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Breathers for a Relativistic Nonlinear Wave Equation

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Abstract

A new class of one-dimensional relativistic nonlinear wave equations with a singular δ-type nonlinear term is considered. The sense of the equations is defined according to the least-action principle. The energy and momentum conservation is established. The main results are the existence of time-periodic finite-energy solutions, the existence of global solutions and soliton-type asymptotics for a class of finite-energy initial data.

1. Introduction

We consider real-valued solutions to a relativistic nonlinear wave equation of the type

$$\ddot{u}(x,t) = u''(x,t) + F(u(x,t)), \quad x \in \mathbb{R}. \tag{1.1}$$

We assume that $u(x,t) \in C(\mathbb{R}^2)$ and the nonlinear term $F(u)$ is a distribution of the type

$$F(u) = \sum_{k\in\mathbb{Z}} F_k \delta(u - z_k), \quad u \in \mathbb{R}, \tag{1.2}$$

where $Z = \{z_k : k \in \mathbb{Z}\}$ is a discrete closed subset of $\mathbb{R}$. Then there exists a piecewise constant potential $V(u)$ such that $F(u) = -V'(u)$. Let us note that any continuous potential can be approximated by piecewise constant functions. Physically, (1.1) describes a string with a nonlinear self-action. This self-action is concentrated at the union of the level sets

$$\Gamma_k = \Gamma_k(u) = u^{-1}(z_k) \equiv \{(x,t) \in \mathbb{R}^2 : u(x,t) = z_k\}. \tag{1.3}$$

We define the sense of (1.1) by the variational Hamilton least-action principle. This sense does not coincide with distributional treatment of (1.1) because the tangent space to the phase manifold differs from the standard space of test functions.

Moreover, the distributional treatment of (1.1) seems to be impossible. Indeed, $\delta(u(x,t)-z_k)$ is a well-defined distribution if $u(x,t)\in C^1$ at the points of $\Gamma_k(u)$. However, this C^1 continuity contradicts (1.1).

Consider the Cauchy problem for (1.1) with the initial conditions

$$u|_{t=0}=u_0(x),\quad \dot u|_{t=0}=v_0(x),\ \ x\in\mathbb{R}. \tag{1.4}$$

We subject the initial data to a monotonicity condition that in particular includes the case of monotone $u_0(x)$ and $v_0(x)\equiv 0$. We restrict ourselves by piecewise C^2 initial data and piecewise C^2 solutions for the simplicity of exposition.

We prove the existence and uniqueness of a global solution to the Cauchy problem (1.1), (1.4). Our main results are the existence of time-periodic solutions to (1.1) and the following soliton-type asymptotics:

$$u(x,t)\sim b(\lambda(x-vt),\lambda(t-vx))+\sum_{\pm}\phi_\pm(x\mp t),\quad t\to\infty. \tag{1.5}$$

Here $b(x,t)$ is a time-periodic solution ("breather") to (1.1), $|v|<1$, $\lambda=1/\sqrt{1-v^2}$ and the "photons" $\phi_\pm(x\mp t)$ are concentrated near the light cone $|x|\sim t$. Similar asymptotics hold as $t\to-\infty$. The results hold for the potentials $V(u)$ which are piecewise constant approximations to the Ginzburg-Landau potential $(u-u_-)^2(u-u_+)^2$ (see Fig. 1). For proofs we reduce (1.1) to a system of ordinary differential equations for the lines $\Gamma_k(u)$ and derive the corresponding properties for the solutions to the system. This derivation is the main part of our analysis.

Soliton-type asymptotics are established for some translation invariant completely integrable one-dimensional equations, [9]. The asymptotics in *local* energy seminorms are established for a translation-invariant three-dimensional system of a scalar field coupled to a particle [8]. The asymptotics of type (1.5) in *global* energy norm are established for small perturbations of soliton-type solutions to one-dimensional nonlinear Schrödinger translation-invariant equations [3] and to a translation-invariant three-dimensional system of a scalar or Maxwell field coupled to a particle [5, 6]. In the present paper the asymptotics of type (1.5) are established for the first time for a relativistic-invariant nonlinear wave equation.

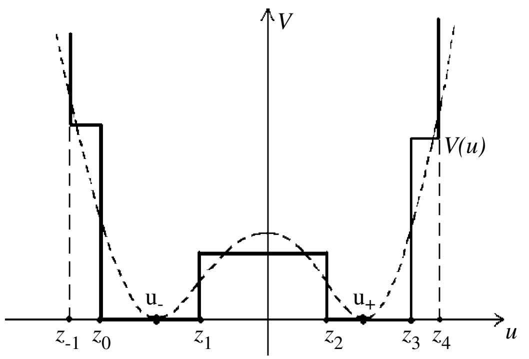

Fig. 1.

The existence of time-periodic solutions is known only for the sine-Gordon equation with $F(u) = \sin u$, [2, 7]. The nonexistence of the breathers for a smooth function $F(u) \not\equiv \sin u$ has been analyzed in [2, 4, 7]. For singular $F(u)$ of type (1.2) we construct infinite sets of the breathers depending on a functional parameter. We suppose that the existence of the breathers in our case is related to the fact that equation (1.1) with the nonlinear term of type (1.2) is degenerate in some sense. For instance, the set of finite-energy stationary solutions is not discrete, because it contains any constant function $u(x,t) = c$ with $c \notin Z$. In this paper we restrict ourselves to a particular class of solutions monotonous in x. We hope that a suitable development of our technique could provide the existence of global solutions and the asymptotics (1.5) for each finite-energy solution.

Remark. Numerical experiments [1, 10] show that the asymptotics of type (1.5) hold for general equations (1.1) with a polynomial $F(u)$. The results were confirmed with high precision by G. Cohen, F. Collino, T. Fouquet, P. Joly, L. Rhaouti and O. Vacus (Project ONDES, INRIA). However, the proof is still an open problem.

2. Main results

Let us describe our results more precisely. We denote by $V(u)$ the potential, $F(u) = -V'(u)$. We choose $u_\pm \in \mathbb{R}$ such that (see Fig. 1)

$$V(u_\pm) = 0, \tag{2.1}$$

$$u_\pm \notin Z. \tag{2.2}$$

Note that the potential $V(u)$ satisfying (2.1) exists if and only if

$$\sum_{k:\, u_- < z_k < u_+} F_k = 0. \tag{2.3}$$

For instance, such potential always exists in the case $u_+ = u_-$. We assume that the potential $V(u)$ is bounded from below,

$$V(u) \geqq -V_0, \quad u \in \mathbb{R}, \tag{2.4}$$

where $V_0 \geqq 0$ due to (2.1). Let us introduce a phase space E of finite-energy states for (1.1). For an open set $\omega \subset \mathbb{R}$ and $l = 0, 1, 2, \ldots$ denote by $C_{\mathrm{b}}^l(\omega)$ the space of functions in ω with bounded derivatives up to the order l. Let $I = [a, b] \subset \mathbb{R}$ be a closed interval. Denote by $C_{\mathrm{pw}}^l(I)$ the space of piecewise C^l-continuous functions $u(x)$. This means that there exists a finite subset $S(u) \subset I$ such that $u \in C_{\mathrm{b}}^l(\bar{\omega})$ for any open interval $\omega \subset I \setminus S(u)$. Denote by $C_{\mathrm{pw}}^l(\mathbb{R})$ the set of functions $u(x)$ such that $u \in C_{\mathrm{pw}}^l(I)$ for each finite interval $I = [a, b] \subset \mathbb{R}$.

Definition 2.1. The *phase space* E is the set of the states $(u(x), v(x)) \in C_{\mathrm{pw}}^2(\mathbb{R}) \times C_{\mathrm{pw}}^1(\mathbb{R})$ such that

(i) $u'(x), v(x) \in L^2(\mathbb{R})$, where $u'(x)$ is the derivative in the sense of the distributions,

(ii) the set $\{x \in \mathbb{R} : u(x) \in Z\}$ is discrete, and

$$u(x) \to u_{\pm}, \;\; v(x) \to 0, \qquad x \to \pm\infty. \tag{2.5}$$

Equation (1.1) is a formal Hamiltonian system with the phase space E and with the Hamiltonian functional H and the total momentum P

$$H(u,v) = \int \Big(\tfrac{1}{2}|v(x)|^2 + \tfrac{1}{2}|u'(x)|^2 + V(u(x))\Big)dx, \tag{2.6}$$

$$P(u,v) = \int \; u'(x)\, v(x)\, dx \tag{2.7}$$

for $(u, v) \in E$. The Lagrangian functional L is defined in the phase space E by

$$L(u,v) = \int \Big(\tfrac{1}{2}|v(x)|^2 - \tfrac{1}{2}|u'(x)|^2 - V(u(x))\Big)dx. \tag{2.8}$$

Remark. The integrands in (2.6)–(2.8) are defined for a.e. $x \in \mathbb{R}$ and the integrals converge due to (2.2), (2.1) and Definition 2.1.

Let us define an appropriate class $\mathcal{E} \subset C(\mathbb{R}^2)$ of solutions $u(x,t)$ to (1.1). We call a set $\mathcal{S} \subset \mathbb{R}^2$ *characteristic* if it is a union of open intervals of some characteristics $x \pm t$ =const. We call a characteristic set $\mathcal{S}$ *locally finite* if for every $B > 0$ the intersection of $\mathcal{S}$ with the set $|x| + |t| < B$ is a union of a finite number of open intervals of the characteristics. Define $\Gamma = \Gamma(u) = \cup_k \Gamma_k(u)$ (see (1.3)).

Definition 2.2. The set $\mathcal{E}$ is the set of functions $u(x,t) \in C(\mathbb{R}^2)$ such that

(i) $(u(\cdot,t), \dot u(\cdot,t)) \in E$, $t \in \mathbb{R}$, where $\dot u(x,t)$ is the derivative in the sense of distributions.

(ii) The following limits hold $\forall\, t \in \mathbb{R}$:

$$u(x,t) \to u_{\pm}, \;\; x \to \pm\infty. \tag{2.9}$$

(iii) If the set $\Gamma_k = \Gamma_k(u)$ is nonempty for a $k \in \mathbb{Z}$, then it is a line $x = x_k(t)$ with $x_k(\cdot) \in C(\mathbb{R}) \cap C^2_{\mathrm{pw}}(\mathbb{R})$.

(iv) We have $u(x,t) \in C^2(\bar\Omega)$ for any connected (open) component Ω of $\mathbb{R}^2 \setminus (\Gamma \cup \overline{\mathcal{S}})$ where $\mathcal{S} = \mathcal{S}(u) \subset \mathbb{R}^2$ is a locally finite characteristic set.

To define the sense of (1.1) for $u(x,t) \in \mathcal{E}$ we introduce a "continuity" function

$$C(x,t) = \tfrac{1}{2}|\dot u(x,t)|^2 - \tfrac{1}{2}|u'(x,t)|^2 + V(u(x,t)), \;\; (x,t) \in \mathbb{R}^2 \setminus (\Gamma(u) \cup \overline{\mathcal{S}(u)}). \tag{2.10}$$

The function is well defined due to Definition 2.2(iv).

Definition 2.3. The function $u(x,t) \in \mathcal{E}$ is a solution to (1.1) if

(i) in the sense of distributions,

$$\ddot u = u'', \;\; (x,t) \in \mathbb{R}^2 \setminus \Gamma(u), \tag{2.11}$$

(ii) The following splicing condition holds on each non-empty $\Gamma_k(u)$:

$$C(x_k(t) - 0, t) = C(x_k(t) + 0, t), \quad a.e.\ t \in \mathbb{R}. \tag{SC}$$

Remarks. (i) The condition (SC) is meaningful due to Definition 2.2(iv). (ii) For $u(x,t) \in \mathcal{E}$, the system (2.11), (SC) is equivalent to the Hamilton least-action principle (see Appendix C).

In Appendix A we prove the following lemma that provides *a priori* estimates for solutions.

Lemma 2.4. *Let* $u(x,t) \in \mathcal{E}$ *be a solution to* (1.1). *Then*

(i) *if* (2.1) *and* (2.2) *hold, the energy is conserved:*

$$H(u(\cdot,t), \dot{u}(\cdot,t)) = \text{const}, \;\; t \in \mathbb{R}, \tag{2.12}$$

(ii) *if* (2.2) *and* (2.3) *hold, the momentum is conserved:*

$$P(u(\cdot,t), \dot{u}(\cdot,t)) = \text{const}, \;\; t \in \mathbb{R}. \tag{2.13}$$

We introduce a class of "monotonic" states.

Definition 2.5. We define

(i) M as the set of the states $(u, v) \in E$ such that

$$|u'(x \pm 0)| > |v(x \pm 0)|, \;\; x \in \mathbb{R}, \tag{2.14}$$

(ii) and $\mathcal{M}$ as the set of the functions $u(x,t) \in \mathcal{E}$ such that $(u(\cdot,t), \dot{u}(\cdot,t)) \in M$, $t \in \mathbb{R}$, where $\dot{u}(\cdot,t)$ is the derivative in the sense of distributions.

Remark. From inequality (2.14) it follows that the functions $u(x) \pm \int_0^x |v(y)|dy$ are strong monotone.

The set Z is discrete. Hence, for $u(x,t) \in \mathcal{E}$ the set $\Gamma(u)$ is a disjoint union of a finite number of the lines $\Gamma_k = \Gamma_k(u) : \; x = x_k(t), k = 1, \ldots, N$. We may assume that $x_k(0)$ increase in k. Then by continuity $x_k(t)$ is also increasing in k for every $t \in \mathbb{R}$,

$$-\infty \equiv x_0(t) < x_1(t) \cdots < x_N(t) < x_{N+1}(t) \equiv +\infty. \tag{2.15}$$

Theorem 2.6. *Let* (2.1) *hold and* $(u_0, v_0) \in M$. *Then*

(i) *There exists a unique solution* $u(x,t) \in \mathcal{M}$ *to the Cauchy problem* (1.1), (1.4).
(ii) *The line* Γ_k *is time-like for any* $k = 1, \ldots, N$ *in the following sense:*

$$|\dot{x}_k(t \pm 0)| < 1, \;\; t \in \mathbb{R}. \tag{2.16}$$

For the proof we derive a finite system of nonlinear ordinary differential equations for the functions $x_k(t)$. Namely, the lines Γ_k divide the plane x, t into $N+1$ "strips",

$$\Pi_k = \{(x,t) \in \mathbb{R}^2 : \; x_{k-1}(t) < x < x_k(t)\}, \;\; k = 1, \ldots, N+1. \tag{2.17}$$

First we prove the following weakened form of (2.16).

Lemma 2.7. *Let* $u(x,t) \in \mathcal{M}$ *be a solution to* (1.1). *Then for any* $k = 1, \ldots, N$,

$$|x_k(t) - x_k(s)| < |t - s|, \;\; s \neq t. \tag{2.18}$$

This lemma and (2.11) imply the d'Alembert representation

$$u(x,t) = f_k(x-t) + g_k(x+t) \tag{2.19}$$

in every strip Π_k, $k = 0, \ldots, N+1$. Definition 2.2(iii) implies

$$f_k, g_k \in C(\mathbb{R}) \cap C^2_{\mathrm{pw}}(\mathbb{R}). \tag{2.20}$$

Then substituting the representation (2.19) into (SC) we obtain the system of differential equations for the functions $x_k(t)$. The system implies the bound (2.16). This allows us to prove the existence and uniqueness of the solution $x_1(t), \ldots, x_N(t)$ and to reconstruct the functions f_k and g_k.

Next we consider *weak* solutions to (1.1). The weak solutions $u(\cdot, t)$ admit the values $u = z_k$ on non-empty open segments. Therefore, we have to distinguish the values $z_k - 0$ and $z_k + 0$ to define $V(u)$ correctly in (2.6), (2.8) and (2.10). This allows us to remove the restriction (2.2). For concreteness, we consider the particular case when the interval $[u_-, u_+]$ contains only two points of the set Z (see Fig. 1):

$$[u_-, u_+] \cap Z = \{z_1, z_2\}, \quad z_1 < z_2 \ . \tag{2.21}$$

We consider the weak solutions $\hat{u}(x,t)$ which take the values in the disjoint union $\hat{I} := [u_-, z_1 - 0] \cup [z_1 + 0, z_2 - 0] \cup [z_2 + 0, u_+]$. We have $F_1 = -F_2$ by (2.3), then (2.1) implies

$$V(\hat{u}) = \left\{ \begin{array}{ll} 0, & \hat{u}_- \leqq \hat{u} \leqq z_1 - 0 \\ F_2, & z_1 + 0 < \hat{u} \leqq z_2 - 0 \\ 0, & z_2 + 0 \leqq \hat{u} \leqq \hat{u}_+ \end{array} \right| \quad \hat{u} \in \hat{I} \tag{2.22}$$

with a natural ordering in $\hat{I}$. The potential with $F_2 > 0$ simulates the features of the Ginzburg-Landau potential $(u - u_-)^2(u - u_+)^2$ (see Fig. 1). We introduce a *weak* phase space $\hat{M}$ and a class $\hat{\mathcal{M}}$ of *weak* solutions instead of M and $\mathcal{M}$ (mainly we replace $>$ by $\geqq$ in (2.14)). For the simplicity of exposition we change the condition (2.5) in Definition 2.1 to

$$\hat{u}(x) = \hat{u}_\pm, \ \ v(x) = 0, \quad \pm x \geqq \mathrm{R} \tag{2.23}$$

with some $R > 0$. We adjust the sense of the weak solution in Section 5.

We prove the existence and uniqueness of the global weak solution $\hat{u}(x,t) \in \hat{\mathcal{M}}$ for $(\hat{u}_0, v_0) \in \hat{M}$ assuming that $F_2 > 0$ in (2.22) (Theorem 5.6). In the next theorem we consider the initial states $(\hat{u}_0, v_0) \in \hat{M}$ with the boundary values $\hat{u}_- = z_1 - 0$, $\hat{u}_+ = z_2 + 0$ in (2.23):

$$\hat{u}(x) = \left\{ \begin{array}{ll} z_1 - 0, & x < -R \\ z_2 + 0, & x > R \end{array} \right| \quad v(x) = 0, \quad |x| \geqq \mathrm{R}. \tag{2.24}$$

Let us note that this is possible since the restriction (2.2) is not necessary for the weak solutions. Then we prove that the corresponding weak solution is a *breather*, i.e., is time-periodic up to a Lorentz transformation:

Theorem 2.8. *Let* (2.22) *hold, and* $(\hat{u}_0, v_0) \in \hat{M}$ *admits the boundary values* (2.24). *Then the weak solution to the Cauchy problem* (1.1), (1.2) *admits the representation*

$$\hat{u}(x,t) = \hat{b}(\lambda(x - vt), \lambda(t - vx)), \quad (x,t) \in \mathbb{R}^2, \tag{2.25}$$

where $|v| < 1$, $\lambda = 1/\sqrt{1-v^2}$, $\hat{b}(x,t)$ *is a time-periodic weak solution to* (1.1), *and*

$$\hat{b}(x,t) = \begin{cases} z_1 - 0, & x \leqq x_1(t), \\ z_2 + 0, & x \geqq x_2(t). \end{cases} \tag{2.26}$$

For the proof we generalize the methods of Section 3 and obtain the uniform bound $|\dot{x}_k(t)| \leqq \gamma < 1$, $k = 1, 2$. Equation (2.25) means that the solution $\hat{u}(x,t)$ is the breather $\hat{b}(x,t)$ moving with speed v.

In the next theorem we assume (2.24) but we do not assume that $\hat{u}_- = z_1 - 0$, $\hat{u}_+ = z_2 + 0$ as in (2.24). We denote by p the natural projection $\hat{I} \to \mathbb{R}$. The following theorem is the main result of the paper.

Theorem 2.9. *Let* (2.21) *and* (2.22) *hold with* $F_2 > 0$, *and* $(\hat{u}_0, v_0) \in \hat{M}$. *Then for the weak solution* $\hat{u}(x,t) \in \hat{\mathcal{M}}$ *to the Cauchy problem* (1.1), (1.4), *the asymptotics* (1.5) *hold in the following sense: there exists a time* $t^* > 0$ *such that*

$$p\hat{u}(x,t) = p\hat{b}(\lambda(x - vt), \lambda(t - vx)) + \sum_{\pm} \phi_\pm(x \mp t), \quad x \in \mathbb{R},\ t \geqq t^*, \tag{2.27}$$

where $|v| < 1$, $\lambda = 1/\sqrt{1-v^2}$, $\hat{b}(x,t)$ *is a time-periodic weak solution to* (1.1), *and* (2.26) *holds;* $\phi_\pm(x) \in C^2_{\mathrm{pw}}(\mathbb{R})$ *and* $\phi'_\pm(x) \in L^2(\mathbb{R})$. *Similar asymptotics hold for negative* $t \leqq t_*$.

For the proof we generalize the method of Section 6 and show that the line Γ_1 or Γ_2 intersects the characteristics $x \pm t = C$ with $C < x_1(0)$ or $C > x_2(0)$ respectively (see Proposition 8.1). This means that the effective speed of the propagation for the segment $[x_1(t), x_2(t)]$ is strictly less than 1. The bound is very natural physically for the relativistic equation (1.1) but its proof is not straightforward. This is a central part of our arguments. The arguments use essentially the *a priori* bounds which follow from the energy conservation for the weak solutions (see Corollary 8.2 and Lemma 8.3). Furthermore, the arguments rely on special features of the potential (2.22) with $F_2 > 0$.

Remarks. (i) We assume (2.23) for the simplicity of exposition. It is possible to consider more general initial data, assuming sufficiently fast convergence

$$\hat{u}(x) \to \hat{u}_\pm, \ v(x) \to 0, \qquad x \to \pm\infty. \tag{2.28}$$

Then (2.27) holds asymptotically, as $t \to \infty$.
(ii) Let us emphasize that the breather $\hat{b}(x,t)$ is not a solution in the sense of Definition 2.3 since (2.26) contradicts (2.2).
(iii) We cannot ignore the weak solutions with the boundary values (2.26) since they combine the attractor of (1.1): even if we start with an initial state $(\hat{u}_0, v_0) \in \hat{M}$ with the boundary values (2.23) where $\hat{u}_- \neq z_1 - 0$ and $\hat{u}_+ \neq z_2 + 0$, the limit breather satisfies (2.26).

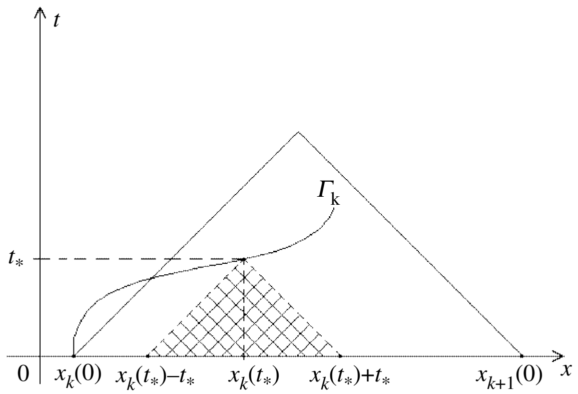

Fig. 2.

In Sections 3, 4 we prove the existence and uniqueness of the solution to the Cauchy problem. Section 5 concerns the weak solutions. In Sections 6 and 7 we construct the breathers. Section 8 concerns the soliton-type asymptotics. Appendix A concerns energy and momentum conservation. In Appendix B we prove the dichotomy of the roots of an algebraic equation, and in Appendix C we analyze the Hamilton least-action priciple.

3. Uniqueness: reconstruction of solution

We prove the uniqueness of the solution in Theorem 2.6(i). The proof is constructive and leads automatically to the existence. We consider $t \geqq 0$ for concreteness. Let $u(x,t) \in \mathcal{M}$ be a solution to the Cauchy problem (1.1), (1.4) with $(u_0, v_0) \in M$. Then for every fixed $t \in \mathbb{R}$ the function $u(x,t)$ is a strong monotonic in x. Let us assume for example that $u_0(x)$ is (strong) monotone increasing. Then by continuity $u(\cdot,t) \in M$ is also a strong monotonically increasing function for every $t \in \mathbb{R}$. Note that $x_k(0)$ are determined uniquely by the initial datum $u_0(x)$. We will derive below ordinary differential equations for the functions $x_k(t)$ defining the lines Γ_k.

Now we start to reconstruct all functions $f_k(x)$, $g_k(x)$ and $x_k(t)$. For $k = 1, \dots, N+1$ let us denote $X_k(t) = (x_{k-1}(t), x_k(t))$, $t \in \mathbb{R}$, and let Δ_k be an open "characteristic triangle" $\{(x,t) \in \mathbb{R}^2 : (x-t, x+t) \subset X_k(0)\}$. Let us introduce $T_0 = \frac{1}{2}\min_{2 \leqq k \leqq N} |x_k(0) - x_{k-1}(0)|$.

Step 1. Uniqueness in characteristic region

Let us substitute (2.19) into initial conditions (1.4). This leads to the standard d'Alembert formulae: up to an additive constant, for $k = 1, \dots, N+1$,

$$f_k(x) = \frac{u_0(x)}{2} - \frac{1}{2}\int_{x_k(0)}^{x} v_0(y)dy, \quad g_k(x) = \frac{u_0(x)}{2} + \frac{1}{2}\int_{x_k(0)}^{x} v_0(y)dy, \quad (3.1)$$

where $x \in \overline{X_k(0)}$. The formulas (2.19) and (3.1) define the solution $u(x,t)$ uniquely in the region Δ_k with any $k = 1, \ldots, N+1$.

Step 2. Proof of Lemma 2.7

Equations (3.1) and Definitions 2.1, 2.5 imply, for $k = 1, \ldots, N+1$,

$$f_k, g_k \in C(\overline{X_k(0)}) \cap C^2_{\mathrm{pw}}(\overline{X_k(0)}), \tag{3.2}$$

$$f_k'(x \pm 0) > 0, \;\; g_k'(x \pm 0) > 0, \quad x \pm 0 \in \overline{X_k(0)}. \tag{3.3}$$

Let us assume, contrary to (2.18), that $|x_k(t_*) - x_k(s_*)| \geqq |t_* - s_*|$ for some k and sufficiently close $s_*, t_* \in \mathbb{R}$, $s_* \neq t_*$. We may assume that $s_* = 0 < t_* < T_0$, and $(x_k(t_*), t_*) \in \overline{\Delta_k}$ for concreteness. Then

$$x_k(0) \leqq x_k(t_*) - t_* < x_k(t_*) + t_* \leqq x_{k+1}(0). \tag{3.4}$$

Moreover, we may assume that the interior of the open triangle $\triangle_*$ with the vertices $(x_k(t_*), t_*)$, $(x_k(t_*) \pm t_*, 0)$ does not contain any points of the line Γ_k (see Fig. 2). Then the d'Alembert representation (2.19) holds in $\overline{\triangle_*}$. Therefore, (3.3) and (3.4) imply

$$\begin{aligned} u(x_k(t_*), t_*) &= f_{k+1}(x_k(t_*) - t_*) + g_{k+1}(x_k(t_*) + t_*) \\ &> f_{k+1}(x_k(0)) + g_{k+1}(x_k(0)) = u(x_k(0), 0) = z_k, \end{aligned} \tag{3.5}$$

which contradicts the assumption $(x_k(t_*), t_*) \in \Gamma_k$. $\square$

Corollary 3.1. *For any $k = 1, \ldots, N$, the line Γ_k is time-like in the following weak sense:*

$$|\dot{x}_k(t \pm 0)| \leqq 1, \;\; t \in \mathbb{R}. \tag{3.6}$$

Step 3. Differential equation for Γ_k

Now we are going to determine the lines Γ_k in the strip $0 \leqq t \leqq T_0$. Namely, we derive a system of ordinary differential equations for the functions $x_k(t)$ in the interval $[0, T_0]$ using the splicing condition (SC) and (2.18). Let us denote $\nabla_k = \nabla_k(T_0) = \{(x,t) \in \mathbb{R}^2 : |x - x_k(0)| \leqq t \leqq T_0\}$ (see Fig. 3). The inequality (2.18) implies $(x_k(t), t) \in \nabla_k$ for $t \in [0, T_0]$.

The definition of $x_k(t)$ and the splicing condition (SC) imply the following three identities for any $k = 1, \ldots, N$:

$$\begin{gathered} u(x_k(t) \pm 0, t) = z_k, \;\; t \in \mathbb{R}, \\ [\tfrac{1}{2}|\dot{u}|^2 - \tfrac{1}{2}|u'|^2 + V(u)]\Big|_{(x_k(t)-0,t)}^{(x_k(t)+0,t)} = 0, \quad \text{a.e. } t \in R. \end{gathered} \tag{3.7}$$

Formula (3.2) implies $f_k(x), g_{k+1}(x) \in C^2(\bar{\omega})$ for any interval $\omega \subset X_k \setminus S_k$ or $\omega \subset X_{k+1} \setminus S_{k+1}$ respectively, where S_k, S_{k+1} are some finite sets, $k = 2, \ldots, N$. Then $f_k(x-t), g_{k+1}(x+t) \in C^2(\bar{\Omega})$ for any connected (open) subset $\Omega \subset \nabla_k \setminus \mathcal{S}_k$, where $\mathcal{S}_k$ is a locally finite characteristic set $\{(x,t) \in \nabla_k : x - t \in S_k$ or $x + t \in S_{k+1}\}$. For simplicity of exposition we assume for a moment that $S_k = S_{k+1} = \emptyset$, hence $\mathcal{S}_k = \emptyset$. Then substituting the representation (2.19) for $u(x, t)$ into (3.7), we get the system

$$\left.\begin{aligned} f_k(x_k(t) - t) + g_k(x_k(t) + t) &= z_k, \\ f_{k+1}(x_k(t) - t) + g_{k+1}(x_k(t) + t) &= z_k, \end{aligned}\right| \quad t \in [0, T_0], \tag{3.8}$$

$$\begin{aligned} &f_k'(x_k(t) - t) g_k'(x_k(t) + t) \\ &\quad - f_{k+1}'(x_k(t) - t) g_{k+1}'(x_k(t) + t) - \tfrac{1}{2} F_k = 0, \text{ a.e. } t \in [0, T_0]. \end{aligned} \tag{3.9}$$

We have used

$$[V(u)]\Big|_{(x_k(t)-0,t)}^{(x_k(t)+0,t)} = -F_k\,,$$

which follows from the fact that $u(\cdot, t)$ is strong increasing.

For $(x, t) \in \nabla_k$, the "ingoing" waves $f_k(x - t)$ and $g_{k+1}(x + t)$ are known from (3.1) since $x - t \in X_k$ and $x + t \in X_{k+1}$ by the definition of T_0. Hence from the bound (3.6) it follows that in the system (3.8), (3.9) two functions $f_k(x_k(t) - t)$ and $g_{k+1}(x_k(t) + t)$ are known. Therefore, we may eliminate two unknown "reflected" waves $g_k(x_k(t) + t)$ and $f_{k+1}(x_k(t) - t)$ to get an equation for $x_k(t)$. Later on, when $x_k(t)$ is determined, we will derive the reflected waves $g_k(x_k(t) + t)$ and $f_{k+1}(x_k(t) - t)$ from (3.8) in the interval $0 \leqq t \leqq T_0$. To eliminate g_k and f_{k+1}, let us differentiate the identities (3.8) in t:

$$\left.\begin{aligned} f_k'(x_k(t) - t)(\dot{x}_k(t) - 1) + \quad g_k'(x_k(t) + t)(\dot{x}_k(t) + 1) &= 0 \\ f_{k+1}'(x_k(t) - t)(\dot{x}_k(t) - 1) + g_{k+1}'(x_k(t) + t)(\dot{x}_k(t) + 1) &= 0 \end{aligned}\right| \quad t \in [0, T_0]. \tag{3.10}$$

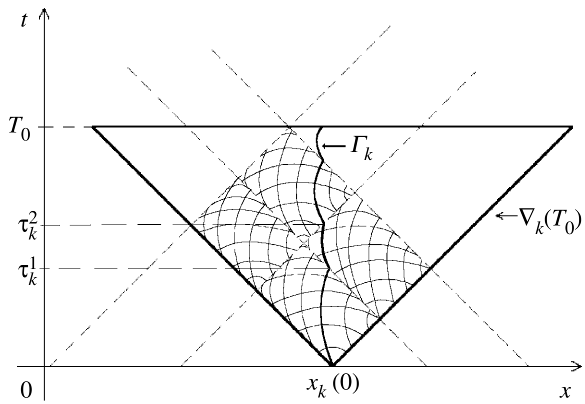

Fig. 3.

Let us assume for a moment that $\dot{x}_k(t) \neq \pm 1$. Then we get the derivatives of the reflected waves:

$$\left.\begin{aligned} g_k'(x_k(t)+t) &= -f_k'(x_k(t)-t)\frac{\dot{x}_k(t)-1}{\dot{x}_k(t)+1} \\ f_{k+1}'(x_k(t)-t) &= g_{k+1}'(x_k(t)+t)\frac{\dot{x}_k(t)+1}{\dot{x}_k(t)-1} \end{aligned}\right| \quad t\in[0,T_0]. \tag{3.11}$$

Substituting these expressions into the splicing condition (3.9), we get

$$-|f_k'(x_k(t)-t)|^2\frac{\dot{x}_k(t)-1}{\dot{x}_k(t)+1}+|g_{k+1}'(x_k(t)+t)|^2\frac{\dot{x}_k(t)+1}{\dot{x}_k(t)-1}-\frac{F_k}{2}=0, \text{ a.e. } t\in[0,T_0]. \tag{3.12}$$

Therefore the differential equation holds

$$a_k(x_k(t),t)\dot{x}_k^2(t)-2b_k(x_k(t),t)\dot{x}_k(t)+c_k(x_k(t),t)=0, \quad \text{a.e. } t\in[0,T_0]. \tag{3.13}$$

Here we denote

$$\left.\begin{aligned} a_k(x,t) &= |f_k'(x-t)|^2-|g_{k+1}'(x+t)|^2+\frac{F_k}{2} \\ b_k(x,t) &= |f_k'(x-t)|^2+|g_{k+1}'(x+t)|^2 \\ c_k(x,t) &= |f_k'(x-t)|^2-|g_{k+1}'(x+t)|^2-\frac{F_k}{2} \end{aligned}\right| \quad (x,t)\in\nabla_k\,. \tag{3.14}$$

Obviously, (3.13) follows from (3.10) without the assumption $\dot{x}_k(t) \neq \pm 1$.

Now we return to the general case when the sets S_k, $S_{k=1}$ and $\mathcal{S}_k$ are not empty. Then (3.13) becomes

$$a_k(x_k(t),t)\dot{x}_k^2(t)-2b_k(x_k(t),t)\dot{x}_k(t)+c_k(x_k(t),t)=0, \quad \text{a.e. } t\in[0,T_0]\setminus\mathcal{T}_k, \tag{3.15}$$

where $\mathcal{T}_k := \{\tau\in[0,T_0] : (x_k(\tau),\tau)\in\mathcal{S}_k\}$: see Fig. 3. The set $\mathcal{T}_k=\{\tau_k^1,\tau_k^2,\ldots\}$ is finite by Lemma 2.7, and the coefficients (3.14) belong to $C^1(\bar{\Omega})$ for any connected (open) component Ω of the set $\nabla_k\setminus\mathcal{S}_k$.

Step 4. Dichotomy for the roots

To express $\dot{x}_k(t)$ from (3.15), we have to analyze the function $v_k(x,t)$ defined by the algebraic equation

$$a_k(x,t)v_k^2(x,t)-2b_k(x,t)v_k(x,t)+c_k(x,t)=0, \quad (x,t)\in\nabla_k\setminus\mathcal{S}_k\,. \tag{3.16}$$

Furthermore, from the inequality (3.6) it follows that we have to choose the roots $v_k(x,t)\in[-1,1]$. The following Lemma states the features of the roots and implies the bound (2.16).

Lemma 3.2. *Let* $(u_0,v_0)\in M$. *Then for any* $k=1,\ldots,N$ *and* $(x,t)\in\nabla_k\setminus\mathcal{S}_k$

(i) *there exists a unique solution* $\bar{v}_k(x,t)\in[-1,1]$ *to* (3.16)*;*
(ii) *moreover,* $\bar{v}_k(x,t)\in(-1,1)$*;*
(iii) $\bar{v}_k(x,t)\in C^1(\bar{\Omega})$ *for any connected* (*open*) *component* Ω *of the set* $\nabla_k\setminus\mathcal{S}_k$.

Proof. *Proof of* (*i*) *and* (*ii*). For $(x, t) \in \nabla_k \setminus \mathcal{S}_k$ we have for the discriminant,

$$d_k(x, t) \equiv b_k^2(x, t) - a_k(x, t)c_k(x, t) = 4|g'_{k+1}(x + t)|^2 |f'_k(x - t)|^2 + \tfrac{1}{4}F_k^2 > 0. \tag{3.17}$$

Hence the roots of (3.16) are real and distinct,

$$v_k(x, t) = \begin{cases} v_k^{\pm}(x, t) \equiv \dfrac{b_k(x, t) \pm \sqrt{d_k(x, t)}}{a_k(x, t)} & \text{if } a_k(x, t) \neq 0, \\ v_k^0(x, t) \equiv \dfrac{c_k(x, t)}{2b_k(x, t)}, & \text{if } a_k(x, t) = 0. \end{cases} \tag{3.18}$$

Using (3.14) and (3.3) it is easy to check the following dichotomy for the roots (see Appendix B),

$$|v_k^-(x, t)| < 1 \ \text{ and } \ |v_k^+(x, t)| > 1 \ \text{ if } \ a_k(x, t) \neq 0, \tag{3.19}$$

$$|v_k^0(x, t)| < 1 \quad \text{if } \ a_k(x, t) = 0. \tag{3.20}$$

Hence,

$$\overline{v}_k(x, t) = \begin{cases} v_k^-(x, t) & \text{if } \ a_k(x, t) \neq 0, \\ v_k^0\ (x, t) & \text{if } \ a_k(x, t) = 0. \end{cases} \tag{3.21}$$

Proof of (*iii*). By (3.17) the functions $a_k(x, t), b_k(x, t), c_k(x, t)$ are C^1 near the points $(x, t) \in \bar{\Omega}$ with $a_k(x, t) \neq 0$. On the other hand, from (3.3) and (3.14) it follows that for every $k = 1, \ldots, N$

$$b_k(x, t) \geqq b > 0, \quad (x, t) \in \bar{\Omega}. \tag{3.22}$$

Therefore $\sqrt{d_k(x, t)} = b_k(x, t)[1 - a_k(x, t)c_k(x, t)/b_k^2(x, t)]^{1/2}$ admits an expansion to a convergent series in $a_k(x, t)c_k(x, t)/b_k^2(x, t)$ in the set $(x, t) \in \overline{\Omega}$ with small $|a_k(x, t)|$. Therefore,

$$\bar{v}_k(x, t) = b_k(x, t)\Big(1 - [1 - a_k(x, t)c_k(x, t)/b_k^2(x, t)]^{1/2}\Big)\Big/ a_k(x, t)$$

is also a C^1 function in this set. □

Step 5. Bounds and uniqueness for $t \in [0, T_0]$

The equation (3.15) for $t \in [0, T_0] \setminus \mathcal{T}_k$, Lemma 3.2(i) and the inequality (3.6) imply

$$\dot{x}_k(t) = \overline{v}_k(x_k(t), t), \ \ t \in [0, T_0] \setminus \mathcal{T}_k. \tag{3.23}$$

Therefore, $x_k(t) \in C[0, T_0] \cap C^2_{\rm pw}(0, T_0)$ by Lemma 3.2(iii). Furthermore, Lemma 3.2(ii) implies

$$|\dot{x}_k(t \pm 0)| < 1, \ \ t \in [0, T_0]. \tag{3.24}$$

Lemma 3.3. *The function* $x_k(t)$ *is determined uniquely for* $0 < t \leqq T_0$ *by* (3.23) *and the initial point* $x_k(0)$.

Proof. Let us note that the function $\overline{v}_k(x,t)$ generally is not Lipschitz continuous in x, hence the uniqueness of $x_k(t)$ does not follow directly from the standard theorem. Consider the finite sequence of times $\tau_k^1, \tau_k^1, \ldots \in \mathcal{T}_k$: $\tau_k^1 = \sup\{t \in [0,T_0] : (x_k(s),s) \notin \mathcal{S}_k$ for $s \in (0,t)\}$, $\tau_k^2 = \sup\{t \in [\tau_k^1, T_0] : (x_k(s),s) \notin \mathcal{S}_k$ for $s \in (\tau_k^1, t)\}$, etc. (see Fig. 3). We have $\tau_k^1 > 0$ due to (3.23), (3.24), and $x_k(t)$ is defined uniquely for $0 \leqq t \leqq \tau_k^1$ by Lemma 3.2(iii). If $\tau_k^1 < T_0$, then repeating this argument with the initial point $x_k(\tau_k^1)$ at the time $t = \tau_k^1$, we determine $x_k(t)$ uniquely for $\tau_k^1 \leqq t \leqq \tau_k^2$ and so on. By induction, we define $x_k(t)$ uniquely for all $t \in [0,T_0]$ since the set $\mathcal{T}_k$ is finite. □

Now we reconstruct $u(x,t)$ for $t \in [0,T_0]$ and all $x \in \mathbb{R}$. Let us define the reflected waves $f_{k+1}(x-t)$, $g_k(x+t)$ from algebraic identities (3.8),

$$\begin{aligned} f_{k+1}(x_k(t)-t) &= z_k - g_{k+1}(x_k(t)+t), \\ g_k(x_k(t)+t) &= z_k - f_k(x_k(t)-t), \end{aligned} \tag{3.25}$$

where $0 \leqq t \leqq T_0$. The functions $g_k(\cdot)$, $f_{k+1}(\cdot)$ are defined uniquely in the segments $[x_k(0), x_k(T_0)+T_0]$ and $[x_k(T_0)-T_0, x_k(0)]$ respectively because the maps $t \mapsto x_k(t) \pm t$ are invertible from $[0,T_0]$ to these segments by (3.24). Therefore, $u(x,t)$ is defined uniquely by (2.19) for all $x \in \mathbb{R}$ and $t \in [0,T_0]$.

Step 6. Bounds and uniqueness for all $t \geqq 0$

We can repeat all the constructions described above with $t = T_0$ instead of $t = 0$ and deduce the uniqueness of $u(x,t)$ and bounds (2.16) for $T_0 < t < T_0 + T_1$, where the step $T_1 = \frac{1}{2}\min_{2\leqq k \leqq N} |x_k(T_0) - x_{k-1}(T_0)|$. The induction implies the uniqueness of $u(x,t)$ and bounds (2.16) for all $t > 0$ because of the *a priori* bound for the step

$$T(t) := \frac{1}{2} \min_{2\leqq k\leqq N} |x_k(t) - x_{k-1}(t)| \geqq \frac{C}{1+t}, \qquad t > 0. \tag{3.26}$$

The bound holds by the following lemma.

Lemma 3.4. *Let* (2.4) *hold. Then*

$$\tfrac{1}{2}\int [|\dot{u}|^2 + |u'|^2]dx \leqq C(u_0,v_0) + 2V_0|t| \ \ \textit{for} \ t \in \mathbb{R}. \tag{3.27}$$

Proof. Equation (2.12) implies

$$\tfrac{1}{2}\int [|\dot{u}|^2 + |u'|^2]dx = H(u_0,v_0) - \int V(u)dx. \tag{3.28}$$

On the other hand, (2.9) and (2.1), (2.2) imply

$$V(u(x,t)) = 0, \qquad \pm x \geqq a + |t| \tag{3.29}$$

with an $a \in \mathbb{R}$. Therefore, (2.4) implies

$$\int V(u(x,t))dx = \int_{|x|\leqq a+|t|} V(u(x,t))dx \geqq -2V_0(a+|t|). \qquad \square$$

This Lemma implies, by Cauchy-Schwarz inequality, that

$$|z_{k+1} - z_k|^2 = |u(x_{k+1}(t), t) - u(x_k(t), t)|^2 \tag{3.30}$$
$$\leqq \int_{x_k(t)}^{x_{k+1}(t)} |u'(x,t)|^2 \, dx \; |x_{k+1}(t) - x_k(t)|$$
$$\leqq C(1 + |t|)|x_{k+1}(t) - x_k(t)|, \quad t \in R. \tag{3.31}$$

Then (3.26) follows. □

4. Existence of global solution

The above proof of the uniqueness gives the explicit algorithm for constructing the solution.

Step 1. Existence in characteristic region
We construct $u(x,t)$ in the characteristic regions Δ_k as described in Step 1 of the previous Section with f_k, g_k satisfying (3.2), (3.3) for $k = 1, \ldots, N+1$.

Step 2. Existence for $0 \leqq t \leqq T_0$
For $k = 1, \ldots, N$ we construct $x_k(t)$ and $u(x,t)$ for $0 < t < T_0$ and all $x \in \mathbb{R}$ as described in Step 4 of the previous Section. We define $x_k(t)$ as a solution to (3.23) for $0 < t < T_0$ with initial value $x_k(0)$ defined by $u_0(x)$. The solution $x_k \in C(0, T_0) \cap C^2_{\text{pw}}(0, T_0)$ exists due to Lemma 3.2(iii). The inequality (3.24) holds as above. Therefore, the formula (3.25) for the reflected waves implies $g_k(\cdot) \in C([x_k(0), x_k(T_0) + T_0]) \cap C^2_{\text{pw}}([x_k(0), x_k(T_0) + T_0])$, and $f_{k+1}(\cdot) \in C([x_k(T_0) - T_0, x_k(0)]) \cap C^2_{\text{pw}}([x_k(T_0) - T_0, x_k(0)])$. Hence, the d'Alembert representation (2.19) defines $u(x,t) \in C(\mathbb{R} \times [0, T_0])$: the continuity along the characteristics $x \pm t = x_k(0)$ follows from (3.8) and the continuity of the initial function $u_0(x)$. Further, $u(x,t) \in C^2(\overline{\Omega})$ for any connected (open) component of $\mathbb{R} \times [0, T_0] \setminus (\Gamma_0 \cup \overline{\mathcal{S}_0})$ where $\Gamma_0 \equiv \{(x,t) \in \mathbb{R} \times [0, T_0] : u(x,t) \in Z\}$ and $\mathcal{S}_0 \subset \mathbb{R} \times [0, T_0]$ is a locally finite characteristic set. Let us check that $(u(\cdot, t), \dot{u}(\cdot, t)) \in M$ for $t \in [0, T_0]$. It suffices to verify that for all $k = 2, \ldots, N$

$$f_k'(x - t \pm 0) > 0, \;\; g_k'(x + t \pm 0) > 0, \qquad x \pm 0 \in \overline{X_k(t)}. \tag{4.1}$$

We prove (4.1) separately for x from every interval of the decomposition (see Fig. 4)

$$\overline{X_k(t)} = [x_{k-1}(t), x_{k-1}(0)+t] \cup [x_{k-1}(0)+t, x_k(0)-t] \cup [x_k(0)-t, x_k(t)]. \tag{4.2}$$

Inequalities (4.1) for $x \in [x_{k-1}(0) + t, x_k(0) - t]$ follow from (3.3). The proofs of (4.1) are identical for the two remaining segments in (4.2). Let us consider for example, $x \in [x_k(0) - t, x_k(t)]$. Then (4.1) for f_k' follows from (3.3). Therefore, (3.11) for g_k' together with (3.24) imply

$$g_k'(x_k(t) + t \pm 0) > 0, \qquad t \in [0, T_0]. \tag{4.3}$$

Then (4.1) follows for g_k' at $x \pm 0 \in [x_k(0) - t, x_k(t)]$.

Step 3. Existence for all $t \geqq 0$

We continue by induction and take into account *a priori* estimate (3.26) for the step. Then (2.16) follows from (3.24), and the limits (2.9) follow by (2.5) and (2.16). Hence, the global solution $u(x,t) \in \mathcal{M}$ exists. Theorem 2.6 is proved. □

5. Weak solutions

We generalize the definition of the solution to the case $u_\pm \in Z$, see (2.2). To do this we need to make a distinction between possible limit values $z_k \pm 0$ of the solution $u(x,t)$ to define $V(u)$ correctly in (2.6), (2.8) and (2.10). Let us assume for simplicity of exposition that the set Z is finite, $\#(Z) = N$, and let us enumerate all points $z \in Z$ in ascending order of $z_1, \ldots, z_N$,

$$-\infty = z_0 < z_1 < \cdots < z_N < z_{N+1} = \infty. \tag{5.1}$$

Let us denote $\Sigma_k = [z_{k-1}, z_k]$ for $k = 1, \ldots, N+1$, and $\hat{\mathbb{R}}$ – a disjoint union of the segments Σ_k. Define $I_k : \Sigma_k \to \hat{\mathbb{R}}$ as the corresponding injection, $p : \hat{\mathbb{R}} \to \mathbb{R}$ as the projection $p|_{\Sigma_k} = I_k^{-1}$. Set $z_k^- = I_{k-1}(z_k)$ and $z_k^+ = I_k(z_k)$.

Remark. The points $z_k^\pm$ represent the "limit values" $z_k \pm 0$.

Let us generalize Definitions 2.1–2.5 for the functions with the values in $\hat{\mathbb{R}}$. Denote by $B(\mathbb{R}, \hat{\mathbb{R}})$ the set of the Borel measurable maps $\mathbb{R} \to \hat{\mathbb{R}}$.

Definition 5.1. The function $(\hat{u}(x), v(x)) \in \hat{E}$ if $\hat{u}(\cdot) \in B(\mathbb{R}, \hat{\mathbb{R}})$ and $(p\hat{u}(x), v(x)) \in E$.

Let us define the potential $\hat{V}$ in $\hat{\mathbb{R}}$ as

$$\hat{V}(\hat{u}) = \begin{cases} V(p\hat{u}) & \text{if } p\hat{u} \notin Z, \\ V(z_k \pm 0) & \text{if } \hat{u} = z_k^\pm, \ k = 1, \ldots N. \end{cases} \tag{5.2}$$

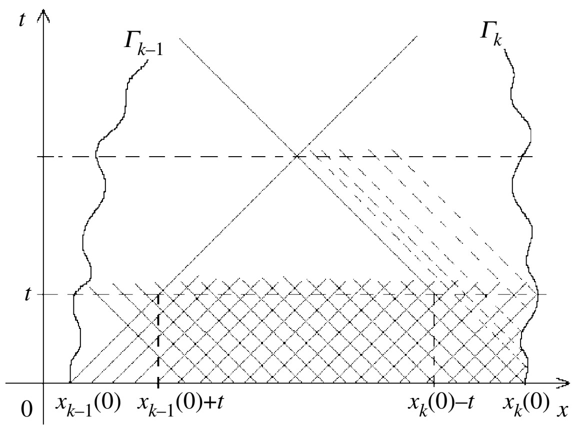

Fig. 4.

Define the Hamiltonian functional and the total momentum for $(\hat{u}(x), v(x)) \in \hat{E}$ as

$$H(\hat{u}, v) = \int \left(\tfrac{1}{2}|v(x)|^2 + \tfrac{1}{2}|u'(x)|^2 + \hat{V}(\hat{u}(x)) \right) dx, \tag{5.3}$$

$$P(\hat{u}, v) = \int u'(x)\, v(x)\, dx, \tag{5.4}$$

where $u(x) = p\hat{u}(x)$ for $x \in \mathbb{R}$. Let us generalize Definitions 2.2, 2.3 for the function $\hat{u}(x,t)$ with the values in $\hat{\mathbb{R}}$. We need an adjustment for the definition of the sets Γ_k and Γ. Let us set

$$\begin{aligned} \hat{\Gamma}_k(\hat{u}) &= \overline{\hat{u}^{-1}(z_k^-)} \cap \overline{\hat{u}^{-1}(z_k^+)}, \\ \hat{\Gamma}(\hat{u}) &= \cup_k \hat{\Gamma}_k(\hat{u}). \end{aligned} \tag{5.5}$$

Definition 5.2. The trajectory $\hat{u}(x,t) \in \hat{\mathcal{E}}$ if $u(x,t) \equiv p\hat{u}(x,t) \in \mathcal{E}$ satisfies all conditions of Definition 2.2 with $\hat{\Gamma}(\hat{u})$ instead of $\Gamma(u)$.

Definition 5.3. The trajectory $\hat{u}(x,t) \in \hat{\mathcal{E}}$ is a weak solution to (1.1) if (2.11) and (SC) hold for $u(x,t) = p\hat{u}(x,t)$ with the set $\hat{\Gamma}(\hat{u})$ instead of $\Gamma(u)$ and with $\hat{V}(\hat{u})$ instead of $V(u)$.

Now we can omit the assumption (2.2) in Theorem 2.6. Let us fix an arbitrary $\hat{u}_\pm \in \hat{\mathbb{R}}$ with

$$\hat{V}(\hat{u}_\pm) = 0. \tag{5.6}$$

A potential $\hat{V}(\hat{u})$ satisfying (5.6) exists if and only if an analog of (2.3) holds:

$$\sum_{k:\ \{\hat{u}_- \leqq z_k^-\} \& \{\hat{z}_k^+ \leqq u_+\}} F_k = 0. \tag{5.7}$$

Lemma 5.4. *Let* $u(x,t) \in \hat{\mathcal{E}}$ *be a solution to* (1.1). *Then*

(i) *if* (5.6) *holds, the energy is conserved:*

$$H(\hat{u}(\cdot,t), \dot{u}(\cdot,t)) = \text{const}, \quad t \in \mathbb{R}, \tag{5.8}$$

(ii) *if* (5.7) *holds, the momentum is conserved:*

$$P(\hat{u}(\cdot,t), \dot{u}(\cdot,t)) = \text{const}, \quad t \in \mathbb{R}. \tag{5.9}$$

The proof coincides with that of Lemma 2.4 given in Appendix A.

For concreteness, we consider below the particular case when the interval $[p\hat{u}_-, p\hat{u}_+]$ contains only two points of the set Z (see (2.21)). Then (5.7) implies $F_1 = -F_2$, and (2.22) holds with $z_1 - 0 \equiv z_1^-$ and $z_2 + 0 \equiv z_2^+$. Define $u(x) = p\hat{u}(x)$ and $u(x,t) = p\hat{u}(x,t)$ as above.

Definition 5.5. We define

(i) $\hat{M}$ as the set of states $(\hat{u}, v) \in \hat{E}$ such that (2.23) holds with an $R > 0$, $u(x)$ is a monotone function, and (cf. (2.14))

$$|u'(x \pm 0)| \geqq |v(x \pm 0)|, \quad x \in \mathbb{R}, \tag{5.10}$$
$$|u'(x \pm 0)| > |v(x \pm 0)|, \quad \hat{u}(x \pm 0) \in [z_1^+, z_2^-]; \tag{5.11}$$

(ii) and $\hat{\mathcal{M}}$ as the set of the functions $\hat{u}(x,t) \in \hat{\mathcal{E}}$ such that $(\hat{u}(\cdot,t), \dot{u}(\cdot,t)) \in M$, $t \in \mathbb{R}$.

For a function $\hat{u}(x,t) \in \hat{\mathcal{M}}$, we have $\hat{\Gamma}(\hat{u}) = \hat{\Gamma}_1(\hat{u}) \cup \hat{\Gamma}_2(\hat{u})$ where $\hat{\Gamma}_k(\hat{u})$ is the trajectory $x = x_k(t)$, $k = 1, 2$.

Theorem 5.6. *Let* (2.21), (2.22) *hold with* $F_2 > 0$, *and* $(\hat{u}_0, v_0) \in \hat{M}$. *Then there exists a unique weak solution* $\hat{u}(x,t) \in \hat{\mathcal{M}}$ *to the Cauchy problem* (1.1), (1.4) *such that*

$$|\dot{x}_k(t \pm 0)| < 1, \quad t \in \mathbb{R}, \quad k = 1, 2. \tag{5.12}$$

Proof. The proof follows the same strategy as the proof of Theorem 2.6. However, the inequality (5.10) is not strong in contrast to (2.14). In this regard, the proof of Theorem 2.6 requires a little modification. For instance, Lemma 2.7 generally does not hold for the weak solutions. This is why we include the bound (5.12) in the statement to provide the uniqueness of the solution. Further, Lemma 3.2 now becomes

Lemma 5.7. *Let* $(u_0, v_0) \in \hat{M}$. *Then for* $k = 1, 2$ *and* $(x,t) \in \nabla_k \setminus \mathcal{S}_k$

(i) *there exists a unique solution* $\bar{v}_k(x,t) \in (-1, 1)$ *to* (3.16)*;*
(ii) $\overline{v}_k(x,t) \in C^1(\bar{\Omega})$ *for any connected* (*open*) *component* Ω *of the set* $\nabla_k \setminus \mathcal{S}_k$.

The proof of this lemma almost coincides with the proof of Lemma 3.2 but the signs of $F_2 > 0$ and $F_1 = -F_2 < 0$ are now important.

Remark. (i) The strong inequality in (5.10) does not hold by (2.23).
(ii) For $F_2 < 0$ the inequalities "$<$" in (3.19) and (3.20) generally become "$\leqq$", and the solution $x_k(t)$ to (3.15) satisfying (5.12) generally does not exist. Likewise, the solution $\hat{u} \in \hat{\mathcal{M}}$ generally does not exist. However, a solution $\hat{u} \in \hat{\mathcal{E}}$ with space-like trajectories Γ_k (i.e., with $|\dot{x}_k(t)| \geqq 1$) may exist.

6. Breathers

We prove Theorem 2.8 by a modification of the "d'Alembert" arguments of Section 3 with another choice of the regions and taking into account the specific features of the potential (2.22) with $F_2 > 0$.

Step 1. According to (2.24) and to the definitions of $\hat{\mathcal{M}}$ and of $x_k(t)$, we have

$$\hat{u}(x,t) = \begin{cases} z_1^-, & x \leqq x_1(t), \\ z_2^+, & x \geqq x_2(t). \end{cases} \tag{6.1}$$

For $k = 1, 2, 3$ define the waves $f_k(x)$ and $g_k(x)$ by the formulas (3.1) in the intervals $\bar{X}_k(0)$. Then $f_1(x) = g_1(x) = z_1^-/2$, $x \leqq x_1(0)$, and $f_3(x) = g_3(x) = z_2^+/2$, $x \geqq x_2(0)$ since $v_0(x) = 0$ for $x \leqq x_1(0)$ and $x \geqq x_2(0)$ by (5.10).

Step 2. Next we start to reconstruct the lines $\Gamma_k, k = 1, 2$, according to the differential equation of type (3.23) in the modified regions Q_k instead of ∇_k. First, consider $k = 1$. We can define the coefficients a_1, b_1, c_1 by the expressions (3.14) in the strip $Q_1 := \{(x,t) \in \mathbb{R}^2 : x - t \leqq x_1(0), x_1(0) \leqq x + t \leqq x_2(0)\}$ since $f_1(x)$ is known for $x \leqq x_1(0)$ and $g_2(x)$ is known for $x_1(0) \leqq x \leqq x_2(0)$. The inequality (5.11) implies that $|g_2'(x+t)|^2 \geqq \varepsilon, x_1(0) \leqq x+t \leqq x_2(0)$, with an $\varepsilon > 0$. Therefore, we can apply Lemma 10.1 to $p := |f_1'(x-t)|^2$, $q := |g_2'(x+t)|^2$ and $r := F_1 < 0$. Then we find that for $(x,t) \in Q_1$ there exists a unique solution $\overline{v}_1(x,t) \in (-1,1)$ to the algebraic equation (3.16), and the uniform bound holds:

$$|\overline{v}_1(x,t)| \leq \gamma < 1, \quad (x,t) \in Q_1\,. \tag{6.2}$$

Hence, the differential equation of type (3.23) holds for $x_1(t)$ until $(x_1(t), t) \in Q_1$. The equation follows by the arguments of Section 3. As in Lemma 3.2(ii), $\overline{v}_1(x,t) \in C^1(\bar{\Omega})$ for any connected (open) component Ω of the set $Q_1 \setminus \mathcal{S}_1$, where $\mathcal{S}_1$ is a locally finite characteristic set. Therefore, the differential equation (3.23) defines the line $x_1(t)$ uniquely for $t \in [0, t_1^1]$, where $t_1^1 := \sup\{t > 0 : (x_1(s), s) \in Q_1 \text{ for } s \in [0,t]\}$

Step 3. Further, the line Γ_1 intersects the boundary of the strip Q_1 at the point $(x_1(0), 0)$ and cannot intersect the boundary in the points (x,t) with $x \pm t = x_1(0)$, $t > 0$, by the bound (5.12). On the other hand, the uniform bound (6.2) implies $t_1^1 < \infty$. Hence, the line Γ_1 intersects the boundary line $x + t = x_2(0)$ of the strip Q_1 at the unique point $(x(t_1^1), t_1^1)$ with the $t_1^1 > 0$:

$$x_1(t_1^1) + t_1^1 = x_2(0). \tag{6.3}$$

Similarly, the line Γ_2 intersects the boundary line $x - t = x_1(0)$ of the strip $Q_2 := \{(x,t) \in \mathbb{R}^2 : x + t \geqq x_2(0), x_1(0) \leqq x - t \leqq x_2(0)\}$ at the unique point $(x(t_2^1), t_2^1)$ with the $t_2^1 > 0$:

$$x_2(t_2^1) - t_2^1 = x_1(0). \tag{6.4}$$

Denote by Γ_k^1 the segment $\{(x,t) \in \mathbb{R}^2 : x = x_k(t), 0 \leqq t \leqq t_k^1\}$ of the line Γ_k, $k = 1, 2$.

Step 4. Further, we define the reflected wave $f_2(x-t)$ at the points of Γ_1^1 (i.e., for $x_1(0) < x - t < x_1(t_1^1) - t_1^1$) from the first equation of (3.25) with $k = 1$:

$$f_2(x_1(t) - t) = z_1 - g_2(x_1(t) + t), \quad t \in [0, t_1^1]. \tag{6.5}$$

Similarly, we define the reflected wave $g_2(x+t)$ at the points of Γ_2^1 (i.e., for $x_2(0) < x+t < x_2(t_2^1)+t_2^1$) from the second equation of (3.25) with $k=2$:

$$g_2(x_1(t)+t) = z_2 - f_2(x_2(t)-t), \quad t \in [0, t_2^1]. \tag{6.6}$$

Step 5. Next we construct the segment $\Gamma_k^2 : x = x_k(t), t_k^1 \leqq t \leqq t_k^2$ of the line Γ_k according to (3.23) with $k = 1, 2$. Here t_k^2 are defined (uniquely) by

$$x_1(t_1^2) + t_1^2 = x_2(t_2^1) + t_2^1, \;\; x_2(t_2^2) - t_2^2 = x_1(t_1^1) - t_1^1. \tag{6.7}$$

The points $A_2^1 = (x_2(t_2^1), t_2^1)$, $A_1^2 = (x_1(t_1^2), t_1^2)$ belong to the common characteristic $x - t$ =const; the points $A_1^1 = (x_1(t_1^1), t_1^1)$, $A_2^2 = (x_2(t_2^2), t_2^2)$ belong to the common characteristic $x+t$ = const (see Fig. 5). The existence (and uniqueness) of the moments t_k^2 follow as above, from the uniform bound of type (6.2). Further, we define the reflected waves $f_2(x-t)$ and $g_2(x+t)$ at the points of Γ_1^2 and Γ_2^2 respectively. By induction we get the segments $\Gamma_k^n, n \in \mathbb{Z}$, of the lines $\Gamma_k : x = x_k(t)$, $t \in \mathbb{R}$, and the waves $f_2(x-t)$, $g_2(x+t)$ with $(x,t) \in \mathbb{R}^2$. Then the d'Alembert representation (2.19) with $k = 2$ gives the solution in the strip Π_2 between the lines Γ_1 and Γ_2. The solution is defined by (6.1) outside the strip. Now we can deduce Theorem 2.8 from the following lemma which we prove in the next section.

Lemma 6.1.
(i) *The segment* $\Gamma_2^n(\Gamma_1^n)$ *is the translation of* $\Gamma_1^{n-1}(\Gamma_2^{n-1})$*:*

$$\Gamma_2^n = \Gamma_1^{n-1} + \mathcal{T}_1, \quad \Gamma_1^n = \Gamma_2^{n-1} + \mathcal{T}_2, \qquad n \in \mathbb{Z}, \tag{6.8}$$

where $\mathcal{T}_1$ *is the vector* $A_1^0 A_2^1$ *and where* $\mathcal{T}_2$ *is the vector* $A_2^0 A_1^1$ *(see Fig. 5).*
(ii) *The solution* $\hat{u}(x,t)$ *is a periodic function with the period* $\mathcal{T} = (S, T) := \mathcal{T}_1 + \mathcal{T}_2$*, i.e.,*

$$\hat{u}(x+S, t+T) = \hat{u}(x,t), \quad (x,t) \in \mathbb{R}^2. \tag{6.9}$$

(iii) *The period* $\mathcal{T}$ *is a time-like vector:*

$$|S| < T. \tag{6.10}$$

From this Lemma it follows that $\hat{u}(x,t)$ is time-periodic with the period T, if $S = 0$. Therefore, Theorem 2.8 is proved in this case. If $S \neq 0$, we apply the Lorentz transformation Λ_v with some $v \in \mathbb{R}$, $|v| < 1$, to the function $\hat{u}(x,t)$

$$\hat{b}(x,t) = \hat{u}(\Lambda_v(x,t)), \quad \Lambda_v(x,t)) = (\lambda(x+vt), \lambda(t+vx)), \quad (x,t) \in \mathbb{R}^2, \tag{6.11}$$

where $\lambda = 1/\sqrt{1-v^2}$. The function $\hat{b}(x,t)$ is the weak solution to (1.1). This follows from the invariance of the d'Alembert equation (2.11) and of the expression (2.10) with respect to the Lorentz transformations. From (6.9) it follows that the function $b(x,t) := p\hat{b}(x,t)$ is periodic with the period

$$\Lambda_v^{-1}\mathcal{T} = (\lambda(S - vT), \lambda(T - vS)). \tag{6.12}$$

Choosing $v = S/T$ we see that the function $\hat{b}$ is time-periodic. This choice is admissible since $|v| < 1$ by (6.10). At last, (6.11) implies (2.25). □

7. Proof of periodicity

We prove Lemma 6.1.

Step 1. Note that $\mathcal{T}_1 := A_1^0 A_2^1 = (x_2(t_2^1) - x_1(0), t_2^1) = (t_2^1, t_2^1)$ and $\mathcal{T}_2 := A_2^0 A_1^1 = (x_1(t_1^1) - x_2(0), t_1^1) = (-t_1^1, t_1^1)$, hence the sum $\mathcal{T} := \mathcal{T}_1 + \mathcal{T}_2$ admits the representation

$$\mathcal{T} = (S, T) = (t_2^1 - t_1^1, t_2^1 + t_1^1), \tag{7.1}$$

which implies (6.10).

Step 2. It suffices to check (6.8) for $n = 2$ and the periodicity (6.9) for (x, t) in a neighborhood of the segment $\{(x, 0) : x_1(0) < x < x_2(0)\}$. First let us construct the segments Γ_k^1 of the lines Γ_k, $k = 1, 2$ as in the previous section, and the reflected (from Γ_k^1) waves by formulas (6.5), (6.6). Next we *define* the segments Γ_k^2 by the translations (6.8) with $n = 2$ (see Fig. 5), and the reflected (from Γ_k^2) waves by the "periodicity" up to an additive constant:

$$\begin{aligned} f_2(x - t) &= f_2((x - S) - (t - T)) + z_1 - z_2, \\ &\qquad x_1(0) < (x - S) - (t - T) < x_2(0), \end{aligned} \tag{7.2}$$

$$\begin{aligned} g_2(x + t) &= g_2((x - S) + (t - T)) + z_2 - z_1, \\ &\qquad x_1(0) < (x - S) + (t - T) < x_2(0). \end{aligned} \tag{7.3}$$

Next we will check the identity $f_2(x-t) + g_2(x+t) = z_k$ and the splicing condition (SC) at the points of the segments Γ_k^2, $k = 1, 2$. Then the arguments of Section 3 show that the line $\Gamma_k^1 \cup \Gamma_k^2$ is a continuous solution to the differential equation (3.23). Hence (6.8) for $n = 2$ follows by the uniqueness of the continuous solution to (3.23) with the fixed $x_k(t_k^1)$. Finally, for (x, t) in a neighborhood of the segment $\{(x, 0) : x_1(0) < x < x_2(0)\}$, the periodicity (6.9) obviously follows from (7.2), (7.3) by the d'Alembert representation (2.19).

Step 3. Consider $k = 2$ for example. By definition, each point $(x, t) \in \Gamma_2^2$ is the translation of the point $(x, t) - \mathcal{T}_1 \in \Gamma_1^1$ by the vector $\mathcal{T}_1 = (t_2^1, t_2^1)$: in other words, $x - t_2^1 = x_1(t - t_2^1)$. Hence,

$$\left.\begin{aligned} f_2(x - t) &= f_2(x_1(t - t_2^1) - (t - t_2^1)) \\ g_2(x + t) &= g_2(x_1(t - t_2^1) + (t - t_2^1)) + z_2 - z_1 \end{aligned}\right| \quad (x, t) \in \Gamma_2^2. \tag{7.4}$$

The last identity follows by (7.3) since $(x - S) + (t - T) = (x - t_2^1) + (t - t_2^1)$ by (7.1). Now the d'Alembert representation (2.19) implies

$$u(x - 0, t) = u((x + 0, t) - \mathcal{T}_1) + z_2 - z_1, \quad (x, t) \in \Gamma_2^2. \tag{7.5}$$

Therefore, the identity $u(x - 0, t) = z_2$ at the points of Γ_2^2 follows by the identity $u((x + 0, t) - \mathcal{T}_1) = z_1$ at the points of Γ_1^1 (see (6.5)). Furthermore, (7.4) implies the identities

$$\dot{u}(x - 0, t) = \dot{u}((x + 0, t) - \mathcal{T}_1) \text{ and } u'(x - 0, t) = u'((x + 0, t) - \mathcal{T}_1), \quad (x, t) \in \Gamma_2^2 \tag{7.6}$$

since the derivatives of $z_2 - z_1$ are zero. Therefore, the condition (SC) at the points of Γ_2^2 follows from the same condition at the points of Γ_1^1 by (6.1) and the identity $F_1 = -F_2$. □

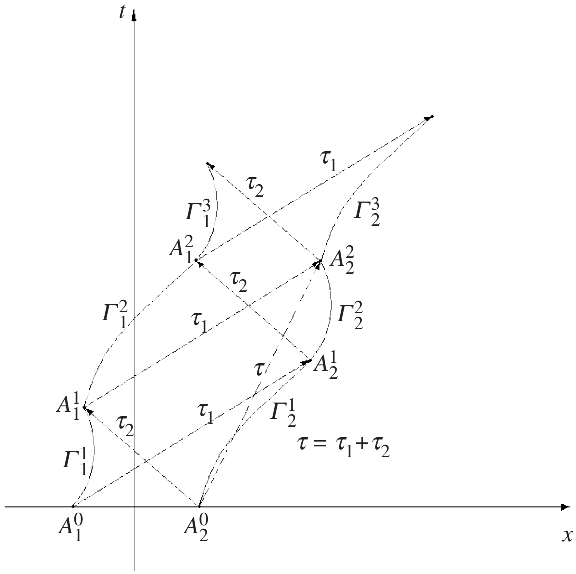

Fig. 5.

Remark. The condition $F_2 > 0$ is "necessary" because it provides (5.12), and therefore, the existence (and uniqueness) of the solution to the equations (6.3), (6.4). For $F_2 \leqq 0$ the lines Γ_k generally are not time-like and generally do not intersect the corresponding characteristics. Then (6.3), (6.4) do not admit the solutions t_1^1, t_2^1.

8. Soliton-type asymptotics

Theorem 2.9 follows by the d'Alembert method similarly to the proof of Theorem 2.8 above. The main difficulty is to prove the following proposition:

Proposition 8.1. *Let all the conditions of Theorem 2.9 hold. Then the line Γ_1 intersects each characteristic $x \pm t = C$ with $C \leqq x_1(0)$, and the line Γ_2 intersects each characteristic $x \pm t = C$ with $C \geqq x_2(0)$.*

Proof of Theorem 2.9. By the definition (2.23) of the space $\hat{M}$, we have

$$\hat{u}_0(x) = \hat{u}_\pm, \;\; v_0(x) = 0, \quad \pm x \geqq \mathrm{R}_0 \tag{8.1}$$

with some $R_0 > 0$. We can assume that $-R_0 \leqq x_1(0) < x_2(0) \leqq R_0$. By Proposition 8.1 the characteristic $x-t = -R_0$ intersects the line Γ_1 at a point $(x_1(t_1^*), t_1^*) \in \Gamma_1$ with a $t_1^* > 0$, and the characteristic $x + t = R_0$ intersects the line Γ_2 at a point $(x_2(t_2^*), t_2^*) \in \Gamma_2$ with a $t_2^* > 0$. By (5.12), the solution $\hat{u}(x, t)$ admits the d'Alembert representation (2.19) with $k = 1$ and $k = 3$ in the strips $\Pi_1 := \{(x, t) : x < x_1(t)\}$ and $\Pi_3 := \{(x, t) : x > x_3(t)\}$, respectively. From (8.1) it follows that the ingoing waves $g_3(x+t)$ and $f_1(x-t)$ are constant functions for $t \geqq t^* := \max(t_1^*, t_2^*)$. Hence, for $t > t^*$ the solution in the regions Π_1 and Π_3 is given by outgoing waves that corresponds to the asymptotics (2.27). Furthermore, the arguments of Sections 6 and 7 are based on the fact that the ingoing waves $g_3(x+t)$ and $f_1(x-t)$ are constant functions. Hence, for $t \geqq t^*$ the solution in the strip $\Pi_2 := \{(x,t) : x_1(t) < x < x_2(t)\}$ coincides with a moving breather of type (2.25), and (2.26) holds. □

Proof of Proposition 8.1.
Step 1. Let us analyze the *a priori* bounds that follow by energy conservation (5.8). First, we have

$$r(t) := |x_2(t) - x_1(t)| = \frac{1}{F_2}\int_{x_1(t)}^{x_2(t)} V(\hat{u}(x,t))dx \leqq r^* < \infty, \quad t \in \mathbb{R}, \tag{8.2}$$

$$|z_2 - z_1| = \left|\int_{x_1(t)}^{x_2(t)} u'(x,t)dx\right| \leqq C\sqrt{r(t)}, \quad t \in \mathbb{R} \tag{8.3}$$

since the potential (2.22) is nonnegative (cf (3.30)). Furthermore, (5.8) implies for $B > 0$ by the Chebyshev inequality,

$$\Lambda(\{x \in [x_1(t), x_2(t)] : |f_2'(x-t)| + |g_2'(x+t)| \geqq B\}) \leqq \frac{a}{B^2}, \quad t \in \mathbb{R} \tag{8.4}$$

where Λ is the Lebesgue measure in $\mathbb{R}$.

Corollary 8.2.
(i) *By* (8.3)*,*

$$r(t) \geqq r_* > 0, \quad t \in \mathbb{R}. \tag{8.5}$$

(ii) *Hence* (8.4) *implies*

$$\Lambda(\{x \in [x_1(t), x_2(t)] : |f_2'(x-t)| + |g_2'(x+t)| \leqq B\}) \geqq r_* - \frac{a}{B^2}, \quad t \in \mathbb{R}. \tag{8.6}$$

Step 2. Let us prove Proposition 8.1 by contradiction. Assume for example that there exists a characteristic $x - t = m_*$ with $m_* < x_1(0)$ that does not intersect Γ_1. Consider the function $s(t) := x_1(t) - t$, $t \in \mathbb{R}$.

Lemma 8.3. *For every* $\delta > 0$ *there exists a* $T_\delta > 0$ *such that* $T_\delta \to \infty$*,* $\delta \to 0$*, and*

$$\Lambda(\{t > T_\delta : |\dot{x}_1(t) - 1| \geqq \delta\}) \leqq \delta, \quad t \in \mathbb{R}. \tag{8.7}$$

Proof. By our assumption, the function $s(t)$ is bounded from below by m_*. On the other hand, $s(t)$ is monotone decreasing since $\dot{s}(t) = \dot{x}_1(t) - 1 < 0$ by (5.12). Therefore,

$$x_1(t) - t \to m, \quad t \to \infty, \tag{8.8}$$

with an $m \geqq m_*$, and there exists a $T_\delta > 0$ such that

$$s(T) - m = -\int_T^\infty (\dot{x}_1(t) - 1)dt \leqq \delta^2, \quad T \geqq T_\delta\,. \tag{8.9}$$

Now (8.7) follows by the Chebyshev inequality. □

Step 3. Next we combine the bounds (8.6) and (8.7) to Lemma 10.1. Namely, let us choose $B, \delta > 0$ such that $r_* - a/B^2 > 2\delta$. Then (8.6) and (8.7) imply that there exists a set of a positive measure $\mathcal{B}_\delta \subset [T_\delta, \infty)$ such that

$$|\dot{x}_1(t) - 1| \leqq \delta, \quad |g_2'(x_1(t) + t)| \leqq B, \quad t \in \mathcal{B}_\delta .$$

On the other hand, we can also assume that $|f_1'(x)| \leqq B, x < x_1(0)$. Therefore, we can apply Lemma 10.1 to $p := f_1'(x_1(t))$, $q := g_2'(x_1(t) + t)$ and $r := F_1/2 < 0$, $t \in \mathcal{B}_\delta$ since the differential equation (3.23) holds with $k = 1$. Then we find that $|g_2'(x_1(t) + t)| \leqq \varepsilon(B, \delta)$, $t \in \mathcal{B}_\delta$, where $\varepsilon(B, \delta) \to 0$, $\delta \to 0$.

Step 4. If $\delta > 0$ is sufficiently small, then T_δ is large, and $x_1(T_\delta) + T_\delta > x_2(0)$ by (8.8). Hence for $t \geqq T_\delta$ there exists a unique $\tau > 0$ such that $x_1(t)+t = x_2(\tau)+\tau$. Denote by $\mathcal{B}_\delta'$ the set of all $\tau > 0$ corresponding to all $t \in \mathcal{B}_\delta$. Let us show that for sufficiently small $\delta > 0$, the splicing condition (SC) with $k = 2$ cannot hold at the points $(x_2(\tau), \tau) \in \Gamma_2$ with $\tau \in \mathcal{B}_\delta'$, i.e., (see (3.9))

$$f_2'(x_2(\tau)-\tau)g_2'(x_2(\tau)+\tau) - f_3'(x_2(\tau)-\tau)g_3'(x_2(\tau)+\tau) \neq \tfrac{1}{2}F_2, \quad \tau \in \mathcal{B}_\delta' . \tag{8.10}$$

This would contradict (SC) with $k = 2$ since the measure of the set $\mathcal{B}_\delta'$ is positive by (5.12).

Step 5. It suffices to check that each product in the left-hand side of (8.10) tends to zero as $\delta \to 0$, uniformly in $\tau \in \mathcal{B}_\delta'$. First, $|g_2'(x_2(\tau) + \tau)| \leqq \varepsilon(B, \delta)$, $\tau \in \mathcal{B}_\delta'$. On the other hand, $|f_2'(x_2(\tau) - \tau)|$ is bounded for $\tau \in \mathcal{B}_\delta'$. Indeed, (8.5) and (8.8) imply

$$\phi_2^- := \lim_{t\to\infty} (x_2(t) - t) \geqq \lim_{t\to\infty} (x_1(t) - t) + r_* , \tag{8.11}$$

where the existence of the first limit follows similarly to (8.8). From the inequality (8.11) it follows that the intersection of Γ_1 with the region $\{(x, t) \in \mathbb{R}^2 : t \geqq 0, x - t \geqq \phi_2^-\}$ is a bounded set. Hence, the ingoing wave $f_2(x_2(\tau) - \tau)$ for $\tau \in \mathcal{B}_\delta'$ can be determined by a finite number of reflections from the lines Γ_1, Γ_2: this follows from the bound (3.26). Therefore, the derivative of the ingoing wave is bounded for $\tau \in \mathcal{B}_\delta'$. Second, by (8.8) we have

$$x_2(\tau) + \tau = x_1(t) + t \geqq R_0$$

for $\tau \in \mathcal{B}_\delta'$ with small $\delta > 0$, hence the ingoing wave $g_3'(x_2(\tau)+\tau)$ in the left-hand side of (8.10) is zero by (8.1). □

9. Appendix A. Energy and momentum conservation

We prove Lemma 2.4.

Energy conservation. The set Γ_k is not empty only for a finite number of $k \in \mathbb{Z}$. Let us enumerate $x_k(0)$ in ascending order of $k = 1, \ldots, N$. Then this order is conserved for all $t \in \mathbb{R}$ by continuity,

$$-\infty \equiv x_0(t) < x_1(t) < \cdots < x_N(t) < x_{N+1}(t) \equiv +\infty \quad \text{for } t \in \mathbb{R}. \tag{9.1}$$

For any $k = 1, \ldots, N$

$$u(x_k(t), t) \equiv z_k, \quad t \in \mathbb{R}. \tag{9.2}$$

Therefore we get after differentiation,

$$u'(x_k(t), t)\dot{x}_k(t) + \dot{u}(x_k(t), t) = 0, \quad \text{a.e. } t \in \mathbb{R}. \tag{9.3}$$

By definition (2.6) and (2.1), (2.2), (2.5),

$$H(t) = \sum_{k=1}^{N-1} \int_{x_k(t)}^{x_{k+1}(t)} [\tfrac{1}{2}|\dot{u}|^2 + \tfrac{1}{2}|u'|^2 + V(u)]dx. \tag{9.4}$$

Let us assume that $\mathcal{S}(u) = \emptyset$ for a moment (see Definition 2.2 (ii)). For a function $p(x, t) \in C(\mathbb{R}^2 \setminus (\Gamma(u) \cup \mathcal{S}(u)))$ let us define

$$\Delta_k p(t) = p(x_k(t) + 0, t) - p(x_k(t) - 0, t), \quad t \in \mathbb{R} \tag{9.5}$$

when the limits exist. Then (9.4) implies,

$$\begin{aligned} \dot{H}(t) = &-\sum_{k=1}^{N} \Delta_k[\tfrac{1}{2}|\dot{u}|^2 + \tfrac{1}{2}|u'|^2 + V(u)](t)\dot{x}_k(t) \\ &+ \sum_{k=1}^{N-1} \int_{x_k(t)}^{x_{k+1}(t)} [\dot{u}\ddot{u} + u'\dot{u}']dx, \quad \text{a.e. } t \in \mathbb{R}. \end{aligned} \tag{9.6}$$

Here $\ddot{u} = u''$ due to (2.11). Therefore rewriting $\dot{u}\ddot{u} + u'\dot{u}' = \dot{u}u'' + u'\dot{u}' = (u'\dot{u})'$ and integrating by parts in every integral in (9.6), we get due to (9.3),

$$\begin{aligned} \sum_{k=1}^{N-1} \int_{x_k(t)}^{x_{k+1}(t)} [\dot{u}\ddot{u} + u'\dot{u}']dx &= -\sum_{k=1}^{N} \Delta_k(\dot{u}u')(t) \\ &= \sum_{k=1}^{N} \Delta_k(|u'|^2\dot{x}_k)(t), \quad \text{a.e. } t \in \mathbb{R}. \end{aligned} \tag{9.7}$$

Hence, (9.6) implies due to (SC)

$$\dot{H}(t) = -\frac{1}{2}\sum_{k=1}^{N} \Delta_k[|\dot{u}|^2 - |u'|^2 + V(u)](t)\dot{x}_k(t) = 0, \quad \text{a.e. } t \in \mathbb{R}. \tag{9.8}$$

Now let us omit the assumption $\mathcal{S}(u) = \emptyset$. For example, let us assume that $\mathcal{S}(u)$ contains only one segment of the characteristic $x - t = c$. Then the right-hand side of (9.8) must be completed with an additional term

$$[\tfrac{1}{2}|\dot{u}|^2 + \tfrac{1}{2}|u'|^2 + u'\dot{u}]|_{x=t+c+0}^{x=t+c-0} = \tfrac{1}{2}|\dot{u} + u'|^2|_{x=t+c+0}^{x=t+c-0}. \tag{9.9}$$

This term is zero a.e. in a neighborhood of t if $(t, t + c) \notin \Gamma$ since then the d'Alembert representation $u(x, t) = f(x - t) + g(x + t)$ implies $\dot{u}(x, t) + u'(x, t) = 2g'(x + t)$. Therefore, (3.2) implies again $\dot{H}(t) = 0$ for a.e. $t \in \mathbb{R}$. □

Momentum conservation. Assuming at first $\mathcal{S}(u) = \emptyset$ as above we get

$$\dot{P}(t) = -\sum_{k=1}^{N} \Delta_k(u'\dot{u})(t)\dot{x}_{j,k}(t) + \sum_{k=0}^{N} \int_{x_k(t)}^{x_{k+1}(t)} [\dot{u}'\dot{u} + u'\ddot{u}]dx, \text{ a.e. } t \in \mathbb{R}. \quad (9.10)$$

Let us substitute again $\ddot{u} = u''$ from (2.11), and rewrite $\dot{u}'\dot{u} + u'\ddot{u} = \dot{u}'\dot{u} + u'u'' = \frac{1}{2}[|\dot{u}|^2 + |u'|^2]'$. Then (9.10) implies due to (9.3)

$$\begin{aligned} \dot{P}(t) &= \sum_{k=1}^{N} \Delta_k(|\dot{u}|^2)(t) - \frac{1}{2}\sum_{k=1}^{N} \Delta_k[|\dot{u}|^2 + |u'|^2](t) \\ &= \frac{1}{2}\sum_{k=1}^{N} \Delta_k[|\dot{u}|^2 - |u'|^2](t), \text{ a.e. } t \in \mathbb{R}. \end{aligned} \quad (9.11)$$

Therefore, (SC) and (2.2), (2.3), (2.5) imply

$$\dot{P}(t) = -\sum_{k=1}^{N} \Delta_k(V(u))(t) = -V(z_+) + V(z_-) = 0, \text{ a.e. } t \in \mathbb{R}. \quad (9.12)$$

Now let us assume for example that $\mathcal{S}(u)$ contains only one segment of the characteristic $x - t = c$. Then the right-hand side of (9.10) must be completed with the same zero term (9.9). □

10. Appendix B. Dichotomy of the roots

We prove (3.19) and (3.20). The inequality (2.14) provides

$$p := |f_k'(x-t)|^2 > 0, \ q := |g_{k+1}'(x+t)|^2 > 0, \ r := \tfrac{1}{2}F_k \neq 0. \quad (10.1)$$

Then (3.14) becomes

$$a_k = p - q + r, \quad b_k = p + q, \quad c_k = p - q - r. \quad (10.2)$$

Proof of (3.20). Since $a_k = 0$, we have $c_k = 2(p-q)$. Therefore $v_k^0 = c_k/2b_k = (p-q)/(p+q)$, hence (3.20) holds by (10.1). □

Proof of (3.19). Equations (3.17), (3.18) and (10.2) imply

$$v_k^\pm = \frac{p + q \pm \sqrt{4pq + r^2}}{p - q + r}. \quad (10.3)$$

We may assume $a := p - q + r > 0$ by symmetry. Then $|v_k^-| < 1$ means $-a < p + q - \sqrt{4pq + r^2} < a$ or equivalently, $p + q + a > \sqrt{4pq + (a - p + q)^2} > p + q - a$. This is true since squaring leads to $ap > 0 > -aq$. Similarly, $|v_k^+| > 1$ means $p + q + \sqrt{4pq + r^2} > a$ or equivalently, $\sqrt{4pq + (a - p + q)^2} > a - p - q$. This is true since squaring leads to $0 > -ap - aq$. □

The arguments above essentially use the fact that $p, q > 0$. More detailed analysis takes into account the sign of r and provides the following extension of (3.19), (3.20) to the case $p, q \geqq 0$ that we use in Sections 6 and 8:

Lemma 10.1. *Let us fix arbitrary $B, \varepsilon > 0$ and assume that* $0 \leqq p, q \leqq B$. *Consider* (i) $p \geqq 0, q \geqq \varepsilon, r < 0$ *or* (ii) $p \geqq \varepsilon, q \geqq 0, r > 0$. *Then the following uniform bounds hold for the the roots* (10.3)*:*

$$\begin{aligned} &|v_k^-(x,t)| \leqq \gamma \ \text{ and } \ |v_k^+(x,t)| \geqq 1 \ \text{ if } \ a \neq 0, \\ &|v_k^0(x,t)| \leqq \gamma \quad \text{if} \ \ a = 0, \end{aligned} \tag{10.4}$$

where $\gamma = \gamma(B, \varepsilon, r) < 1$.

Proof. It suffices to consider the case (i). If $p \geqq \varepsilon_1 > 0$, then (10.4) follows from the above proof. Hence, by the compactness arguments it remains to consider $p = 0$. Then $a = -q + r < 0$ as $r < 0$. Finally, (10.3) for $p = 0$ becomes $v_k^{\pm} = \frac{q \pm |r|}{-q+r}$ that implies (10.4) as $r < 0$. □

11. Appendix C. Least action principle

We derive (2.11) and (SC) from the least-action principle. For a $T > 0$ and a trajectory $u(x,t) \in \mathcal{E}$ we define the action as follows:

$$S_T = \int\limits_{|t|<T} L(u(t), \dot u(t))dt, \tag{11.1}$$

where $u(t) = u(\cdot, t)$ and the Lagrangian L is defined according to (2.8). The Hamilton least-action principle means that for all $T > 0$,

$$\delta S_T = 0 \quad \text{if} \ \ \delta u|_{t=\pm T} = 0 \tag{11.2}$$

for a class of admissible variations δu. Let us define the class of admissible variations, that is a "tangent bundle" to $\mathcal{E}$. Define $I = (-1, 1)$.

Definition 11.1. The space $\mathcal{TE}$ is the set of the functions $u_\varepsilon(x,t) = U(\varepsilon, x, t)$ on $I \times \mathbb{R}^2$, such that

(i) $U(\cdot,\cdot,\cdot) \in C(I \times \mathbb{R}^2)$ and for some $a > 0$

$$u_\varepsilon(x,t) = u_\pm, \ \text{ for } \ \pm x > a + |t|, \ \varepsilon \in I. \tag{11.3}$$

(ii) The characteristic set $\mathcal{S} = \mathcal{S}(u_\varepsilon)$ does not depend on $\varepsilon \in I$, $U(\varepsilon, x, t) \in C^2(I \times \mathbb{R}^2 \setminus (\tilde\Gamma(U) \cup I \times \overline{\mathcal{S}})$ where $\tilde\Gamma(U) = \{(\varepsilon, x, t) \in I \times \mathbb{R}^2 : (x,t) \in \Gamma(u_\varepsilon)\}$, and

$$\sup_{(\varepsilon,x,t)\in\tilde\Omega} |\partial_x^\alpha \partial_t^\beta \partial_\varepsilon^\gamma u_\varepsilon(x,t)| < \infty, \ |\alpha| + |\beta| + |\gamma| \leqq 2 \tag{11.4}$$

for any open bounded subset $\tilde\Omega \subset I \times \mathbb{R}^2 \setminus (\tilde\Gamma(U) \cup I \times \overline{\mathcal{S}})$;

(iii) For any $k \in \mathbb{Z}$ the set $\Gamma_k(u_\varepsilon)$ (if non-empty) is the line $x = x_k(\varepsilon, t)$ with $x_k(\cdot,\cdot) \in C(I \times \mathbb{R})$, $\dfrac{\partial x_k}{\partial \varepsilon}(\cdot, t)$ is piecewise continuous in ε for any fixed t, and $\dot{x}_k(\varepsilon, \cdot)$ is piecewise continuous in t for each fixed $\varepsilon \in I$.

Definition 11.2. The least-action principle (11.2) for $u(x,t) \in \mathcal{E}$ means that for every $U(\varepsilon, x, t) \in \mathcal{TE}$ such that $U(0,x,t) \equiv u(x,t)$, we have, for all $T > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} S_T(U(\varepsilon,\cdot,\cdot))|_{\varepsilon=0} = 0 \;\; if \;\; U(\varepsilon,\cdot,\pm T) = u(\cdot,\pm T) \;\; for \;\; \varepsilon \in I. \tag{11.5}$$

Lemma 11.3. *The least-action principle for $u(x,t) \in \mathcal{E}$ is equivalent to the system of the equation* (2.11) *and of the splicing condition* (SC).

Proof. Let $U(\varepsilon, x, t) \in \mathcal{TE}$ be such that $U(0,x,t) = u(x,t)$. We may assume without loss of generality that the set $\Gamma(u_\varepsilon)$ is a finite union of the lines $x = x_k(\varepsilon, t)$, $1 \leqq k \leqq N$, where $x_k \in C^1(I \times \mathbb{R})$. Then for every k,

$$U(\varepsilon, x_k(\varepsilon,t), t) \equiv z_k \;\text{ for } (\varepsilon, t) \in I \times \mathbb{R}, \tag{11.6}$$

where $z_k \in Z$. Therefore we get after differentiation in ε and t (cf. (9.3)),

$$U'\frac{\partial x_k}{\partial \varepsilon} + \frac{\partial U}{\partial \varepsilon} = 0 \quad \text{for } \; x = x_k(\varepsilon,t) \pm 0 \;\text{ and }\; \text{ a.e. } (\varepsilon,t) \in I \times \mathbb{R}, \tag{11.7}$$

$$U'\dot{x}_k + \dot{U} = 0 \quad \text{for } \; x = x_k(\varepsilon,t) \pm 0 \;\text{ and }\; \text{ a.e. } (\varepsilon,t) \in I \times \mathbb{R}. \tag{11.8}$$

Let us arrange $x_k(0,0)$ in increasing order. Then this order is conserved for all $(\varepsilon, t) \in I \times \mathbb{R}$ by continuity,

$$\begin{aligned} -\infty \equiv x_0(\varepsilon,t) < x_1(\varepsilon,t) < \cdots < x_N(\varepsilon,t) < x_{N+1}(\varepsilon,t) \\ \equiv +\infty \;\text{ for } (\varepsilon,t) \in I \times \mathbb{R}. \end{aligned} \tag{11.9}$$

Then by definitions (2.8), (11.1)

$$S_T(U(\varepsilon,\cdot,\cdot)) = \int_0^T \left(\sum_{k=0}^N \int_{x_k(\varepsilon,t)}^{x_{k+1}(\varepsilon,t)} [\tfrac{1}{2}\dot{U}^2 - \tfrac{1}{2}|U'|^2 - V(U)]dx \right) dt \tag{11.10}$$

For a function $p(\varepsilon,x,t) \in C(I \times \mathbb{R}^2 \setminus (\tilde{\Gamma} \cup I \times \mathcal{S}))$ let us define

$$\Delta_k p(\varepsilon,t) = p|_{x=x_k(\varepsilon,t)-0}^{x=x_k(\varepsilon,t)+0} \quad \text{ for } (\varepsilon,t) \in I \times \mathbb{R}, \tag{11.11}$$

when this expression exists. Then (11.5) is equivalent to

$$\begin{aligned} 0 = &-\int_0^T \left(\sum_{k=1}^N \Delta_k[\tfrac{1}{2}\dot{u}^2 - \tfrac{1}{2}|u'|^2 - V(u)](t)\; v_k(t) \right) dt \\ &+\int_0^T \left(\sum_{k=0}^N \int_{x_k(t)}^{x_{k+1}(t)} [\dot{u}\dot{h} - u'h']dx \right) dt \end{aligned} \tag{11.12}$$

where $v_k(t) = \frac{\partial x_k}{\partial \varepsilon}(0,t)$ and $h(x,t) = \frac{\partial U}{\partial \varepsilon}(0,x,t)$. Let us assume for a moment that each line $x = x_k(t)$ admits the global representation $t = t_k(x)$ with

$t_k(\cdot) \in C([x_k(0), x_k(T)])$ and let us integrate by parts in the last integral in (11.12). Then taking into account (11.3) and boundary values in (11.5), we get for the integral the expression

$$\sum_{k=1}^{N} \left(\int_{x_k(0)}^{x_k(T)} (\dot{u}h)|_{t=t_k(x)+0}^{t=t_k(x)-0} |dx| + \int_0^T \Delta_k(u'h)(t)dt \right) \qquad (11.13)$$
$$- \int_0^T \left(\sum_{k=0}^{N} \int_{x_k(t)}^{x_{k+1}(t)} [\ddot{u} - u'']h \, dx \right) dt.$$

Let us take into account that $h(x_k(t) \pm 0, t) = -u'(x_k(t) \pm 0, t)v_k(t)$ due to (11.7), and $u'(x_k(t) \pm 0, t)\dot{x}_k(t) = -\dot{u}(x_k(t) \pm 0, t)$ due to (11.8). Therefore,

$$\begin{aligned} \int_{x_k(0)}^{x_k(T)} (\dot{u}h)|_{t=t_k(x)+0}^{t=t_k(x)-0} |dx| &= \int_0^T \Delta_k(\dot{u}h)(t)\dot{x}_k(t) \, dt \\ &= - \int_0^T \Delta_k(\dot{u}u'v_k)(t)\dot{x}_k(t) \, dt \\ &= \int_0^T \Delta_k(\dot{u}^2)(t) \; v_k(t) \, dt. \end{aligned} \qquad (11.14)$$

Similarly,

$$\int_0^T \Delta_k(u'h)(t)dt = - \int_0^T \Delta_k(|u'|^2)(t) \; v_k(t) \, dt. \qquad (11.15)$$

Therefore, (11.12) becomes, due to (11.13)–(11.15),

$$0 = \int_0^T \left(\sum_{k=1}^{N} \Delta_k[\tfrac{1}{2}|\dot{u}|^2 - \tfrac{1}{2}|u'|^2 + V(u)](t) \; v_k(t) \right) dt$$
$$- \int_0^T \left(\sum_{k=0}^{N} \int_{x_k(t)}^{x_{k+1}(t)} [\ddot{u} - u'']h \, dx \right) dt. \qquad (11.16)$$

The same result can be obtained without assuming the global representation $t = t_k(x)$ for the lines $x = x_k(t)$ since the local representations are sufficient. Finally, (11.16) implies (2.11) because h can be chosen arbitrary in any compact subset of $\mathbb{R}^2 \setminus \Gamma(u)$. Then the last integral in (11.16) is zero. Therefore (11.16) implies also (SC) because $v_k(t)$ can be chosen arbitrarily in every compact subset of Γ_k for $k = 1, \ldots, N$. $\square$

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