Energy-Momentum Relation for Solitary Waves of Relativistic Wave Equations

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Abstract. Solitary waves of relativistic invariant nonlinear wave equation with symmetry group $U(1)$ are considered. We prove that the energy-momentum relation for spherically symmetric solitary waves coincides with the Einstein energy-momentum relation for point particles.

1. INTRODUCTION

The paper concerns the old problem of mathematically describing elementary particles in field theory. The problem was raised in classical electrodynamics after the discovery of the electron by Thomson in 1897. Abraham realized that a point electron would be unstable due to infinite self-energy, and introduced the model of the "extended" electron with finite electrostatic energy. However, the electrostatic model is also unstable due to electrostatic repulsion. The corresponding tension of the extended electron was analyzed by Poincaré in [13]. To avoid the instability problem for point particles, Einstein suggested that particles could be described as singularities of solutions to the field equations [5]. To obtain stationary finite-energy solutions, Born and Infeld introduced a "unitary field," which is a nonlinear modification of the Maxwell equations [14, 15]. The last approach was not developed further because the relation to the corresponding quantum version was not clarified.

Rosen [17] was the first who proposed a description of particles for the coupled Klein–Gordon–Maxwell equations, which are invariant with respect to the Lorentz group and to the global gauge group $U(1)$: the particle at rest is described by a finite-energy solution that has "Schrödinger's" form $\psi(x)e^{i\omega t}$ ("nonlinear eigenfunctions" or "solitary waves"). The particle with nonzero velocity v, $|v| < 1$, is obtained by the corresponding Lorentz (or Poincaré) transformation. Some numerical analysis has been done, showing the existence of radial solitary waves for a finite range of ω. The particle in the "normal" state is identified with the solitary wave of minimal energy.

The existence of radial solitary waves ("ground states") and nonradial solitary waves ("excited states" with nonzero angular momentum) has been analyzed numerically by many authors for diverse Lagrangian field theories [8, 9, 17–24, 26]. For the complex scalar field, most general results were achieved by Berestycki and Lions [3]. For the nonlinear Dirac equation, the existence is proved in [1, 7]. For the Maxwell–Dirac field, existence was proved by Esteban, Georgiev, and Séré in [6].

The next step after establishing the existence of solitons would be to study their stability as one of the main features of elementary particles. A nonrigorous analysis in [17] suggests an instability of solitons with negative total energy treated as a negative mass of the soliton. The first rigorous proof of instability of solitons with $\omega = 0$ was given by Derrick in [4]. The proof also concerns negative energy and is based on the virial Derrick–Pokhozhaev identity [4, 12].

These instability results hindered the development of the approach to elementary particles as solitons. On the other hand, solitons with $\omega \neq 0$ can be stable; this follows from the general criterion discovered by Grillakis, Shatah, Strauss, and other authors (see [11] and the references therein). Note that in [2] the asymptotic stability of solitons was proved for the 1D nonlinear Schrödinger equation when $\omega \neq 0$.

The key role of elementary particles suggests that the set of all moving solitons forms a global attractor for all finite-energy solutions to the dynamical field equations. However, this is still an unproved conjecture.

Finally, it would be of importance to develop a particle-like dynamics for moving solitons. We make a step in this direction for relativistic-invariant scalar Klein-Gordon equations. Namely, we prove that, for spherically symmetric solitary waves (constructed in [3]), the energy-momentum relation coincides with that of a relativistic particle. Furthermore, we suggest that this fact fails for non-spherically-symmetric solitary waves of some angular structure if such solitary waves exist indeed.

A further step would be a justification of effective dynamics for a solitary wave in an external slowly varying potential. A result of this type was obtained in [16] for the solitons of the Abraham model of classical electrodynamics and in [25] for localized wave packets of the Dirac equation. Recently, an effective dynamics was announced for the solitons of the nonrelativistic nonlinear Hartree equation [10].

2. STANDING SOLITARY WAVES

Consider the relativistic-invariant nonlinear wave equation

$$\ddot{\psi}(x,t) = \triangle \psi(x,t) + f(\psi(x,t)), \qquad x \in \mathbb{R}^n, \tag{2.1}$$

where $n \geqslant 1$ and $\psi \in \mathbb{C}^d$ for $d \geqslant 1$. Assume that

$$f(\psi) = -\nabla_\psi V(\psi), \qquad \psi \in \mathbb{C}^d,$$

where V is a real potential, and ∇_ψ stands for the gradient with respect to $u = \operatorname{Re}\psi$ and $v = \operatorname{Im}\psi$; in other words, $\nabla_\psi = \nabla_u + i\nabla_v$. Then Eq. (2.1) formally becomes a Hamiltonian system with the Hamiltonian functional

$$H(\psi, \dot{\psi}) = \int \left[\frac{|\dot{\psi}|^2}{2} + \frac{|\nabla\psi|^2}{2} + V(\psi) \right] dx. \tag{2.2}$$

Further, we assume that $V(\psi) = \mathcal{V}(|\psi|)$. Then Eq. (2.1) is $U(1)$-invariant, i.e.,

$$f(e^{i\theta}\psi) = e^{i\theta}f(\psi), \qquad \theta \in \mathbb{R}.$$

Consider a (standing) solitary wave $\psi_0(x,t) = a(x)e^{-i\omega t}$ with the energy

$$H(\psi_0, \dot{\psi}_0) = \int \left(\frac{\omega^2|a|^2}{2} + \frac{|\nabla a|^2}{2} + \mathcal{V}(|a|) \right) dx. \tag{2.3}$$

The amplitude $a(x)$ is a solution of the Helmholtz stationary nonlinear equation

$$-\omega^2 a(x) = \triangle a(x) + f(a(x)), \qquad x \in \mathbb{R}^n. \tag{2.4}$$

The existence of nonzero solitary waves was proved in [3] under the following assumptions:

S0 $\qquad d = 1, \quad f \in C(\mathbb{C}), \quad f(0) = 0,$

S1 $\qquad -\infty < \lim\limits_{a \to 0+} f(a)/a + \omega^2 < 0,$

S2 $\qquad \exists a_0 > 0 : \mathcal{V}(a_0) - \dfrac{\omega^2 a_0^2}{2} < 0,$

S3 $\qquad \exists \alpha \geqslant 0 : f(a) = -\alpha a^l + o(a^l), \ a \to \infty, \quad l := \dfrac{n+2}{n-2}.$

Further, the solution $a(x)$ to (2.4) is real and satisfies the following properties:

A1 $$a(x) = R(|x|),$$

A2 $$\exists C, \delta > 0 : |R^{(k)}(r)| \leqslant Ce^{-\delta r}, \quad k = 0, 1, 2.$$

Property **A1** means that the solitary wave thus constructed is a radial (spherically symmetric) "ground state." Below we discuss the excited states with higher angular momentum for $n = 2$.

3. MOVING SOLITARY WAVES

Consider a (moving) solitary wave with velocity $v \in \mathbb{R}^n$:

$$\psi_v(x, t) = \psi_0(\Lambda_v(x, t)).$$

Here $|v| < 1$ and Λ_v is a Lorentz transformation with velocity v:

$$\Lambda_v(x, t) = (\gamma(x^{\parallel} - vt) + x^{\perp}, \gamma(t - vx)),$$

where $x^{\parallel} + x^{\perp} = x$, $x^{\parallel} \| v$, $x^{\perp} \perp v$, and $\gamma = 1/\sqrt{1 - v^2}$. In other words,

$$\psi_v(x, t) = \psi_0(\gamma(x^{\parallel} - vt) + x^{\perp}, \gamma(t - vx)) = a(\gamma(x^{\parallel} - vt) + x^{\perp})e^{-i\omega\gamma(t - vx)}.$$

Note that $\psi_v(x, t)$ is a solution of (2.1).

4. ENERGY-MOMENTUM RELATION FOR RADIAL STATES

For $v \in \mathbb{R}^n$, $|v| < 1$, we denote by

$$E_v := H(\psi_v, \dot{\psi}_v), \qquad P_v := -\operatorname{Re} \int \dot{\psi}_v \overline{\nabla \psi_v}\, dx$$

the energy and momentum of the moving solitary wave, respectively. Let us assume in what follows that

S4 $$\mathcal{V}(a) + \frac{\omega^2 a^2}{2} \geqslant 0, \qquad a \geqslant 0.$$

In this section, we consider spherically symmetric nonzero solitary waves (i.e., waves with zero angular momentum). This spherical symmetry is a typical property of a "ground state" with minimal energy.

Theorem 4.1. *Let $a(x)e^{-i\omega t}$ be a standing solitary wave for (2.1), and let **A1**, **A2** hold. Then the following "particle-like" energy-momentum relation holds*:

$$E_v = \frac{E_0}{\sqrt{1 - v^2}}, \qquad P_v = \frac{E_0 v}{\sqrt{1 - v^2}}. \tag{4.1}$$

Here $E_0 > 0$ for $\omega \neq 0$.

Remark. For a nonzero solitary wave, we have $E_0 := H(\psi_0, \ldots, \psi_0) > 0$. Indeed, if $H(\psi_0, \dot{\psi}_0) = 0$, then $a(x) \equiv \text{const}$ by **S4** and (2.3). However, this implies $a(x) \equiv 0$ by **A2**.

Proof of Theorem 4.1. *Step* 1. We choose the coordinates in such a way that $v = (|v|, 0, \ldots, 0)$. Below we write everywhere v instead of $|v|$ to simplify the notations. Let us write

$$y_1 = \gamma(x_1 - vt), \quad y_k = x_k; \quad y = (y_1, \ldots, y_n).$$

Then

$$\psi_v(x, t) = a(y)e^{-i\omega\gamma(t-vx_1)},$$

$$\dot{\psi}_v(x, t) = \left(-\gamma v(\nabla_1 a)(y) - i\gamma\omega a(y)\right)e^{-i\omega\gamma(t-vx_1)},$$

$$\nabla_1\psi_v(x, t) = \left(\gamma(\nabla_1 a)(y) + i\gamma\omega v a(y)\right)e^{-i\omega\gamma(t-vx_1)},$$

$$\nabla_k\psi_v(x, t) = (\nabla_k a)(y)e^{-i\omega\gamma(t-vx_1)}, \quad k = 2, \ldots, n.$$

Substituting these expressions into E_v, we obtain

$$E_v = \int \left[\frac{\dot{\psi}_v^2}{2} + \frac{|\nabla\psi_v|^2}{2} + V(|\psi_v|)\right] dx$$

$$= \int \left(\frac{1}{2}\left[|\nabla_1 a(y)|^2\gamma^2(v^2+1) + |a(y)|^2\omega^2\gamma^2(v^2+1) + \sum_{k=2}^{n}|\nabla_k a(y)|^2\right] + V(|a(y)|)\right) dx.$$

Then

$$E_v = \int \left(\frac{1}{2}\sum_{k=2}^{n}|\nabla_k a|^2 + \frac{\gamma^2}{2}(v^2+1)\left[|\nabla_1 a|^2 + |a|^2\omega^2\right] + V(|a|)\right)\frac{1}{\gamma}\,dy.$$

Write

$$I_0 = \frac{1}{2}\int |a(y)|^2\,dy, \qquad V_0 = \int V(|a(y)|)\,dy, \tag{4.2}$$

$$I_k = \frac{1}{2}\int |\nabla_k a(y)|^2\,dy, \qquad k = 1, \ldots, n. \tag{4.3}$$

Hence, in the notation (4.2) and (4.3), we represent E_v as

$$E_v = \left(\sum_{k=2}^{n}I_k + \gamma^2(v^2+1)(I_1 + I_0\omega^2) + V_0\right)\Big/\gamma, \tag{4.4}$$

$$E_0 = \sum_{k=1}^{n}I_k + I_0\omega^2 + V_0. \tag{4.5}$$

Step 2. We now derive (4.1) from this representation for E_v with the help of the following lemma.

Lemma 4.2 [3, 4, 12]. *The following Derrick–Pokhozhaev identity holds:*

$$-(n-2)\frac{1}{2}\int |\nabla a(y)|^2\,dy = n\int \left[V(|a(y)|) - \frac{1}{2}\omega^2 a^2(y)\right] dy. \tag{4.6}$$

Proof. Let us explain the proof formally. Relation (2.4) implies the variational identity

$$-\delta\frac{1}{2}\int |\nabla a|^2\,dx = \delta\int \left[V(|a|) - \frac{1}{2}\omega^2 a^2\right] dx.$$

Therefore,

$$-\frac{d}{d\sigma}\Big|_{\sigma=1}\frac{1}{2}\int \left|\nabla_x a\left(\frac{x}{\sigma}\right)\right|^2 dx = \frac{d}{d\sigma}\Big|_{\sigma=1}\int \left[V\left(a\left(\frac{x}{\sigma}\right)\right) - \frac{1}{2}\omega^2 a^2\left(\frac{x}{\sigma}\right)\right] dx.$$

Equivalently,

$$-\frac{d}{d\sigma}\Big|_{\sigma=1}\sigma^{n-2}\frac{1}{2}\int|\nabla_y a(y)|^2\,dy = \frac{d}{d\sigma}\Big|_{\sigma=1}\sigma^n\int\left[V(a(y))-\frac{1}{2}\omega^2 a^2(y)\right]dy.$$

This gives the Pokhozhaev identity (4.6), or (in the notation of (4.2) and (4.3))

$$-(n-2)\sum_{k=1}^n I_k = n\left[V_0 - \omega^2 I_0\right]. \qquad \square \tag{4.7}$$

Step 3. To derive (4.1), let us eliminate V_0 and ω^2 from (4.4) and (4.7) obtaining

$$\omega^2 I_0 = \frac{n-2}{n}(I_1+\cdots+I_n)+V_0. \tag{4.8}$$

Hence, E_0 and E_v (see (4.4) and (4.5)) become

$$E_0 = \frac{2n-2}{n}\sum_{k=1}^n I_k + 2V_0, \tag{4.9}$$

$$\begin{aligned}
E_v &= \frac{1}{\gamma}\left[\sum_{k=2}^n I_k + \gamma^2(v^2+1)\left(I_1 + \frac{n-2}{n}\sum_{k=1}^n I_k + V_0\right) + V_0\right] \\
&= \frac{1}{\gamma}\left[I_1\gamma^2(v^2+1)\left(1+\frac{n-2}{n}\right) + \sum_{k=2}^n I_k\left(1+\gamma^2(v^2+1)\frac{n-2}{n}\right) + V_0(\gamma^2(v^2+1)+1)\right].
\end{aligned} \tag{4.10}$$

Note that

$$\gamma^2(v^2+1)+1 = 2\gamma^2 \qquad \text{and} \qquad 1+\gamma^2(v^2+1)\frac{n-2}{n} = \gamma^2\frac{2n-2}{n} - \gamma^2\frac{2v^2}{n}.$$

Therefore, by using (4.9), we can represent E_v as

$$E_v = \gamma\frac{2n-2}{n}\sum_{k=1}^n I_k + \gamma\frac{2v^2}{n}\left(I_1(n-1)-\sum_{k=2}^n I_k\right) + \gamma 2V_0 = \gamma E_0 + \gamma\frac{2v^2}{n}\left(I_1(n-1)-\sum_{k=2}^n I_k\right). \tag{4.11}$$

Using assumption **A1**, we obtain
$$I_1 = I_2 = \cdots = I_n,$$

which implies identity (4.1) for E_v.

Step 4. We have

$$\begin{aligned}
P_v &= -\operatorname{Re}\int\dot\psi_v\overline{\nabla\psi_v}\,dx \\
&= -\operatorname{Re}\int\left(-\gamma v(\nabla_1 a)(y) - i\omega\gamma a(y)\right)\left(\overline{\gamma(\nabla_1 a)(y)+i\omega\gamma va(y)},\overline{\nabla_2 a(y)},\ldots,\overline{\nabla_n a(y)}\right)\frac{1}{\gamma}\,dy.
\end{aligned}$$

Since $a(x)$ is a real function, we obtain (in the notation of (4.2) and (4.3))

$$P_v = \left(\gamma v\int(|\nabla_1 a(y)|^2 + \omega^2 a^2(y))\,dy, 0,\ldots,0\right) = \left(\gamma v 2(I_1+\omega^2 I_0), 0,\ldots,0\right), \tag{4.12}$$

because $\nabla_1 a(y)$ is an even function with respect to any variable $y_2, \ldots, y_n$, while $\nabla_k a(y)$ is an odd function of y_k, $k = 2, \ldots, n$. Using (4.8) and (4.5), we obtain

$$\gamma v 2 (I_1 + \omega^2 I_0) = \gamma v E_0 + \frac{2\gamma v}{n} \left((n-1) I_1 - \sum_{k=2}^{n} I_k \right). \tag{4.13}$$

Therefore, relations (4.12) and (4.13), together with $I_1 = I_2 = \cdots = I_n$, imply

$$P_v = (\gamma v E_0, 0, \ldots, 0),$$

which yields (4.1) for P_v. $\square$

Remarks. i) Formulas (4.11) and (4.8) show that identity (4.1) holds for general nonradial solitary waves if and only if

$$I_2 + \cdots + I_n = I_1 (n-1). \tag{4.14}$$

ii) Relation (4.1) implies the Einstein identity $E_0 = m_0$ with $m_0 > 0$ (see the remark after Theorem 4.1).

iii) For $\omega = 0$, condition **S4** contradicts **S2**. Hence, in this case we have $E_0 \leqslant 0$, i.e., the "mass" is negative. This corresponds to the instability of solitons with $\omega = 0$, which was proved in [4]. This fact forces us to consider solitary waves with $\omega \neq 0$ for $U(1)$-invariant equations.

5. NONRADIAL EXCITED STATES

In the previous section we have considered a spherically symmetric solitary wave ("ground state") with zero angular momentum. For $n = 1$, the condition (4.14) obviously holds (and condition (4.1) as well) for all solitary waves. Let us prove that (4.1) holds for $n = 2$ for nonradial solitary waves that describe excited states with nonzero angular momentum. Introduce polar coordinates by setting $x_1 = r \cos \varphi$ and $x_2 = r \sin \varphi$.

Lemma 5.1. *Let* $\psi_0(x, t) = a(x) e^{-i\omega t}$ *be a solitary wave for* (2.1) *with*

$$a(x) := R(r) e^{ik\varphi},$$

where k is an integer, $R(r)$ is real-valued, $R(0) = 0$, and **A2** *holds. Then relation* (4.1) *is satisfied.*

Proof. In the polar coordinates,

$$\nabla_1 a = R'(r) \cos k\varphi - ik \frac{R(r)}{r} \sin k\varphi, \qquad \nabla_2 a = R'(r) \sin k\varphi + ik \frac{R(r)}{r} \cos k\varphi.$$

Therefore,

$$I_1 \equiv \frac{1}{2} \int |\nabla_1 a|^2 \, dy_1 dy_2 = \frac{1}{2} \int_0^\infty r dr \int_0^{2\pi} d\varphi \left[|R'(r)|^2 \cos^2 k\varphi + k^2 \frac{|R(r)|^2}{r^2} \sin^2 k\varphi \right]$$

$$= \frac{\pi}{2} \int_0^\infty \left[|R'(r)|^2 + k^2 \frac{|R(r)|^2}{r^2} \right] r \, dr,$$

$$I_2 \equiv \frac{1}{2} \int |\nabla_2 a|^2 \, dy_1 dy_2 = \frac{1}{2} \int_0^\infty r dr \int_0^{2\pi} d\varphi \left[|R'(r)|^3 \sin^2 k\varphi + k^2 \frac{|R(r)|^2}{r^2} \cos^2 k\varphi \right]$$

$$= \frac{\pi}{2} \int_0^\infty \left[|R'(r)|^2 + k^2 \frac{|R(r)|^2}{r^2} \right] r \, dr.$$

Hence, $I_1 = I_2$, i.e., (4.14) holds, which implies (4.1) for E_v. $\square$

Now let us prove identity (4.1) for P_v:

$$P_v := - \operatorname{Re} \int_{\mathbb{R}^2} \dot{\psi}_v \overline{\nabla \psi_v} \, dx$$

$$= - \operatorname{Re} \int_{\mathbb{R}^2} \left(-\gamma v (\nabla_1 a)(y) - i\omega \gamma a(y) \right) \left(\overline{\gamma (\nabla_1 a)(y) + i\omega \gamma v a(y)}, \overline{\nabla_2 a(y)} \right) \frac{1}{\gamma} \, dy. \qquad (5.1)$$

First,

$$\operatorname{Re} \int_{\mathbb{R}^2} (\nabla_1 a)(y) \overline{\nabla_2 a(y)} \, dy$$

$$= \operatorname{Re} \int_0^{+\infty} r \, dr \int_0^{2\pi} \left(R'(r) \cos\varphi - ik \frac{R(r)}{r} \sin k\varphi \right) \left(R'(r) \sin k\varphi - ik \frac{R(r)}{r} \cos k\varphi \right) d\varphi$$

$$= \operatorname{Re} \int_0^{+\infty} \left[R'^2(r) - k^2 \frac{R^2(r)}{r^2} \right] r \, dr \int_0^{2\pi} \frac{\sin 2k\varphi}{2} \, d\varphi = 0. \qquad (5.2)$$

Similarly,

$$\operatorname{Re} \int_{\mathbb{R}^2} a(y) \overline{\nabla_2 a(y)} \, dy = \operatorname{Re} \int_0^{+\infty} r \, dr \int_0^{2\pi} R(r) e^{ik\varphi} \left(R'(r) \sin k\varphi - ik \frac{R(r)}{r} \cos k\varphi \right) d\varphi$$

$$= \operatorname{Re} \int_0^{+\infty} \left[R(r) R'(r) + k \frac{R^2(r)}{r} \right] r \, dr \int_0^{2\pi} \frac{\sin 2k\varphi}{2} \, d\varphi = 0. \qquad (5.3)$$

Hence, relations (5.1)–(5.3) imply

$$P_v := - \operatorname{Re} \int_{\mathbb{R}^2} \left(-\gamma v (\nabla_1 a)(y) - i\omega \gamma a(y) \right) \left(\overline{\gamma (\nabla_1 a)(y) + i\omega \gamma v a(y)}, 0 \right) \frac{1}{\gamma} \, dy$$

$$= \left(\gamma v \int_{\mathbb{R}^2} \left(|\nabla_1 a(y)|^2 + \omega^2 a^2(y) \right) dy, 0 \right) = \left(\gamma v 2 (I_1 + \omega^2 I_0), 0 \right). \qquad (5.4)$$

as in (4.12). Further, relation (4.13) and the equality $I_1 = I_2$, which is already proved, imply the validity of (4.1) for P_v. $\square$

Remark. The existence of solitary waves

$$\psi_0(x, t) = R(r) e^{ik\phi} e^{-i\omega t}$$

with $k \neq 0$ was not proved in [3]. However, in this case, equation (2.4) reduces to an ordinary differential equation (as for radial solutions). Therefore, it is natural to think that existence can be proved by modifying the method in [3].

REFERENCES

1. Balabane, M., Cazenave, T., Douady, A., and Merle, F., Existence of Excited States for a Nonlinear Dirac Field, *Commun. Math. Phys.*, 1988, vol. 119, pp. 153–176.
2. Buslaev, V. S. and Perelman, G. S., On the Stability of Solitary Waves for Nonlinear Schrödinger Equations, *Amer. Math. Soc. Trans. (2)*, 1995, vol. 164, pp. 75–98.
3. Berestycki, H. and Lions, P. L., Nonlinear Scalar Field Equations, *Arch. Rational Mech. Anal.*, 1983, vol. 82, pp. 313–345; 347–375.
4. Derrick, G. H., Comments on Nonlinear Wave Equations as Models for Elementary Particles, *J. Mathematical Phys.*, 1964, vol. 5, pp. 1252–1254.
5. Einstein, A. and Grommer, G., Allgemeine Relativitätstheorie und Bewegungsgesetz, *Sitzungsber. preuss. Akad. Wiss., phys.-math Kl.*, 1927, pp. 2–13.

6. Esteban, M. J., Georgiev, V., and Séré, E., Stationary Solutions of the Maxwell–Dirac and the Klein–Gordon–Dirac Equations, *Calc. Var. Partial Differential Equations*, 1996, vol. 4, no. 3, pp. 265–281.

7. Esteban, M. J. and Séré, E., Stationary States of the Nonlinear Dirac Equation: A Variational Approach, *Commun. Math. Phys.*, 1995, vol. 171, no. 2, pp. 323–350.

8. Finkelstein, R., Fronsdal, K., and Kaus, P., Nonlinear Spinor Field, *Phys. Rev*, 1956, vol. 103, pp. 1571–1580.

9. Finkelstein, R., LeLevier, R., and Ruderman, M., Nonlinear Spinor Fields, *Phys. Rev*, 1951, vol. 83, pp. 326–332.

10. Frohlich, J., Tsai, T.-P., and Yau, H.-T., On a Classical Limit of Quantum Theory and the Non-Linear Hartree Equation, Conference Moshe Flato 1999, Vol. I (Dijon), Kluwer: Dordrecht, 2000, pp. 189–207.

11. Grillakis, M., Shatah, J., and Strauss, W., Stability Theory of Solitary Waves in the Presence of Symmetry, I; II, *J. Funct. Anal.*, 1987, vol. 74, no. 1, pp. 160–197; 1990, vol. 94, no. 2, pp. 308–348.

12. Pohozaev [Pokhozhaev], S. I., On the Eigenfunctions of the Equation $\Delta u + \lambda f(u) = 0$, *Soviet Math. Doklady*, 1965, vol. 5, pp. 1408–1411.

13. Poincaré, H., Sur la dynamique de l'électron, *CRAS*, 1905, vol. 140, pp. 1504–1508.

14. Born, M., On the Quantum Theory of the Electromagnetic Field, *Proc. Roy. Soc.*, 1934, vol. A143, pp. 410–437.

15. Born, M. and Infeld, L., Foundations of the New Field Theory, *Proc. Roy. Soc.*, 1934, vol. A144, pp. 425–451.

16. Komech, A., Kunze, M., and Spohn, H., Effective Dynamics for a Mechanical Particle Coupled to a Wave Field, *Comm. Math. Phys.*, 1999, vol. 203, pp. 1–19.

17. Rosen, N., A Field Theory of Elementary Particles, *Phys. Rev*, 1939, vol. 55, pp. 94–101.

18. Rosen, G., Existence of Particle-Like Solutions to Nonlinear Field Theories, *J. Math. Phys.*, 1966, vol. 7, pp. 2066–2070.

19. Rosen, G., Nonexistence of Localized Periodic Solutions to Nonlinear Field Theories, *J. Math. Phys.*, 1966, vol. 7, p. 2071.

20. Rosen, G., Nonexistence of finite-energy stationary quantum states in nonlinear field theories, *J. Math. Phys.*, 1968, vol. 9, pp. 804–805.

21. Rosen, G., Particle-Like Solutions to Nonlinear Complex Scalar Field Theories with Positive-Definite Energy Densities, *J. Math. Phys.*, 1968, vol. 9, pp. 996–998.

22. Rosen, G., Charged Particle-Like Solutions to Nonlinear Complex Scalar Field Theories with Positive-Definite Energy Densities, *J. Math. Phys.*, 1968, vol. 9, pp. 999–1002.

23. Rosen, G., A Necessary Condition for the Existence of Singularity-Free Global Solutions to Nonlinear Ordinary Differential Equations, *Q. Appl. Math.*, 1969, vol. 27, pp. 133–134.

24. Soler, M., Clasical, Stable, Nonlinear Spinor Field with Positive Rest Energy, *Phys. Rev.*, 1970, vol. D 1, no. 10, p. 2766.

25. Spohn, H., Semiclassical Limit of the Dirac Equation and Spin Precession, *Ann. Physics*, 2000, vol. 282, no. 2, pp. 420–431.

26. Wakano, M., Intensely Localized Solutions of the Classical Dirac–Maxwell Field Equations, *Progr. Theor. Phys.*, 1964, vol. 31, p. 879.