to 0$.

Hence it remains to verify that

$$\rho \equiv \int \left| \frac{r(x, t)}{\Delta t} \right| dx \to 0 \quad \text{as } \Delta t \to 0. \tag{68}$$

This may be proved as Lemma 3.2. In fact, by (67),

$$\rho \leqq C \int | \max_{0 < \theta < 1} f'_u(x, u + \theta \Delta u) | \cdot | \Delta u | \cdot \left| \frac{\Delta u}{\Delta t} \right| dx.$$

Let $n \geqq 2$. As in (44) we get

$$\rho \leqq C \left\| \max_{0 < \theta < 1} f'_u(x, \hat{u}) \right\|_{L^{2q}(B_a)} \cdot \| \Delta u \|_{L^{2p}(B_a)} \cdot \left\| \frac{\Delta u}{\Delta t} \right\|_{L^2(B_a)}, \tag{69}$$

where $p = p(n)$, $q = \frac{p}{p-1}$, $\hat{u} \equiv u + \theta \Delta u$. Now as in (48) and (49) we respectively obtain

$$\left\| \max_{0 < \theta < 1} f'_u(x, \hat{u}) \right\|_{L^{2q}(B_a)} \leqq B\left(\sup_{t < \tau < t + \Delta t} \| u(\cdot, \tau) \|_{1,a} \right), \tag{70}$$

$$\| \Delta u \|_{L^{2p}(B_a)} \leqq C \| \Delta u \|_{1,a}, \tag{71}$$

where the function $B = B(s)$ in the right-hand side of (70) is bounded for bounded s.

Let us take into account that $\|\Delta u\|_{1,a} \to 0$ and $\|\Delta u/\Delta t\|_{L^2(B_a)}$ is bounded as $\Delta t \to 0$ for $u \in \mathscr{E}$. Then (68) follows from (69) in virtue of (70) and (71).

The case $n = 1$ may be considered similarly by using (20).

Assertion (ii) of the Theorem 3.1 is proved, and therefore the proof of Theorem 3.1 is complete.

Remark 5.1. By the same Hölder estimates used in this section, one may check assertion (ii) of Proposition 3.1.

Let us note that it is also possible to prove the energy conservation law (34) in the same way as in [18], where only property (63) of the nonlinear term is used.

6. Asymptotic stability of stationary solutions

Now we formulate the main result on the asymptotic stability of a stationary solution to (1).

Theorem 6.1. *Let all the assumptions* A1–A5 *and* (33) *of Theorem 3.1 hold, and let* $u(x, t) \in \mathscr{E}$ *be a solution to the Cauchy problem* (27). *Then for any* $b > 0$ *there exists an* $\varepsilon = \varepsilon(b)$ *such that for each* $R > 0$

$$\|\gamma_t(u - S)\|_{E,R} \leqq C_{R,b}\mu(t)\|\gamma_0(u - S)\|_{E,b} \quad \text{for } t > 0 \tag{72}$$

if $\|\gamma_0(u - S)\|_{E,b} \leqq \varepsilon$. *Here* $b > 0$ *is the parameter defined in* (33).

Proof. We introduce a new unknown function $h(t) \in C(0, \infty; E)$ by the identity

$$\gamma_t(u - S) \equiv \gamma_t v \equiv \mu(t)h(t). \tag{73}$$

Then (72) is equivalent to the boundedness of $h(t)$ in E_{loc}, i.e., for any $R > 0$,

$$\|h(t)\|_{E,R} \leqq C_{R,b}\|\gamma_0(u - S)\|_{E,b} \quad \text{for } t > 0.$$

Let us substitute $\gamma_t v \equiv \mu(t)h(t)$ into (53). Then we get

$$h(t) = \mu^{-1}(t)\int_0^t U(t - \tau)J(\mu(\tau)h(\tau))\,d\tau + \mu^{-1}(t)U(t)v_0, \quad t > 0.$$

Therefore from (12) and Lemma 3.1 we derive

$$\begin{aligned}\|h(t)\|_{E,R} \leqq{}& C_{R,b}\mu^{-1}(t)\int_0^t \mu(t - \tau)(\mu^p(\tau)\|h(\tau)\|^p_{E,a} + \mu^2(\tau)\|h(\tau)\|^2_{E,a})\,d\tau \\ &+ C_{R,b}\|v_0\|_{E,b}, \quad t > 0\end{aligned} \tag{74}$$

(the second term in the integrand may be omitted in the case $n \geqq 4$).

Let us denote $\rho_R(s) \equiv \max_{0 \leqq t \leqq s}\|h(t)\|_{E,R}$ and $\rho(s) = \rho_a(s)$. Inequality (74) leads to

$$\rho_R(s) \leqq C_{R,b}(m_p(s)\rho^p(s) + m_2(s)\rho^2(s) + \|v_0\|_{E,b}), \quad s \geqq 0. \tag{75}$$

Here we denote

$$m_p(s) \equiv \max_{0 \leqq t \leqq s} \mu^{-1}(t) \int_0^t \mu(t-\tau)\mu^p(\tau)\, d\tau.$$

One can easily check that for any $p > 1$,

$$m_p(s) \leqq C_p < \infty \quad \text{for } s > 0. \tag{76}$$

This is a crucial point of the proof. From here and (75) with $R = a$ it follows that

$$\rho(s) \leqq K(\rho^p(s) + \rho^2(s) + \|v_0\|_{E,b}), \quad s \geqq 0, \tag{77}$$

where $K > 0$ is a constant.

Let us show that $\rho(s)$ is bounded if $\|v_0\|_{E,b}$ is small enough. We choose $\varepsilon > 0$ such that

$$K(((2+K)\varepsilon)^p + ((2+K)\varepsilon)^2 + \varepsilon) \leqq (1+K)\varepsilon, \tag{78}$$

The existence of such an ε is obvious. Let $\|v_0\|_{E,b} \leqq \varepsilon$. Since $\mu(0) = 1$, relation (73) leads to the following estimate: $\rho(0) \leqq \varepsilon$. Now from (77) and (78) it follows that the inequality $\rho(s) \leqq (2+K)\varepsilon$ leads to $\rho(s) \leqq (1+K)\varepsilon$. Since function $\rho(s)$ is continuous and $\rho(0) \leqq \varepsilon \leqq (1+K)\varepsilon$, it follows that $\rho(s) \leqq (1+K)\varepsilon$ for all $s \geqq 0$. The boundedness of $\rho(s)$ gives the boundedness of $\rho_R(s)$ due to (75). The proof of Theorem 6.1 is completed.

Remark 6.1. Asymptotic stability of solutions $u(x, t)$ to equation (1) is absent in the phase space E endowed with norm (28). Indeed, we may take $\gamma_0 u \in E$ with $\|\gamma_0 u - (S, 0)\|_E$ arbitrarily small, but at the same time we may provide

$$\mathscr{H}(\gamma_0 u) \neq \mathscr{H}(S, 0) \tag{79}$$

(we may choose for instance $u^0(x) \equiv S(x)$ with $\|u^1\|_1$ being very small and $\neq 0$). Then by Theorem 3.1 there exists the corresponding solution $u \in \mathscr{E}$. However, the convergence $\|\gamma_t u - (S, 0)\|_E \to 0$ as $t \to \infty$ is impossible due to (79), (34) and to assertion (ii) of Proposition 3.1.

7. Asymptotic expansion for solutions to the nonlinear three-dimensional wave equation

In this section we consider the equation (1) in the case when $n = 3$ and $m = 0$, and we assume, in addition to assumptions A1–A5 listed in Section 2, that the following new one is fulfilled.

A6. The nonlinear term $f(x, u)$ in (1) is a polynomial in u of degree no more than 3:

$$f(x, u) = \sum_{|\beta| \leqq 3} f_\beta(x) u^\beta \quad \text{for } (x, u) \in (\mathbf{R}^n \times \mathbf{R}^d). \tag{80}$$

Here $\beta = (\beta_1, \ldots, \beta_d)$, where $\beta_k \geqq 0$ are integers, $|\beta| = \beta_1 + \cdots + \beta_d$, and as usual

$$u^\beta \equiv \prod_1^d u_k^{\beta_k}. \tag{81}$$

Remark 7.1. (i) It follows from our assumptions (15) and (6) that for each β

$$f_\beta(x) \in C_0^\infty(\mathbf{R}^n) \otimes \mathbf{R}^d, \quad f_\beta(x) \equiv 0 \quad \text{for } |x| \geqq a. \tag{82}$$

(ii) By our main assumptions AS-1 and AS-2 the truncated resolvent $R_\chi(\omega^2)$ of the linearized equation has a meromorphic continuation with respect to ω in the whole complex plane (see assertion (iii) of Remark 2.3), and the poles ω_j^0 of the continuation fit (23), (24):

$$0 > \operatorname{Im} \omega_j^0 \to -\infty \quad \text{as } j \to \infty. \tag{83}$$

Definition 7.1. The sequence ω_k is the set of all finite sums of the ω_j^0 with positive integer coefficients, ordered as follows:

$$0 > \operatorname{Im} \omega_1 \geqq \operatorname{Im} \omega_2 \geqq \cdots. \tag{84}$$

Then by (83),

$$0 > \operatorname{Im} \omega_k \to -\infty \quad \text{as } k \to \infty. \tag{85}$$

We use the notation $\alpha_j^0 \equiv -\operatorname{Im} \omega_j^0$ and similarly $\alpha_k \equiv -\operatorname{Im} \omega_k$.
The main result of the section is the following theorem.

Theorem 7.1. *Let all the assumptions* A1–A6 *hold, and let* $\varepsilon \equiv \varepsilon(b) > 0$ *be the number from the conclusion of Theorem 6.1. Let relations* (33) *be fulfilled and*

$$\| \gamma_0(u - S(x)) \|_E \leqq \varepsilon. \tag{86}$$

Then for each $M = 0, 1, 2, \ldots$ *the solution* $u(x, t)$ *admits the asymptotic expansion*

$$u(x, t) - S(x) = \sum_{k=1}^{M} \sum_{r=0}^{r_k} a_{k,r}(x) t^r \exp(-i\omega_k t) + \rho_M(x, t) \tag{87}$$

Here the r_k *are integers,* $r_k < \infty$, $a_{k,r}(x) \in C^\infty(\mathbf{R}^n)$, *and the remainder term* $\rho_M(x, t)$ *in* (87) *has the following estimate: For each* R, $\delta > 0$,

$$\| \gamma_t \rho_M \|_{E,R} = o(\exp((-\alpha_{M+1} + \delta)t) \quad \text{as } t \to +\infty. \tag{88}$$

Proof. We prove the theorem by induction on M. For $M = 0$ the expansion is given by Theorem 6.1. Indeed, from the last assertion of Remark 2.3 it follows that $\mu(t)$ in the basic estimate (61) has the form $\exp(-\alpha t)$ with $\alpha = \alpha_1^0 - \delta > 0$ and arbitrarily small $\delta > 0$. Hence we may take the same $\mu(t)$ in (72), and so we get (87), (88) for $M = 0$.

Let us assume that (87) holds for some value of $M = L \geqq 1$. We then prove (87) for $M = L + 1$. We may perform the transfer with the help of the following refinement of Proposition 4.1.

Proposition 7.1 (VAINBERG [31, 32, 34–36]). *Let assumptions* AS-1 *and* AS-2 *hold. Then for each* $M > 0$ *the following expansion is valid:*

$$U(t) = \sum_{j=1}^{M} \sum_{r=0}^{r_j^0} A_{j,r} t^r \exp(-i\omega_j^0 t) + Q_M(t) \quad \textit{for } t > 0. \tag{89}$$

Here, $j, r, r_j^0 < \infty$ *are integers; the* $A_{j,r}$ *are bounded operators from* E_b *to* E_{loc} *and the remainder term* $Q_M(t)$ *is rapidly decreasing: For each* $R, b, \delta > 0$ *and for all* $\psi_0(x) \in E_b$,

$$\| \gamma_t Q_M \psi_0 \|_{E,R} \leqq C_{R,b} \exp((-\alpha_{M+1}^0 + \delta)t) \| \psi_0 \|_E \quad \textit{for } t > 0. \tag{90}$$

Remark 7.2. (i) The main terms in (89) are residues of $R_\chi(\omega^2) \exp(-i\omega t)$ (where $R_\chi(\lambda)$ is the meromorphic continuation of the truncated resolvent) at poles $\omega = \omega_j^0$, and the $r_j^0 + 1$ are orders of the poles.

(ii) Under some other assumptions expansion (89) was obtained earlier by MORAWETZ, LAX and PHILLIPS [17–19].

Now let us perform the transformation $M = L \mapsto M = L + 1$ in (87). For this purpose we substitute expansion (87) with $M = L$ for v and expansion (89) with $M = L + 1$ for U in the right-hand side of the Duhamel representation (53). We are going to get the expansion (87) with $M = L + 1$ from this representation.

Indeed, all exponential terms are evident. We need only to verify the estimates for all remainder terms. This is possible by (80), (82), (21) and the following lemma.

Lemma 7.1. *There exist finite constants* C_2 *and* C_3 *such that for* $v_i(x) \in H^1(\mathbf{R}^3)$, $i = 1, 2, 3$,

$$\| v_1(x) v_2(x) \|_{H^0(\mathbf{R}^3)} \leqq C_2 \| v_1(x) \|_{H^1(\mathbf{R}^3)} \| v_2(x) \|_{H^1(\mathbf{R}^3)}, \tag{91}$$

and similarly

$$\| v_1(x) v_2(x) v_3(x) \|_{H^0(\mathbf{R}^3)} \leqq C_2 \| v_1(x) \|_{H^1(\mathbf{R}^3)} \| v_2(x) \|_{H^1(\mathbf{R}^3)} \| v_3(x) \|_{H^1(\mathbf{R}^3)}. \tag{92}$$

The proof follows easily from the Hölder inequality and the Sobolev imbedding theorem (19) with $n = 3$ and $p = 3$.

We now expose the transformation $M \mapsto M + 1$ in detail in the case when $M = 0$ and all poles are simple, i.e., $r_j^0 = 0$. The general case of arbitrary M and r_j^0 is very similar.

So let us write (89) for $M = 1$ (with all $r_j^0 = 0$):

$$U(t) = A_1 \exp(-i\omega_1^0 t) + Q_1(t) \quad \text{for } t > 0. \tag{93}$$

Substituting this expansion into (53), we get

$$\gamma_t v = A_1 v_0 \exp(-i\omega_1^0 t) + Q_1(t) v_0 + \exp(-i\omega_1^0 t) \int_0^t \exp(i\omega_1^0 \tau) A_1(0, F(\cdot, v(\cdot, \tau)))\, d\tau$$
$$+ \int_0^t Q_1(t-\tau)(0, F(\cdot, v(\cdot, \tau)))\, d\tau \quad \text{for } t > 0. \tag{94}$$

Hence we have the desired expansion (87) with $M=1$:

$$\gamma_t v = a_{1,0}(x)\exp(-i\omega_1^0 t) + \rho_M(x,t) \quad \text{for } t>0, \tag{95}$$

where

$$a_{1,0}(x) = A_1 v_0 + \int_0^\infty \exp(i\omega_1^0\tau)\, A_1(0, F(\cdot, v(\cdot,\tau)))\, d\tau, \tag{96}$$

$$\rho_1(x,t) = Q_1(t)v_0 - \exp(-i\omega_1^0 t)\int_t^\infty \exp(i\omega_1^0\tau)\, A_1(0, F(\cdot, v(\cdot,\tau)))\, d\tau$$

$$+ \int_0^t Q_1(t-\tau)(0, F(\cdot, v(\cdot,\tau)))\, d\tau \quad \text{for } t>0. \tag{97}$$

It remains only to verify estimate (88), which in our case has the form

$$\|\gamma_t\rho_1\|_{E,R} = o(\exp((-\alpha_2+\delta)t)) \quad \text{as } t\to+\infty, \tag{98}$$

with $\alpha_2 \equiv \min(\alpha_2^0, 2\alpha_1^0)$.

Let us estimate each term in the right-hand side of (97) separately. First we get from (90) for each $\delta>0$ that

$$\|Q_1(t)v_0\|_{E,R} = o(\exp((-\alpha_2^0+\delta)t)) \quad \text{as } t\to+\infty. \tag{99}$$

Further, from (36) it follows that $F(x,v)$ is a polynomial in v which consists of monomials of order 2 and 3 only. Then from the induction assumption (i.e., (88) with $M=0$) we get by Lemma 7.1 that

$$\|(0, F(\cdot, v(\cdot,\tau)))\|_{E,a} = o(\exp(-2(\alpha_1^0+\delta)t)) \quad \text{as } t\to+\infty. \tag{100}$$

Since the operator $A_1^0: E_b \to E_{\text{loc}}$ is bounded, it follows from (100) that

$$\left\|\exp(-i\omega_1^0 t)\int_0^\infty \exp(i\omega_1^0\tau) A_1(0, F(\cdot, v(\cdot,\tau)))\, d\tau\right\|_{E,R} = o(\exp(-2(\alpha_1^0+\delta)t))$$

$$\text{as } t\to+\infty. \tag{101}$$

Finally from (90) with $M=0$ and (100) we get the following estimate for the last summand in (97):

$$\left\|\int_0^t Q_1(t-\tau)(0, F(\cdot, v(\cdot,\tau)))\, d\tau\right\|_{E,R} = o(\exp(-(\alpha_2^0+\delta)t)) \quad \text{as } t\to+\infty. \tag{102}$$

Estimate (98) then follows from (97), (99), (101), (102).

This completes the proof of Theorem 7.1.

Remark 7.3. It is easy to see by induction that the set of frequencies $\{\omega_k\}$ is an additive semigroup, generated by the poles ω_j^0.

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