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GLOBAL ATTRACTION TO SOLITARY WAVES FOR A NONLINEAR DIRAC EQUATION WITH MEAN FIELD INTERACTION*

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Abstract. We consider a $\mathbf{U}(1)$-invariant nonlinear Dirac equation in dimension $n \geq 1$, interacting with itself via the mean field mechanism. We analyze the long-time asymptotics of solutions and prove that, under certain generic assumptions, each finite charge solution converges as $t \to \pm\infty$ to the two-dimensional set of all "nonlinear eigenfunctions" of the form $\phi(x)e^{-i\omega t}$. This global attraction is caused by the nonlinear energy transfer from lower harmonics to the continuous spectrum and subsequent dispersive radiation. The research is inspired by Bohr's postulate on quantum transitions and Schrödinger's identification of the quantum stationary states to the nonlinear eigenfunctions of the coupled $\mathbf{U}(1)$-invariant Maxwell–Schrödinger and Maxwell–Dirac equations.

Key words. Dirac equation, solitary waves, unitary invariance, global attractor, solitary manifold, nonlinear spectral analysis, Titchmarsh convolution theorem

AMS subject classifications. 35B41, 35Q41, 37K40, 37L30, 37N20, 81Q05

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1. Introduction.

In the present paper we continue our research [24, 21, 22, 23] on the global attraction to solitary waves in $\mathbf{U}(1)$-invariant dispersive nonlinear systems. Our long-term aim is a dynamical interpretation of "quantum jumps" postulated by Bohr. We conjecture that the global convergence to solitary waves is an inherent property of a generic nonlinear $\mathbf{U}(1)$-invariant dispersive system (cf. the "soliton resolution conjecture" [42]).

The long-time asymptotics for nonlinear wave equations have been the subject of intensive research, starting with the pioneering papers on existence and nonlinear scattering by Segal [32, 33, 34], Strauss [36], and Morawetz and Strauss [27]. Some of these results, in the context of defocusing nonlinear Schrödinger and Klein–Gordon equations without an external potential, state that any finite energy solution weakly converges to a solution to the free linear equation and thus locally converges to zero. This convergence to zero is in the agreement with the soliton resolution conjecture, since the only solitary wave solution to such defocusing equations is the zero solution. For the review of these results, see [37, Chapter 6].

Local attraction to solitary waves, or *local asymptotic stability*, in $\mathbf{U}(1)$-invariant dispersive systems was addressed in [39, 4, 40, 5] and then developed in [28, 41, 8, 9, 6, 10].

The attraction of any finite energy solution to the set of all solitary waves was proved in [20, 21] in the context of the $\mathbf{U}(1)$-invariant Klein–Gordon equation coupled to a nonlinear oscillator. In [23], we generalized this result for the Klein–Gordon field coupled to several oscillators. In [22], a similar result was generalized to a higher-

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dimensional setting for the Klein–Gordon equation with the mean field interaction. Let us note that the question of similar asymptotics in the context of the Schrödinger equation coupled to a nonlinear oscillator remains open.

Bohr's transitions as global attraction to solitary waves. According to Bohr's postulates [2], an unperturbed electron runs forever along certain *stationary orbit* denoted by $|E\rangle$ and called *quantum stationary state*. Once in such a state, the electron has a fixed value of energy E, with the energy not being lost via emitted radiation. Under a perturbation, the electron can jump from one quantum stationary state to another,

$$|E_-\rangle \longmapsto |E_+\rangle, \tag{1.1}$$

radiating (or absorbing) a quantum of light. Bohr's postulate suggests the dynamical interpretation of the transitions (1.1) as a long-time asymptotics

$$\psi(t) \longrightarrow |E_\pm\rangle, \qquad t \to \pm\infty \tag{1.2}$$

for any trajectory $\psi(t)$ of the corresponding dynamical system, where the limiting states $|E_\pm\rangle$ depend on the trajectory. Then the *quantum stationary states* should be viewed as points of the *global attractor*.

At first glance, the global attraction (1.2) seems incompatible with the energy conservation and time reversibility of Hamiltonian systems. Yet there is no contradiction, since the attraction takes place uniformly on arbitrary compact sets but not in the global norm. We intend to verify that such asymptotics in principle are possible for nonlinear Hamiltonian field equations. In this paper, we verify this asymptotics for equations of Dirac type.

Schrödinger's quantum stationary states. Developing de Broglie's ideas, Schrödinger identified the quantum stationary states of energy E with the solutions of type

$$\psi(x,t) = \phi(x)e^{-iEt/\hbar}, \qquad x \in \mathbb{R}^3. \tag{1.3}$$

Then the Schrödinger equation

$$i\hbar\partial_t\psi = H\psi := -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi, \qquad \psi = \psi(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}^3, \tag{1.4}$$

becomes the *eigenvalue problem*

$$E\phi(x) = H\phi(x). \tag{1.5}$$

This was one of the original Schrödinger's ideas [29] to identify the integers in the Debye–Sommerfeld–Wilson quantum rules with the integers arising in the eigenvalue problems for PDEs.

For the case of the *free particles*, this identification agrees with de Broglie's wave function $\psi(x,t) = e^{ip\cdot x/\hbar}e^{-iEt/\hbar}$, where $p \in \mathbb{R}^3$ is the momentum of the particle. For the *bound particles* in an external potential, the identification (1.3) reflects the fact that the space is no longer homogeneous due to the presence of the external field. Thus, Schrödinger (1.3) identified Bohr's *stationary orbits* with *solitary waves* $\psi(x,t) = \phi_\omega(x)e^{-i\omega t}$, where $\omega = E/\hbar$.

Thus, the reason for Schrödinger's choice of quantum stationary states in the form (1.3) seems to be rather algebraic. At the same time, the attraction (1.2) suggests the dynamical interpretation of the quantum stationary states as asymptotic states in the long time limit, that is,

$$\psi(x,t) \sim \phi_{\omega_\pm}(x) e^{-i\omega_\pm t}, \qquad t \to \pm\infty. \tag{1.6}$$

This asymptotics should hold for each finite charge solution. The asymptotics of type (1.6) are generally impossible for the linear autonomous equation (1.4) because of the superposition principle. On the other hand, one could expect that the asymptotics (1.6) hold for the coupled nonlinear Maxwell–Schrödinger and Maxwell–Dirac systems.

Maxwell–Schrödinger and Maxwell–Dirac systems and nonlinear Dirac equation. An adequate description of "quantum jumps" in an atom requires that we consider the equation for the electron wave function (Schrödinger or Dirac equation) coupled to the Maxwell system which governs the time evolution of the four-potential $A^\mu(x,t)$. The Maxwell–Schrödinger system was initially introduced by Schrödinger in [30]; its global well-posedness was considered in [15].

The Maxwell–Dirac system can be written as

$$\begin{cases} (i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu)\psi - m\psi = 0, \qquad \psi = \psi(t,x), \quad A_\mu = g_{\mu\nu}A^\nu, \\ \partial^\mu\partial_\mu A^\nu = 4\pi J^\nu, \quad \partial_\mu A^\mu = 0, \quad g_{\mu\nu} = \operatorname{diag}(1,-1,-1,-1). \end{cases} \tag{1.7}$$

Above, $\gamma^0 = \left[\begin{smallmatrix} I_2 & 0 \\ 0 & -I_2 \end{smallmatrix}\right]$ and $\gamma^j = \gamma^0\alpha^j = \left[\begin{smallmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{smallmatrix}\right]$ are the Dirac matrices (I_2 being the 2×2 unit matrix and σ_j, $1 \le j \le 3$, the Pauli matrices); $J^\mu = (\rho, \mathbf{j})$, with ρ and $\mathbf{j}$ the charge and current density generated by the spinor field:

$$J^\mu = -e\bar\psi\gamma^\mu\psi, \qquad \text{with} \quad \bar\psi = \psi^*\gamma^0. \tag{1.8}$$

The results on local existence of solutions to the Maxwell–Dirac system were obtained in [3]. The existence of global small amplitude solutions of the Maxwell–Dirac equations was considered in [14]. The existence of solitary waves in this system was proved in [12, 1]. The stability properties of these solitary waves are presently not understood.

The coupled Maxwell–Dirac system (as well as the Maxwell–Schrödinger system) is $\mathbf{U}(1)$-invariant with respect the (global) gauge group $\big(\psi(x), A^\mu(x)\big) \mapsto \big(\psi(x)e^{-i\theta}, A^\mu(x)\big)$, $\theta \in \mathbb{R}$. Respectively, one might expect the following natural generalization of asymptotics (1.6) for solutions to these systems:

$$\big(\psi(x,t), A^\mu(x,t)\big) \sim \big(\phi_\pm(x)e^{-i\omega_\pm t}, A^\mu_\pm(x)\big), \qquad t \to \pm\infty. \tag{1.9}$$

The asymptotics (1.9) would mean that the set of all solitary waves forms a global attractor for the coupled system. The asymptotics of this type are not available yet in the context of the coupled systems.

Let us also mention the nonlinear Dirac equation, which is a simplified version of the Maxwell–Dirac system with nonlinearity of local type, known as the Soler model [35]:

$$(i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu)\psi - g(\bar\psi\psi)\psi = 0, \qquad \psi = \psi(t,x) \in \mathbb{C}^4, \quad x \in \mathbb{R}^3, \tag{1.10}$$

where g is a smooth function of $\bar\psi\psi = \psi^*\gamma^0\psi$. The existence of standing waves in the nonlinear Dirac equation was proved in [11, 26, 13]. The stability of solitary waves with respect to a particular class of perturbations was analyzed in [38]. The spectral stability of small amplitude solitary waves of the nonlinear Dirac equation in one dimension has been confirmed numerically in [7]. The orbital stability of solitary waves and the global attraction to solitary waves is presently not understood.

Model. In the present paper we establish the global attraction to the set of all solitary waves for the Dirac field $\psi(x,t)\in\mathbb{C}^N$ with the mean field interaction

$$i\partial_t\psi(x,t)=\big(-i\boldsymbol{\alpha}\cdot\nabla+\beta m\big)\psi+\rho(x)F(\langle\psi(\cdot,t),\rho\rangle),\qquad x\in\mathbb{R}^n,\quad n\geq 1. \tag{1.11}$$

Here α_j and β are the Dirac matrices, $\boldsymbol{\alpha}\cdot\nabla:=\sum_{j=1}^n\alpha_j\partial_j$ and $F(z)=a(|z|^2)z$ for $z\in\mathbb{C}$, where $a(\cdot)$ is a nonconstant real polynomial. Further, $\langle\cdot,\cdot\rangle$ stands for the Hermitian scalar product in the complex Hilbert space $L^2(\mathbb{R}^n,\mathbb{C}^N)$, and ρ is a spinor-valued coupling function from the Schwartz class. See section 2 for the details.

Remark 1.1. Let us emphasize that the system (1.11) is not the physical model aimed to explain particular experimental observations. Rather, it is a mathematical model which demonstrates behavior reminiscent of Bohr's quantum jumps. The nonlinear coupling in (1.11) is of a simplistic form, which could informally be interpreted as the interaction of a classical field with a nonlinear oscillator [21, 22].

We will prove that under fairly general assumptions on the coupling function ρ, any solution of finite charge converges as $t\to\pm\infty$ to the set of all solitary waves, which we denote by $\mathbf{S}$. The convergence holds in the topology of $H^{-\varepsilon}_{loc}(\mathbb{R}^n,\mathbb{C}^N)$ for any $\varepsilon>0$:

$$\psi(t)\xrightarrow[t\to\pm\infty]{H^{-\epsilon}_{loc}(\mathbb{R}^n,\mathbb{C}^N)}\mathbf{S}. \tag{1.12}$$

The modeling of the interaction of the matter with gauge fields in terms of local self-interaction has already been employed in [31], where the relativistically invariant nonlinear Klein–Gordon equation originally appeared. As of yet, we can not consider the relativistically invariant case and retreat to the non-translation-invariant mean field interaction.

Let us comment on our methods. We follow the path developed in [21, 22, 23] to prove the absolute continuity of the spectral density for large frequencies, to extract the omega-limit trajectories via the compactness argument, and then to apply the Titchmarsh convolution theorem [43] (see also [25, p. 119] and [16, Theorem 4.3.3]) to pinpoint the spectrum of each omega-limit trajectory to a single frequency.

The absolute continuity is a nonlinear version of Kato's theorem on the absence of the embedded eigenvalues and provides the dispersive decay for the high energy component. The Titchmarsh theorem controls the inflation of spectrum by the nonlinearity. Physically, these arguments justify the following "binary" mechanism of the energy radiation, which is responsible for the attraction to the solitary waves: (i) the nonlinear energy transfer from the lower to higher harmonics, and (ii) the subsequent dispersive decay caused by the energy radiation to infinity.

The realization of the path [21, 22, 23] for the case of the Dirac equation requires new ideas due to the well-known nonpositivity of the energy for the Dirac equation. First, this fact does not allow us to obtain an a priori estimate from the energy conservation, contrary to the case of the Klein–Gordon equation. To circumvent this problem, we rely on the charge conservation and work in the L^2 setting, proving the well-posedness in L^2. As a result, the convergence to the attractor is in local $H^{-\varepsilon}$-norm with $\varepsilon>0$, which is weaker than in the case of the Klein–Gordon equation. Second, the corresponding coupling function $\sigma(\omega)$ from (3.4) necessarily vanishes at least at one frequency. Respectively, we need novel arguments in the proof of the absolute continuity of the spectral density for large frequencies (see Lemma 6.2). The arguments rely on certain assumptions on the form-factor $\rho(x)$. We construct examples which demonstrate that our assumptions are not restrictive. Finally, the vanishing of $\sigma(\omega)$ at certain points relaxes the inflation of spectrum by the nonlinearity; see section 7 on the nonlinear spectral analysis how to overcome this difficulty.

2. Existence of dynamics.

2.1. Notations and functional spaces. We consider the Cauchy problem for (1.11):

$$\begin{cases} i\partial_t\psi(t) = \big(-i\boldsymbol{\alpha}\cdot\nabla+\beta m\big)\psi+\rho(x)F(\langle\psi(\cdot,t),\rho\rangle), & x\in\mathbb{R}^n, \\ \psi(x,0)=\psi_0(x), \qquad \psi_0\in L^2(\mathbb{R}^n,\mathbb{C}^N). \end{cases} \tag{2.1}$$

Above, $F:\ \mathbb{C}\to\mathbb{C}$ is a nonlinear function, and $\langle\psi(\cdot,t),\rho\rangle=\int_{\mathbb{R}^n}\psi(x,t)\cdot\rho(x)\,d^nx$, where "$\cdot$" is the Hermitian inner product and ρ is a spinor-valued coupling function from the Schwartz space:

$$\rho\in\mathscr{S}(\mathbb{R}^n,\mathbb{C}^N), \qquad \rho\not\equiv\mathbf{0}.$$

We assume that ρ is of Schwartz type for the simplicity of exposition; this assumption could be weakened. Hermitian $N\times N$ matrices α_j, $1\le j\le n$, and $\alpha_0:=\beta$ satisfy

$$\{\alpha_j,\alpha_k\}=2\delta_{jk}\boldsymbol{I}, \qquad j,\,k=0,\,1,\,\dots,\,n, \tag{2.2}$$

where $\{\cdot,\cdot\}$ stands for the anticommutator. In the cases $n=1$ and $n=2$, one can take $N=2$, with any choice of the Pauli matrices for α_j and β. In the case $n=3$, one can take $N=4$ and the standard Dirac matrices.

Assumption A. The nonlinearity $F(z)$ has the following structure:

$$F(z)=a(|z|^2)z, \qquad z\in\mathbb{C}, \tag{2.3}$$

where $a(\cdot)$ is a nonconstant polynomial with real coefficients

$$a(s)=\sum_{k=0}^{p}a_k\,s^k, \qquad a_k\in\mathbb{R}, \quad p\ge 1, \quad a_p\ne 0. \tag{2.4}$$

Remark 2.1.

(i) Assumption A implies that (2.1) is $\mathbf{U}(1)$-invariant:

$$F(e^{i\theta}z)=e^{i\theta}F(z), \qquad z\in\mathbb{C}, \quad \theta\in\mathbb{R}. \tag{2.5}$$

(ii) The assumption that the nonlinearity is polynomial is crucial in our argument; it will allow us to apply the Titchmarsh convolution theorem.

(iii) The requirement that $a(\cdot)$ is nonconstant means that (1.11) is *strictly nonlinear*. It is essential since generally there is no global attraction in *linear* models because of the superposition principle.

Equation (2.1) can be written as a Hamiltonian system,

$$\dot\Psi(t)=J\,D\mathcal{H}(\Psi), \qquad \Psi=(\operatorname{Re}\psi,\operatorname{Im}\psi), \qquad J=\begin{bmatrix}0 & I_N\\ -I_N & 0\end{bmatrix}, \tag{2.6}$$

where $D\mathcal{H}$ is the Fréchet derivative of the Hamilton functional

$$\mathcal{H}(\Psi)=\frac12\int_{\mathbb{R}^n}\psi(x)\cdot\big(-i\boldsymbol{\alpha}\cdot\nabla+\beta m\big)\psi(x)\,d^nx+\frac12 W(|\langle\psi,\rho\rangle|^2), \tag{2.7}$$

where $W(s)=\displaystyle\int_0^s a(t)\,dt$, $s\in\mathbb{R}$.

Denote by $\|\cdot\|_{L^2}$ the norm in $L^2(\mathbb{R}^n,\mathbb{C}^N)$. Let $H^s(\mathbb{R}^n,\mathbb{C}^N)$, $s\in\mathbb{R}$, be the Sobolev space with the norm

$$\|\psi\|_{H^s}=\|(m^2-\Delta)^{s/2}\psi\|_{L^2}. \tag{2.8}$$

For $s\in\mathbb{R}$ and $R>0$, $H^s_0(\mathbb{B}^n_R,\mathbb{C}^N)$ is the space of distributions from $H^s(\mathbb{R}^n,\mathbb{C}^N)$ supported in $\mathbb{B}^n_R$ (the ball of radius R in $\mathbb{R}^n$). We denote by $\|\cdot\|_{H^s(\mathbb{B}_R,\mathbb{C}^N)}$ the norm in $H^s(\mathbb{B}^n_R,\mathbb{C}^N)$, which is defined as the dual to $H_0^{-s}(\mathbb{B}^n_R,\mathbb{C}^N)$.

Definition 2.2.

(i) *The space of states is $\mathscr{X}=L^2(\mathbb{R}^n,\mathbb{C}^N)$, with the standard norm.*

(ii) *$\mathscr{Y}$ is the vector space $\mathscr{X}$ endowed with the norm $\|\psi\|_{\mathscr{Y}}:=\sum_{R\in\mathbb{N}}2^{-R}$ $\|\psi\|_{H^{-\varepsilon}(\mathbb{B}^n_R,\mathbb{C}^N)}$, with fixed $\varepsilon>0$.*

Remark 2.3. The Sobolev embedding theorem implies that the embedding $\mathscr{X}\subset\mathscr{Y}$ is compact.

2.2. Global well-posedness.

Proposition 2.4. *Let $\rho\in\mathscr{S}(\mathbb{R}^n,\mathbb{C}^N)$, and let $F(z)$ satisfy Assumption* A. *Then*

(i) *for every $\psi_0\in\mathscr{X}$ the Cauchy problem (2.1) has a unique solution $\psi\in C_b(\mathbb{R},\mathscr{X})$ in the sense of distributions.*

(ii) *for any $t\in\mathbb{R}$, the map $\mathcal{W}(t):\ \psi_0\mapsto\psi(t)$ defines continuous mappings $\mathscr{X}\to\mathscr{X}$ and $\mathscr{Y}\to\mathscr{Y}$.*

(iii) *for $\psi_0\in\mathscr{X}$, the value of the charge functional is conserved: $\|\psi(t)\|^2_{L^2}=\mathcal{Q}(\psi(t))=\mathcal{Q}(\psi_0)$, $t\in\mathbb{R}$.*

(iv) *for any $T\ge 0$, the map $\mathcal{W}(t):\psi_0\mapsto\psi(t)$ is continuous as a map $\mathscr{Y}\to C([-T,T],\mathscr{Y})$.*

Proof. The global existence stated in Proposition 2.4 is obtained by standard arguments from the contraction mapping principle. To achieve this, we use the integral representation for the solutions to the Cauchy problem (2.1):

$$\psi(t)=\mathcal{W}_0(t)\psi_0+\mathcal{I}[\psi](t),\qquad \mathcal{I}[\psi](t):=\int_0^t\mathcal{W}_0(t-s)\rho\,F(\langle\psi(\cdot,s),\rho\rangle)\,ds,\qquad t\ge 0. \tag{2.9}$$

Here $\mathcal{W}_0(t)=e^{-i(-i\boldsymbol{\alpha}\cdot\nabla+\beta m)t}$ is the dynamical group for the linear Dirac equation. $\mathcal{W}_0(t)$ is a unitary operator in the space $H^{-\varepsilon}(\mathbb{R}^n,\mathbb{C}^N)$ for any $\varepsilon\ge 0$. The bound

$$\|\mathcal{I}[\psi_1](t)-\mathcal{I}[\psi_2](t)\|_{H^{-\varepsilon}}\le C|t|\sup_{s\in[0,t]}\|\psi_1(s)-\psi_2(s)\|_{H^{-\varepsilon}},\qquad C>0,\quad |t|\le 1,\quad \varepsilon\ge 0, \tag{2.10}$$

which holds for any two functions ψ_1, $\psi_2\in C(\mathbb{R},\mathscr{X})$, shows that $\mathcal{I}[\psi]$ is a contraction operator in $C([0,t],H^{-\varepsilon})$, $\varepsilon\ge 0$, if $t>0$ is sufficiently small. Hence, the existence and uniqueness of a local solution is proved via the contraction principle. This way one proves the continuity of the mappings $\mathcal{W}(t):\mathscr{X}\to\mathscr{X}$ and $\mathscr{Y}\to\mathscr{Y}$.

The charge conservation is obtained first for smooth initial data with compact support. Then, by the Duhamel integral equation (2.9), $\psi(t)\in C^1(\mathbb{R},C_0^\infty(\mathbb{R}^3,\mathbb{C}^N))$; hence the charge conservation follows by direct differentiation:

$$\dot{\mathcal{Q}}(\psi(t))=\langle\dot\psi(t),\psi(t)\rangle+\langle\psi(t),\dot\psi(t)\rangle. \tag{2.11}$$

Here $\dot\psi(t)=i\mathscr{D}\psi(t)-i\rho F(\langle\psi(t),\rho\rangle)$ by (2.1), where $\mathscr{D}=-i\boldsymbol{\alpha}\cdot\nabla+\beta m$ is the free Dirac operator. Hence, (2.11) reads

$$\dot{\mathcal{Q}}(\psi(t))=i\Big\langle\mathscr{D}\psi(t)-\rho F(\langle\psi(t),\rho\rangle),\psi(t)\Big\rangle-i\Big\langle\psi(t),\mathscr{D}\psi(t)-\rho F(\langle\psi(t),\rho\rangle)\Big\rangle=0 \tag{2.12}$$

since the operator $\mathscr{D}$ is symmetric and $F(\langle\psi(t),\rho\rangle) = a\big(|\langle\psi(t),\rho\rangle|^2\big)\langle\psi(t),\rho\rangle$ by (2.3), where $a(\cdot)$ is real-valued. For general initial states, the charge conservation follows by the density argument and the continuity of $\mathcal{W}(t)$ in $\mathscr{X}$ for small $|t| < \varepsilon$, which is proved above. The charge conservation $\|\psi\|_{L^2} = \text{const}$ allows us to extend the existence results for all times, proving the global well-posedness in the "finite charge" space $\mathscr{X}$.

Finally, the continuity of the map $\mathcal{W}(t) : \mathscr{X} \to C([-T,T],\mathscr{X})$ follows from the contraction mapping principle (based on (2.10)). Similarly one proves the continuity of the mapping $\mathcal{W}(t) : H^{-\varepsilon} \to C([-T,T],H^{-\varepsilon})$. This continuity, together with the finite speed of propagation, yields the continuity of the mapping $\mathcal{W}(t) : \mathscr{Y} \to C([-T,T],\mathscr{Y})$. ∎

3. Main results.

Definition 3.1.

(i) *The solitary waves are solutions of the form* $\psi(x,t) = \phi_\omega(x)e^{-i\omega t}$, *where* $\omega \in \mathbb{R}$ *and* $\phi_\omega \in \mathscr{X}$.

(ii) *The solitary manifold is the set* $\mathbf{S} = \big\{\phi_\omega\colon \phi_\omega(x)e^{-i\omega t}$ *is a solitary wave*$\big\} \subset \mathscr{X}$.

We construct solitary waves in Appendix A.

Remark 3.2. The identity (2.5) implies that (2.1) is $\mathbf{U}(1)$-invariant; hence the set $\mathbf{S}$ from Definition 3.1 is invariant under multiplication by $e^{i\theta}$, $\theta \in \mathbb{R}$. Generically, the solitary manifold $\mathbf{S}$ is two-dimensional. It may contain disconnected components. Since $F(0) = 0$ by (2.3), the zero solitary wave is an element of $\mathbf{S}$, corresponding to any $\omega \in \mathbb{R}$.

Let us formulate our second assumption, which will be used for the proof of our main result on attraction (1.12) for the case $n \geq 3$ in the simplest situation. First, let us denote

$$\hat{\mathscr{D}}(\xi) = \boldsymbol{\alpha}\cdot\xi + \beta m, \qquad \xi \in \mathbb{R}^n. \tag{3.1}$$

Since $\hat{\mathscr{D}}(\xi)$ is self-adjoint and $\hat{\mathscr{D}}^2(\xi) = \xi^2 + m^2$, the eigenvalues of $\hat{\mathscr{D}}(\xi)$ are $\pm\sqrt{\xi^2+m^2}$, and

$$\Pi_\pm(\xi) = \frac{1}{2}\left(1 \pm \frac{\hat{\mathscr{D}}(\xi)}{\sqrt{\xi^2+m^2}}\right) \tag{3.2}$$

are the orthogonal projectors onto the corresponding eigenspaces. Denote by $\hat\rho(\xi) = \mathcal{F}_{x\to\xi}\rho := \int_{\mathbb{R}^n} e^{-i\xi x}\rho(x)\,d^n x$ the Fourier transform of the coupling function $\rho(x) \in \mathscr{S}(\mathbb{R}^n,\mathbb{C}^N)$. We decompose

$$\hat\rho(\xi) = \hat\rho_-(\xi) + \hat\rho_+(\xi), \qquad \hat\rho_\pm(\xi) := \Pi_\pm(\xi)\hat\rho(\xi). \tag{3.3}$$

Further, define

$$\sigma(\omega) = \int\limits_{\mathbb{R}^n} \frac{[(\omega + \boldsymbol{\alpha}\cdot\xi + \beta m)\hat\rho(\xi)]\cdot\hat\rho(\xi)}{(\omega+i0)^2 - \xi^2 - m^2}\,\frac{d^n\xi}{(2\pi)^n}, \qquad \omega \in \mathbb{R}\backslash\{\pm m\}. \tag{3.4}$$

Note that for $|\omega| \leq m$ the integral converges in dimensions $n \geq 3$, while for $|\omega| > m$ the limit exists by the Sokhotsky–Plemelj formula. Moreover, the function $\sigma(\omega)$ is continuous at $\omega = \pm m$ when $n \geq 3$. Indeed, by the Parseval identity,

$$\sigma(\pm m + z) = (2\pi)^{-n}\langle R(z^2 \pm 2mz)[(\pm m + z - i\boldsymbol{\alpha}\cdot\nabla + \beta m)\rho],\rho\rangle, \tag{3.5}$$

where $R(\zeta) := (\Delta - \zeta)^{-1}$ is the resolvent of the Laplacian. Now the continuity of $\sigma(\omega)$ at $\omega = \pm m$ follows from the continuity of the resolvent at $\zeta = 0$ in $\mathbb{C} \setminus (-\infty, 0)$ in the Agmon weighted norms (see [19] for the dimension $n = 3$ and [18] and [17] for the dimensions $n = 4$ and $n \geq 5$, respectively) since the function ρ is from the Schwartz space.

Since $\sigma(\omega)$ is a continuous function which becomes positive for $\omega \to +\infty$ and negative for $\omega \to -\infty$, we conclude that there is at least one point where $\sigma(\omega)$ vanishes.

Assumption B.

(i) For any $\lambda > 0$, $\hat{\rho}_\pm(\xi)$ do not vanish identically on the sphere $|\xi| = \lambda$;

(ii) $\sigma(\omega)$ does not vanish on $[-m, m]$ except perhaps at one point which will be denoted ω_σ.

In section 8, we will relax this assumption and also will extend the result to the dimensions $n = 1$, 2.

Let us give examples of such functions $\rho \in \mathscr{S}(\mathbb{R}^n, \mathbb{C}^N)$. We take a spherically symmetric function ρ such that $\hat{\rho}(\xi) \neq 0$ for $\xi \neq 0$, with $\hat{\rho}(\xi)$ being an eigenvector of β (since β is Hermitian and $\beta^2 = 1$, its eigenvalues are ± 1). Then $|\hat{\rho}_\pm(\xi)|^2 = \frac{1}{2}\hat{\rho}(\xi) \cdot (1 \pm \frac{\boldsymbol{\alpha}\cdot\xi+\beta m}{\sqrt{\xi^2+m^2}})\hat{\rho}(\xi)$; hence

$$\int_{|\xi|=\lambda} |\hat{\rho}_\pm(\xi)|^2 \, dS \geq \int_{|\xi|=\lambda} \hat{\rho}(\xi) \cdot \left(1 - \frac{m}{\sqrt{\xi^2 + m^2}}\right) \hat{\rho}(\xi) \, dS > 0, \qquad \lambda > 0,$$

where dS stands for the Lebesgue measure on the sphere. The integral of the term with $\boldsymbol{\alpha} \cdot \xi$ dropped out since $\hat{\rho}(\xi)$ is spherically symmetric. Let us now consider $\sigma(\omega)$. Due to $\hat{\rho}$ being spherically symmetric, $\boldsymbol{\alpha} \cdot \xi$-term cancels out from the integration in (3.4). Since we require that $\hat{\rho}(\xi)$ be an eigenvector of β, one has $\hat{\rho}(\xi) \cdot (\omega + \beta m)\hat{\rho}(\xi) = (\omega \pm m)\hat{\rho}(\xi) \cdot \hat{\rho}(\xi)$; hence the expression under the integral in (3.4) is sign-definite for all $\omega \in [-m, m]$ except at $\omega = m$ or at $\omega = -m$. Therefore, $\sigma(\omega)$ does not vanish on $[-m, m]$ except at one of the endpoints.

Our main result is the convergence (1.12) of finite charge solutions to the solitary manifold $\mathbf{S}$ introduced in Definition 3.1. First we formulate and prove the result under Assumptions A and B for $n \geq 3$ to clarify main ideas. The generalizations with weakened Assumption B will be done in section 8 where we also extend the result to the dimensions $n = 1$, 2.

THEOREM 3.3 (main theorem). *Let $n \geq 3$. Assume that the nonlinearity $F(z)$ satisfies Assumption* A *and that the coupling function $\rho(x)$ satisfies Assumption* B. *Then for any $\psi_0 \in \mathscr{X}$ the corresponding solution $\psi(t) \in C(\mathbb{R}, \mathscr{X})$ to the Cauchy problem* (2.1) *converges to* $\mathbf{S}$ *in the space* $\mathscr{Y}$:

$$\lim_{t\to\pm\infty} \operatorname{dist}_{\mathscr{Y}}(\psi(t), \mathbf{S}) = 0, \tag{3.6}$$

where $\operatorname{dist}_{\mathscr{Y}}(\psi, \mathbf{S}) := \inf_{\boldsymbol{s}\in\mathbf{S}} \|\psi - \boldsymbol{s}\|_{\mathscr{Y}}$. *See Figure* 1.

Since the Hamiltonian system is time reversible, it suffices to prove Theorem 3.3 for $t \to +\infty$.

Because of the lack of orbital stability results for systems based on the Dirac equation, we can not easily give an example of the initial condition such that the corresponding solution would converge for large times to a nonzero limit, except a trivial example of the initial data corresponding to the solitary waves themselves. See Appendix A for the sufficient condition for the existence of nonzero solitary waves.

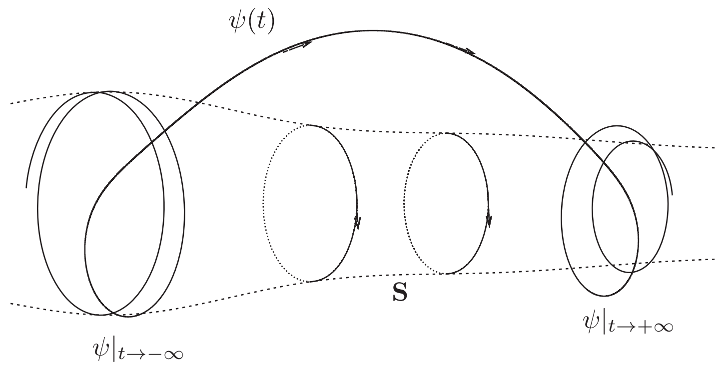

FIG. 1. *For $t \to \pm\infty$, a finite charge solution $\psi(t)$ approaches the global attractor (formed by all solitary waves $\mathbf{S}$).*

As we mentioned, Assumption B could be relaxed. This will be done in section 8. See Assumption C and Theorem 8.3, which also covers the dimensions $n = 1, 2$.

4. Omega-limit trajectories.

4.1. Compactness. Fix $\psi_0 \in \mathscr{X}$, and let $\psi \in C_b(\mathbb{R}, \mathscr{X})$ be the solution to the Cauchy problem (1.11) with the initial data $\psi(x, 0) = \psi_0(x)$. Let $t_j > 0$, $j \in \mathbb{N}$, be a sequence such that $t_j \to \infty$. Since $\psi(t_j)$ are bounded in $\mathscr{X}$, the compactness stated in Remark 2.3 allows us to choose a subsequence of $\{t_j\}$, also denoted $\{t_j\}$, such that

$$\psi(t_j) \xrightarrow[t_j\to\infty]{} \mathfrak{z}_0 \qquad \text{in } \mathscr{Y}, \tag{4.1}$$

where $\mathfrak{z}_0 \in \mathscr{X}$. By Proposition 2.4, there is a solution $\mathfrak{z}(x, t) \in C_b(\mathbb{R}, \mathscr{X})$ to (1.11) with the initial data $\mathfrak{z}|_{t=0} = \mathfrak{z}_0 \in \mathscr{X}$:

$$i\partial_t \mathfrak{z}(x, t) = (-i\boldsymbol{\alpha}\cdot\nabla + \beta m)\mathfrak{z}(x, t) + \rho(x)F(\langle \mathfrak{z}, \rho\rangle), \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R}; \qquad \mathfrak{z}|_{t=0} = \mathfrak{z}_0 \in \mathscr{X}. \tag{4.2}$$

Due to Proposition 2.4, this solution satisfies the bound

$$\sup_{t\in\mathbb{R}} \|\mathfrak{z}(\cdot, t)\|_{\mathscr{X}} < \infty. \tag{4.3}$$

Let S_τ be the time shift operators

$$S_\tau \psi(t) = \psi(\tau + t).$$

By (4.1) and Proposition 2.4(iv), there is the convergence

$$S_{t_j}\psi(t) \xrightarrow[t_j\to\infty]{C(\mathbb{R},\mathscr{Y})} \mathfrak{z}(t), \tag{4.4}$$

since a solution $\psi(t)$ satisfies $S_{t_j}\psi(t) = \mathcal{W}(t)\psi(t_j)$. The convergence (4.4) stands for the uniform convergence on each compact interval $|t| \le R$, $R > 0$.

COROLLARY 4.1. *Let $\psi_0 \in \mathscr{X}$ and $\psi \in C_b(\mathbb{R}, \mathscr{X})$ be the corresponding solution to the Cauchy problem* (1.11). *Then the family of trajectories $\{S_{t_j}\psi(t) = \psi(t_j + t)\}$ is relatively compact in $C(\mathbb{R}, \mathscr{Y})$ for any sequence $t_j \to \infty$.*

DEFINITION 4.2. *A function $\mathfrak{z}(t)$ appearing as the limit in* (4.4) *for some sequence $t_j \to \infty$ is called the* omega-limit trajectory *of the solution ψ.*

To conclude the proof of Theorem 3.3, it suffices to check that every omega-limit trajectory of any solution $\psi \in C(\mathbb{R}, \mathscr{X})$ belongs to the set of solitary waves; that is, for some $\omega_+ \in \mathbb{R}$,

$$\mathfrak{z}(t) = \phi_{\omega_+} e^{-i\omega_+ t}, \qquad t \in \mathbb{R}, \tag{4.5}$$

where $\phi_{\omega_+} \in \mathscr{X}$.

4.2. Splitting off the dispersive component. First we split the solution $\psi(x,t)$ into $\psi(x,t) = \chi(x,t) + \varphi(x,t)$, where χ and φ are defined as solutions to the following Cauchy problems:

$$i\partial_t \chi(x,t) = (-i\boldsymbol{\alpha} \cdot \nabla + \beta m)\chi(x,t), \qquad \chi|_{t=0} = \psi_0, \tag{4.6}$$

$$i\partial_t \varphi(x,t) = (-i\boldsymbol{\alpha} \cdot \nabla + \beta m)\varphi(x,t) + \rho(x) f(t), \qquad \varphi|_{t=0} = 0, \tag{4.7}$$

where ψ_0 is the initial data from (2.1) and

$$f(t) := F(\langle \psi(\cdot,t), \rho \rangle). \tag{4.8}$$

Note that $\langle \psi(\cdot,t), \rho \rangle$ belongs to $C_b(\mathbb{R})$ since $\psi \in C_b(\mathbb{R}, \mathscr{X})$ by Proposition 2.4. Hence,

$$f(\cdot) \in C_b(\mathbb{R}). \tag{4.9}$$

On the other hand, since $\chi(t)$ is a finite charge solution to the free Dirac equation, we also have

$$\chi \in C_b(\mathbb{R}, \mathscr{X}). \tag{4.10}$$

Hence, the function $\varphi(t) = \psi(t) - \chi(t)$ also satisfies

$$\varphi \in C_b(\mathbb{R}, \mathscr{X}). \tag{4.11}$$

Proposition 4.3 (dispersive decay). *For any $\varrho \in \mathscr{S}(\mathbb{R}^n)$, one has* $\lim_{t\to\infty} \|\varrho(\cdot)\chi(\cdot,t)\|_{L^2} = 0$.

Proof. For the Fourier transform of $\chi(x,t)$ in x, we have

$$\hat{\chi}(\xi,t) = \hat{\mathcal{W}}(\xi,t)\hat{\psi}_0(\xi) = e^{-i(\boldsymbol{\alpha}\cdot\xi + \beta m)t}\hat{\psi}_0(\xi).$$

Pick $\epsilon > 0$. We split the initial data ψ_0 into $\psi_0 = \psi_1 + \psi_2$ so that

$$\|\psi_1\|_{L^2} < \epsilon, \qquad \psi_2 \in \mathscr{S}(\mathbb{R}^n, \mathbb{C}^N), \qquad \text{and} \qquad \hat{\psi}_2|_U \equiv 0, \tag{4.12}$$

where $\hat{\psi}_2(\xi)$ is the Fourier transform of $\psi_2(x)$ and U is an open neighborhood of $\xi = 0$. (The size of U depends on ϵ.) Let χ_1 and χ_2 be the solutions to the linear Dirac equation with the initial data

$$\chi_1|_{t=0} = \psi_1, \qquad \chi_2|_{t=0} = \psi_2.$$

Due to (4.12) and the charge conservation, $\|\chi_1(t)\|_{L^2} \le \epsilon$ for $t \in \mathbb{R}$. It suffices to show that

$$\lim_{t\to\infty} \|\varrho(\cdot)\chi_2(\cdot,t)\|_{L^2} = 0. \tag{4.13}$$

We have

$$\|\varrho\chi_2(\cdot,t)\|_{L^2}^2 \le \|\varrho\chi_2(\cdot,t)\|_{L^1}\|\varrho\chi_2(\cdot,t)\|_{L^\infty} \le \|\varrho\|_{L^2}\|\chi_2(\cdot,t)\|_{L^2}\|\varrho\chi_2(\cdot,t)\|_{L^\infty}. \tag{4.14}$$

The first two factors in the right-hand side of (4.14) are bounded uniformly in time. For the last factor in the right-hand side of (4.14), we have

$$\|\varrho(\cdot)\chi_2(\cdot,t)\|_{L^\infty} \le \|\hat\varrho(\cdot)*\hat\chi_2(\cdot,t)\|_{L^1} = \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\hat\varrho(\eta-\xi)\Big(e^{-i(\boldsymbol{\alpha}\cdot\xi+\beta m)t}\hat\psi_2(\xi)\Big)\,d^n\xi\right|\,d^n\eta. \tag{4.15}$$

Using the projectors $\Pi_\pm(\xi)$ onto the eigenspaces of the eigenvalues $\pm\sqrt{\xi^2+m^2}$ of $\hat{\mathscr{D}}(\xi)=\boldsymbol{\alpha}\cdot\xi+\beta m$ and introducing the operator $L=\frac{i\sqrt{\xi^2+m^2}}{t\xi^2}\xi\cdot\nabla_\xi$, we get the identity

$$\begin{aligned} e^{-i(\boldsymbol{\alpha}\cdot\xi+\beta m)t} &= \Pi_+(\xi)e^{-it\sqrt{\xi^2+m^2}}+\Pi_-(\xi)e^{it\sqrt{\xi^2+m^2}} \\ &= \Pi_+(\xi)Le^{-it\sqrt{\xi^2+m^2}}-\Pi_-(\xi)Le^{it\sqrt{\xi^2+m^2}}, \quad \xi\ne 0. \end{aligned}$$

Since $\hat\psi_2\equiv 0$ in an open neighborhood U of $\xi=0$, the denominator of L is never zero on the support of (4.15); hence we can use L to integrate by parts in ξ, gaining the factor t^{-1}. This shows that $\lim_{t\to\infty}\|\varrho\chi_2(\cdot,t)\|_{L^\infty}=0$. Together with (4.14), this yields (4.13), finishing the proof. ∎

Proposition 4.3 yields the decay

$$S_{t_j}\chi(t)\xrightarrow[t_j\to\infty]{C(\mathbb{R},\mathscr{Y})}0. \tag{4.16}$$

5. Complex Fourier–Laplace transform. Let us calculate the Fourier–Laplace transform of $\varphi(x,t)$. Define

$$\Sigma(x,\omega)=\mathcal{F}^{-1}_{\xi\to x}\big[\hat\Sigma(\xi,\omega)\big],\qquad \hat\Sigma(\xi,\omega)=\frac{(\omega+\boldsymbol{\alpha}\cdot\xi+\beta m)\hat\rho(\xi)}{\omega^2-\xi^2-m^2},\qquad \omega\in\mathbb{C}^+, \tag{5.1}$$

where $\mathbb{C}^+=\{\omega\in\mathbb{C}\colon \operatorname{Im}\omega>0\}$. Note that $\Sigma(\cdot,\omega)$ is an analytic function of $\omega\in\mathbb{C}^+$ with the values in $\mathscr{S}(\mathbb{R}^n,\mathbb{C}^N)$.

For $\varepsilon\ge 0$, we denote $\Omega_\varepsilon=\{\omega\in\mathbb{R}\colon |\omega-m|>\epsilon,\quad |\omega+m|>\epsilon\}$, and in particular $\Omega_0=\mathbb{R}\backslash\{\pm m\}$.

Lemma 5.1.

(i) *The smooth function $\Sigma(x,\omega)$ could be extended from $\mathbb{R}^n\times\mathbb{C}^+$ to $\mathbb{R}^n\times\Omega_0$ as the boundary trace*

$$\Sigma(x,\omega)=\lim_{\epsilon\to 0+}\Sigma(x,\omega+i\epsilon),\qquad (x,\omega)\in\mathbb{R}^n\times\Omega_0, \tag{5.2}$$

where the convergence holds in the topology of the space $C^\infty(\mathbb{R}^n\times\Omega_0)$.

(ii) *For any $\varepsilon>0$, each derivative $\partial_x^\alpha\Sigma(x,\omega)$ is a bounded function of $(x,\omega)\in\mathbb{R}^n\times\Omega_\varepsilon$.*

Proof. For each $\omega\in(-m,m)$, one could use the formula (5.1) to define $\Sigma(\cdot,\omega)\in C^\infty(\mathbb{R}^n)$, and the convergence is immediate in dimensions $n\ge 3$. For $|\omega|>m$, it suffices to rewrite $\Sigma(x,\omega)$ as

$$\Sigma(x,\omega)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{e^{i\xi x}(\omega+\boldsymbol{\alpha}\cdot\xi+\beta m)\hat\rho(\xi)\,d^n\xi}{(\omega+i0)^2-\xi^2-m^2}=\int_0^\infty\frac{R(x,\lambda)\,d\lambda}{(\omega+i0)^2-\lambda^2-m^2}, \tag{5.3}$$

where the limits exist by the Sokhotsky–Plemelj formula and

$$R(x,\lambda)=\frac{1}{(2\pi)^n}\int_{|\xi|=\lambda} e^{i\xi x}(\omega+\boldsymbol{\alpha}\cdot\xi+\beta m)\hat\rho(\xi)\,dS. \tag{5.4}$$

The derivatives in x can be considered similarly since the function $\hat\rho(\xi)$ is from the Schwartz class. □

Remark 5.2. Note that $\sigma(\omega)$ defined in (3.4) could be expressed via Σ as $\sigma(\omega)=\langle\Sigma(\cdot,\omega),\rho\rangle$, $\omega\in\Omega_0$.

Further, let us note that $f(t)$ is bounded by (4.9); hence its Fourier–Laplace transform

$$\tilde f(\omega)=\mathcal{F}_{t\to\omega}[\Theta(t)f(t)]:=\int_0^\infty e^{i\omega t}f(t)\,dt$$

is a tempered distribution of $\omega\in\mathbb{R}$. Similarly, by (4.11), the Fourier–Laplace transform $\tilde\varphi(x,\omega)=\mathcal{F}_{t\to\omega}[\Theta(t)\varphi(x,t)]$ is a tempered $\mathscr{X}$-valued distribution of $\omega\in\mathbb{R}$.

Proposition 5.3. *The following relation holds in the sense of distributions:*

$$\tilde\varphi(x,\omega)=\Sigma(x,\omega)\tilde f(\omega),\qquad \omega\in\Omega_0. \tag{5.5}$$

Proof. Abusing notations, we denote by $\tilde f(\omega)$ and $\tilde\varphi(x,\omega)$, $\omega\in\mathbb{C}^+$, the complex Fourier–Laplace transforms of f and φ:

$$\tilde f(\omega)=\mathcal{F}_{t\to\omega}[\Theta(t)f(t)]=\int_0^\infty e^{i\omega t}f(t)\,dt,\qquad \omega\in\mathbb{C}^+,$$

$$\tilde\varphi(x,\omega)=\mathcal{F}_{t\to\omega}[\Theta(t)\varphi(x,t)]:=\int_0^\infty e^{i\omega t}\varphi(x,t)\,dt,\quad \omega\in\mathbb{C}^+,\quad x\in\mathbb{R}^n. \tag{5.6}$$

Due to (4.9), the function $\tilde f(\omega)$ is analytic for $\omega\in\mathbb{C}^+$. Similarly, due to (4.11), $\tilde\varphi(\cdot,\omega)$ is an $\mathscr{X}$-valued analytic function for $\omega\in\mathbb{C}^+$. Equation (4.7) for φ implies that

$$\omega\tilde\varphi(x,\omega)=(-i\boldsymbol{\alpha}\cdot\nabla+\beta m)\tilde\varphi(x,\omega)+\rho(x)\tilde f(\omega),\quad \omega\in\mathbb{C}^+. \tag{5.7}$$

Using the function $\Sigma(x,\omega)$ from (5.1), we can write the solution $\tilde\varphi(x,\omega)$ as follows:

$$\tilde\varphi(x,\omega)=\Sigma(x,\omega)\tilde f(\omega),\qquad \omega\in\mathbb{C}^+. \tag{5.8}$$

Let us justify that the representation (5.8) for $\tilde\varphi(x,\omega)$ is also valid when $\omega\in\mathbb{R}$, $\omega\ne\pm m$, if the multiplication in (5.8) is understood in the sense of distributions. The distribution $\tilde f(\omega)$, $\omega\in\mathbb{R}$ is the boundary trace of the analytic function $\tilde f(\omega)$, $\omega\in\mathbb{C}^+$:

$$\tilde f(\omega)=\lim_{\epsilon\to0+}\tilde f(\omega+i\epsilon),\quad \omega\in\mathbb{R}, \tag{5.9}$$

where the convergence holds in the space of tempered distributions $\mathscr{S}'(\mathbb{R})$. Indeed, $\tilde f(\omega+i\epsilon)=\mathcal{F}_{t\to\omega}[\Theta(t)\varphi(t)e^{-\epsilon t}]$, while $\Theta(t)f(t)e^{-\epsilon t}\underset{\epsilon\to0+}{\longrightarrow}\Theta(t)f(t)$, with the convergence taking place in $\mathscr{S}'(\mathbb{R})$ since the function $\Theta(t)f(t)$ is bounded. Therefore, (5.9) holds by the continuity of the Fourier transform $\mathcal{F}_{t\to\omega}$ in $\mathscr{S}'(\mathbb{R})$. Similarly, $\tilde\varphi(\cdot,\omega)$, $\omega\in\mathbb{R}$ is the boundary trace of the analytic function $\tilde\varphi(\cdot,\omega)$, $\omega\in\mathbb{C}^+$:

$$\tilde\varphi(\cdot,\omega)=\lim_{\epsilon\to0+}\tilde\varphi(\cdot,\omega+i\epsilon),\qquad \omega\in\mathbb{R}, \tag{5.10}$$

where the convergence holds in $\mathscr{S}'(\mathbb{R},\mathscr{X})$. This follows from the convergence $\Theta(t)\varphi(\cdot,t)e^{-\epsilon t} \underset{\epsilon\to 0+}{\longrightarrow} \Theta(t)\varphi(\cdot,t)$ in $\mathscr{S}'(\mathbb{R},\mathscr{X})$ and the continuity of the Fourier transform $\mathcal{F}_{t\to\omega}$ in $\mathscr{S}'(\mathbb{R},\mathscr{X})$. Finally, by Lemma 5.1, for each $x\in\mathbb{R}^n$, $\varSigma(x,\omega+i\epsilon)$ converges in the space of smooth functions of $\omega\in\varOmega_0$ as $\epsilon\to 0+$. Thus, we may pass to the limit $\operatorname{Im}\omega\to 0+$ in (5.8), arriving at (5.5). ∎

Remark 5.4. The relation (5.5) corresponds to the limiting absorption principle in the diffraction theory.

6. Absolutely continuous spectrum. Recall that $\tilde f(\omega)$ on the real line is defined by (5.9) as the trace distribution $\tilde f(\omega)=\tilde f(\omega+i0)$, $\omega\in\mathbb{R}$. We are going to prove that $\tilde f$ is locally integrable for $|\omega|>m$.

PROPOSITION 6.1. $\tilde f\in L^1_{loc}(\mathbb{R}\backslash[-m,m])$.

Proof. We need to prove that

$$\int_I |\tilde f(\omega)|^2\,d\omega<\infty \tag{6.1}$$

for any compact interval I such that $I\cap[-m,m]=\emptyset$. The Parseval identity implies that

$$\int_{\mathbb{R}} \|\tilde\varphi(\cdot,\omega+i\epsilon)\|^2_{L^2}\frac{d\omega}{2\pi}=\int_0^\infty \|\varphi(\cdot,t)\|^2_{L^2}e^{-2\epsilon t}\,dt,\qquad \epsilon>0.$$

Since $\sup_{t\ge 0}\|\varphi(\cdot,t)\|_{L^2}<\infty$ by (4.11), we may bound the right-hand side by c_0/ϵ, with some $c_0>0$. Taking into account (5.8), we arrive at the key inequality

$$\int_{\mathbb{R}} |\tilde f(\omega+i\epsilon)|^2\|\varSigma(\cdot,\omega+i\epsilon)\|^2_{L^2}\,d\omega\le\frac{c_0}{\epsilon}. \tag{6.2}$$

LEMMA 6.2. *Assume that I is a compact interval such that $I\cap[-m,m]=\emptyset$. Then there exists $c_I>0$ such that*

$$\|\varSigma(\cdot,\omega+i\epsilon)\|^2_{L^2}\ge\frac{c_I}{\epsilon},\qquad \omega\in I,\quad 0<\epsilon\le|I|/2. \tag{6.3}$$

Proof. For definiteness, we will consider the case $I\subset(m,+\infty)$. In terms of $\hat\rho_\pm=\Pi_\pm(\xi)\hat\rho(\xi)$, which are eigenvectors of $\omega+\boldsymbol{\alpha}\cdot\xi+\beta m$, the function $\hat\varSigma(\xi,\omega)$, $\omega\in\mathbb{C}^+$, could be expressed as

$$\begin{aligned}\hat\varSigma(\xi,\omega)&=\frac{(\omega+\sqrt{\xi^2+m^2})\hat\rho_+(\xi)+(\omega-\sqrt{\xi^2+m^2})\hat\rho_-(\xi)}{\omega^2-\xi^2-m^2}\\&=\frac{\hat\rho_+(\xi)}{\omega-\sqrt{\xi^2+m^2}}+\frac{\hat\rho_-(\xi)}{\omega+\sqrt{\xi^2+m^2}}.\end{aligned} \tag{6.4}$$

Using the relation (6.4) and mutual orthogonality of $\hat\rho_+$ and $\hat\rho_-$ with respect to the L^2-product, we obtain

$$\begin{aligned}\|\varSigma(\cdot,\omega)\|^2_{\mathscr{X}}&=\int_{\mathbb{R}^n}|\hat\varSigma(\xi,\omega)|^2\frac{d^n\xi}{(2\pi)^n}\\&=\int_{\mathbb{R}^n}\left(\frac{|\hat\rho_+(\xi)|^2}{|\omega-\sqrt{\xi^2+m^2}|^2}+\frac{|\hat\rho_-(\xi)|^2}{|\omega+\sqrt{\xi^2+m^2}|^2}\right)\frac{d^n\xi}{(2\pi)^n},\qquad\omega\in\mathbb{C}^+.\end{aligned} \tag{6.5}$$

Hence, for $\omega \in \mathbb{R}$ and $\epsilon > 0$,

$$\|\Sigma(\cdot,\omega+i\epsilon)\|^2_{\mathscr{X}} \ge \int_0^\infty \left[\int_{|\xi|=\lambda} \frac{|\hat\rho_+(\xi)|^2}{(\omega-\sqrt{\xi^2+m^2})^2+\epsilon^2} dS\right] d\lambda = \int_m^\infty \frac{\mathscr{R}_+(\nu)\,d\nu}{|\omega-\nu|^2+\epsilon^2}, \tag{6.6}$$

where $\nu = \sqrt{\lambda^2+m^2}$, $\nu\,d\nu = \lambda\,d\lambda$, and $\mathscr{R}_+(\nu)$ is a continuous function of $\nu \in (m,+\infty)$ defined by

$$\mathscr{R}_+(\nu) := \frac{\nu}{\sqrt{\nu^2-m^2}} \int_{|\xi|=\sqrt{\nu^2-m^2}} |\hat\rho_+(\xi)|^2 dS, \qquad \nu > m.$$

Note that

$$r_+(I) := \inf_{\nu\in I} \mathscr{R}_+(\nu) > 0. \tag{6.7}$$

Indeed, $r_+(I) = 0$ would imply that $\hat\rho_+(\xi) = 0$ for $|\xi| = \sqrt{\nu^2-m^2}$, with some $\nu \in I$, contradicting Assumption B(i).

For any $\epsilon \in (0, \frac{1}{2}|I|)$, the inequality (6.6) yields

$$\|\Sigma(\cdot,\omega+i\epsilon)\|^2_{\mathscr{X}} \ge r_+(I) \int_I \frac{d\nu}{|\omega-\nu|^2+\epsilon^2} \ge r_+(I) \int_{I\cap[\omega-\epsilon,\omega+\epsilon]} \frac{d\nu}{\epsilon^2} \ge \frac{r_+(I)}{\epsilon}, \qquad 0 < \epsilon \le |I|/2. \tag{6.8}$$

In the first inequality we used (6.7). The last inequality is due to $|I\cap[\omega-\epsilon,\omega+\epsilon]| \ge \epsilon$, which follows from $\omega \in I$ and $\epsilon < |I|/2$. $\square$

Substituting (6.3) into (6.2), we obtain the bound

$$\int_I |\tilde f(\omega+i\epsilon)|^2\,d\omega \le c_0/c_I, \qquad 0 < \epsilon \le \epsilon_I. \tag{6.9}$$

We conclude that the set of functions $g_\epsilon(\omega) = \tilde f(\omega+i\epsilon)$, $0 < \epsilon \le \epsilon_I$, defined for $\omega \in I$, is bounded in the Hilbert space $L^2(I)$ and, by the Banach theorem, is weakly compact. Hence, the convergence of the distributions (5.9) implies the weak convergence $g_\epsilon \xrightarrow[\epsilon\to 0+]{} g$ in the Hilbert space $L^2(I)$. The limit function $g(\omega)$ coincides with the distribution $\tilde f(\omega)$ restricted onto I. This proves the bound (6.1) and finishes the proof of the proposition. $\square$

7. Nonlinear spectral analysis.

Compactness of the spectrum. We denote $g(t) = F(\langle \mathfrak{z}(t),\rho\rangle)$, where $\mathfrak{z}(t)$ is an omega-limit trajectory from (4.4).

PROPOSITION 7.1. $\operatorname{supp} \tilde g \subset [-m, m]$.

Proof. According to (4.4), the time shifts $S_{t_j}\psi(t) = \psi(t_j+t)$ converge to $\mathfrak{z}$ in the topology of $C(\mathbb{R},\mathscr{Y})$ (that is, uniformly on each compact set $|t| \le R$). Hence (4.9) implies that

$$S_{t_j} f(t) = F(\langle S_{t_j}\psi(t),\rho\rangle) \xrightarrow[t_j\to\infty]{\mathscr{S}'(\mathbb{R})} F(\langle \mathfrak{z}(t),\rho\rangle) := g(t). \tag{7.1}$$

By the continuity of the Fourier transform in $\mathscr{S}'(\mathbb{R})$, we also have

$$e^{-i\omega t_j}\tilde{f}(\omega) \xrightarrow[t_j\to\infty]{\mathscr{S}'(\mathbb{R})} \tilde{g}(\omega). \tag{7.2}$$

Since the distribution $\tilde{f}$ is absolutely continuous for $\omega \in \mathbb{R}\backslash[-m,m]$ by Proposition 6.1, while $t_j \to \infty$, the Riemann–Lebesgue lemma shows that $\operatorname{supp}\tilde{g} \subset [-m,m]$. □

Spectral inclusion. Recall that ω_σ is the only zero of $\sigma(\omega)$ on $[-m,m]$ (if such zero exists at all).

PROPOSITION 7.2. $\operatorname{supp}\tilde{g} \subset \operatorname{supp}\langle\tilde{\mathfrak{z}}(\cdot,\omega),\rho\rangle \cup \omega_\sigma$.

Proof. The convergence (4.4), together with (4.16) prove that

$$S_{t_j}\varphi \xrightarrow[t_j\to\infty]{C(\mathbb{R},\mathscr{Y})} \mathfrak{z}. \tag{7.3}$$

Hence, taking the Fourier transform,

$$e^{-i\omega t_j}\tilde{\varphi}(\omega) \xrightarrow[t_j\to\infty]{\mathscr{S}'(\mathbb{R},\mathscr{Y})} \tilde{\mathfrak{z}}(\omega). \tag{7.4}$$

On the other hand, multiplying both sides of (5.5) by $e^{-i\omega t_j}$, we get

$$e^{-i\omega t_j}\tilde{\varphi}(x,\omega) = \Sigma(x,\omega)e^{-i\omega t_j}\tilde{f}(\omega), \qquad (x,\omega)\in\mathbb{R}^n\times\Omega_0.$$

Sending j to infinity and taking into account (7.2), (7.3), and the fact that $\Sigma(x,\omega)$ is smooth in ω away from $\omega\neq\pm m$ (and hence a multiplicator in the space of distributions of ω away from $\pm m$), we obtain the following relation, which holds in the sense of distributions:

$$\tilde{\mathfrak{z}}(x,\omega) = \Sigma(x,\omega)\tilde{g}(\omega), \qquad (x,\omega)\in\mathbb{R}^n\times\Omega_0. \tag{7.5}$$

Taking the pairing of (7.5) with ρ and using Remark 5.2, we get

$$\langle\tilde{\mathfrak{z}}(\cdot,\omega),\rho\rangle = \sigma(\omega)\tilde{g}(\omega), \qquad \omega\in\Omega_0. \tag{7.6}$$

First we will prove Proposition 7.2 modulo the discrete set $\{\pm m\}$.

LEMMA 7.3. $\operatorname{supp}\tilde{g} \subset \{\pm m\} \cup \operatorname{supp}\langle\tilde{\mathfrak{z}}(\cdot,\omega),\rho\rangle \cup \omega_\sigma$.

Proof. By Proposition 7.1, $\operatorname{supp}\tilde{g}\subset[-m,m]$. Thus, the statement of the lemma follows from (7.6) and from Assumption B that $\sigma(\omega)$ is nonzero for $\omega\in[-m,m]\backslash\omega_\sigma$. □

It remains to prove the following lemma.

LEMMA 7.4. *If* $\omega_0\in\{\pm m\}\backslash\omega_\sigma$ *belongs to* $\operatorname{supp}\tilde{g}$*, then also* $\omega_0\in\operatorname{supp}\langle\tilde{\mathfrak{z}},\rho\rangle$.

Proof. In the case when $\omega_0\in\{\pm m\}$ is not an isolated point in $\operatorname{supp}\tilde{g}$ (this implies that $\omega_0\in\{\pm m\}$), the relation (7.6) together with $\sigma(\omega)$ being nonzero for $\omega\in[-m,m]\backslash\omega_\sigma$ show that $\omega_0\in\operatorname{supp}\langle\tilde{\mathfrak{z}},\rho\rangle$ since the support is a closed set.

We are left to consider the case when $\omega_0\in\{\pm m\}\backslash\omega_\sigma$ is an isolated point of $\operatorname{supp}\tilde{g}$. In this case our proof relies on the Fourier transform of (4.2):

$$\omega\tilde{\mathfrak{z}}(x,\omega) = (-i\boldsymbol{\alpha}\cdot\nabla+\beta m)\tilde{\mathfrak{z}}(x,\omega)+\rho(x)\tilde{g}(\omega). \tag{7.7}$$

We pick an open neighborhood $U \subset \mathbb{R}$ of ω_0 such that $U \cap \operatorname{supp} \tilde{g} = \omega_0$ and consider (7.7) for $\omega \in U$. Pick $\zeta \in C_0^\infty(\mathbb{R})$, $\operatorname{supp} \zeta \subset U$, such that $\zeta(\omega_0) = 1$. First we note that

$$\zeta(\omega)\tilde{g}(\omega) = G_0\, \delta(\omega - \omega_0), \qquad G_0 \in \mathbb{C}\backslash\{0\}, \tag{7.8}$$

where the derivatives of the $\delta(\omega - \omega_0)$ are prohibited in the right-hand side because the left-hand side is the Fourier transform of the bounded function $\check{\zeta} * g(t)$, where $\check{\zeta}(t)$ denotes the inverse Fourier transform of $\zeta(\omega)$. By (7.5), we have $U \cap \operatorname{supp}_\omega \tilde{\mathfrak{z}} \subset \omega_0$; hence

$$\zeta(\omega)\tilde{\mathfrak{z}}(x, \omega) = \delta(\omega - \omega_0)b(x), \qquad b \in \mathscr{X}. \tag{7.9}$$

Again, the terms with the derivatives of $\delta(\omega - \omega_0)$ are prohibited since the left-hand side is the Fourier transform of the bounded $\mathscr{X}$-valued function by (4.3), while the inclusion $b \in \mathscr{X}$ is due to $\tilde{\mathfrak{z}} \in \mathscr{S}'(\mathbb{R}, \mathscr{X})$. Multiplying the Fourier transform of (7.7) by $\zeta(\omega)$ and taking into account (7.8) and (7.9), we get

$$\omega_0 b(x) = (-i\boldsymbol{\alpha} \cdot \nabla + \beta m)b(x) + G_0\rho(x).$$

The solution is given by

$$b(x) = G_0 \varSigma(x, \omega_0), \tag{7.10}$$

where $\varSigma(x, \omega_0)$ is given by (5.1):

$$\varSigma(x, \omega_0) = \mathcal{F}^{-1}_{\xi\to x}\big[\hat{\varSigma}(\xi, \omega_0)\big], \qquad \hat{\varSigma}(\xi, \omega_0) = -\frac{(\omega + \boldsymbol{\alpha} \cdot \xi + \beta m)\hat{\rho}(\xi)}{\xi^2}, \tag{7.11}$$

since $\omega_0^2 = m^2$. Hence, $\varSigma(x, \omega_0) \in \mathscr{X}$ if $n \geq 5$. For $n = 3$ and 4 , we should consider two different cases: $\varSigma(x, \omega_0) \in \mathscr{X}$ and $\varSigma(x, \omega_0) \notin \mathscr{X}$. The latter takes place when $\hat{\rho}(0) \neq 0$. In our situation, the case $\varSigma(x, \omega_0) \notin \mathscr{X}$ is impossible since the formula (7.10) would then be in contradiction with the inclusion $b \in \mathscr{X}$.

Thus, $\varSigma(x, \omega_0) \in \mathscr{X}$, and hence $\sigma(\omega_0) = \langle \varSigma(\cdot, \omega_0), \rho\rangle$ is finite. Coupling (7.9) with ρ and then using (7.10), we obtain

$$\begin{aligned} \zeta(\omega)\big\langle \tilde{\mathfrak{z}}(\cdot, \omega), \rho\big\rangle &= \delta(\omega - \omega_0)\big\langle b, \rho\big\rangle \\ &= G_0\delta(\omega - \omega_0)\big\langle \varSigma(\cdot, \omega), \rho\big\rangle = G_0\delta(\omega - \omega_0)\sigma(\omega_0). \end{aligned} \tag{7.12}$$

For $\omega_0 \in \{\pm m\}\backslash\omega_\sigma$, one has $\sigma(\omega_0) \neq 0$ by Assumption B; hence we see that $\omega_0 \in \operatorname{supp}\langle\tilde{\mathfrak{z}}, \rho\rangle$. ◻

Lemmas 7.3 and 7.4 show that $\operatorname{supp} \tilde{g}(\omega) \subset \operatorname{supp}\langle\tilde{\mathfrak{z}}(\cdot, \omega), \rho\rangle \cup \omega_\sigma$, finishing the proof of Proposition 7.2. ◻

Reduction to one frequency. Finally, we reduce the spectrum of $\gamma(t) := \langle\mathfrak{z}(\cdot, t), \rho\rangle$ to one point using the nonlinear inflation of spectrum.

Proposition 7.5. $\operatorname{supp} \tilde{\gamma}(\omega) \subset \omega_+$ *for some* $\omega_+ \in [-m, m]$.

Proof. By (7.1) and (2.3), (2.4), $g(t) = F(\gamma(t)) = \sum_{k=0}^p a_k|\gamma(t)|^{2k}\gamma(t)$. Then, by the Titchmarsh convolution theorem [43],

$$\begin{aligned} \sup \operatorname{supp} \tilde{g} &= \max_{k\in\{k\leq p,\, a_k\neq 0\}} \sup \operatorname{supp} \underbrace{(\bar{\tilde{\gamma}} * \tilde{\gamma}) * \cdots * (\bar{\tilde{\gamma}} * \tilde{\gamma})}_{k} * \tilde{\gamma} \\ &= p \cdot \sup \operatorname{supp} \tilde{\gamma} + (p - 1) \cdot \sup \operatorname{supp} \bar{\tilde{\gamma}}. \end{aligned} \tag{7.13}$$

Remark 7.6. The Titchmarsh convolution theorem applies because $\operatorname{supp}\tilde\gamma$ is compact by Proposition 7.1.

Noting that $\sup\operatorname{supp}\tilde{\bar{\gamma}} = -\inf\operatorname{supp}\tilde\gamma$, we rewrite (7.13) as

$$\sup\,\operatorname{supp}\tilde g = \sup\,\operatorname{supp}\tilde\gamma + (p-1)(\sup\operatorname{supp}\tilde\gamma - \inf\operatorname{supp}\tilde\gamma). \tag{7.14}$$

Similarly,

$$\inf\,\operatorname{supp}\tilde g = \inf\,\operatorname{supp}\tilde\gamma - (p-1)(\sup\operatorname{supp}\tilde\gamma - \inf\operatorname{supp}\tilde\gamma). \tag{7.15}$$

Taking into account Proposition 7.2, we see that if $\sup\operatorname{supp}\tilde\gamma > \inf\operatorname{supp}\tilde\gamma$, then, by (7.14) and (7.15), the support of $\tilde g$ contains at least two points which do not belong to the support of $\tilde\gamma$. This contradicts Proposition 7.2, which allows at most one point, ω_σ, to be in the support of $\tilde g$ but not in the support of $\tilde\gamma$.

Thus, $\sup\operatorname{supp}\tilde\gamma = \inf\operatorname{supp}\tilde\gamma$, showing that $\operatorname{supp}\tilde\gamma \subset \omega_+$ for some $\omega_+ \in [-m,m]$. ◻

Conclusion of the proof of Theorem 3.3. We need to prove (4.5). As follows from Proposition 7.5, $\tilde\gamma(\omega)$ is a finite linear combination of $\delta(\omega-\omega_+)$ and its derivatives. As the matter of fact, the derivatives could not be present because of the boundedness of $\gamma(t) := \langle \mathfrak{z}(\cdot,t),\rho\rangle$ that follows from (4.3). Therefore, $\tilde\gamma = C\,\delta(\omega-\omega_+)$. This implies the following identity:

$$\gamma(t) = C_1 e^{-i\omega_+ t}, \qquad t\in\mathbb{R}. \tag{7.16}$$

It follows that $\tilde g(\omega) = C_2\,\delta(\omega-\omega_+)$, and the representation (7.5) implies that $\mathfrak{z}(x,t) = \mathfrak{z}(x,0)e^{-i\omega_+ t}$. Due to (4.2) and the bound (4.3), $\mathfrak{z}(x,t)$ is a solitary wave solution. Note that $\omega_+\in[-m,m]$ since the corresponding solitary wave is to be of finite charge (cf. Lemma A.1 in Appendix A). This completes the proof of Theorem 3.3.

8. Generalizations. Here we show how one could relax Assumption B.

DEFINITION 8.1. *The set $M\subset\{\pm m\}$ is defined as follows:*

For $n\le 4$, $\pm m\in M$ if $\hat\rho_\pm(\xi)$ vanishes at $\xi=0$ of order at least $\left[3-\frac{n}{2}\right]$.

For $n\ge 5$, $M=\{\pm m\}$.

DEFINITION 8.2. *The set $Z_\rho\subset\mathbb{R}\backslash[-m,m]$ is defined as follows: $\omega>m$ (respectively, $\omega<-m$) belongs to Z_ρ if $\hat\rho_+(\xi)$ (respectively, $\hat\rho_-(\xi)$) vanishes identically on the sphere $|\xi|=\sqrt{\omega^2-m^2}$.*

Let us note that frequencies of all finite charge solitary waves lie in $(-m,m)\cup M\cup Z_\rho$; see Appendix A. Let us also note that $M=\{\pm m\}$ and $Z_\rho=\emptyset$ if Assumption B is satisfied.

Here is the relaxed version of Assumption B.

Assumption C.

(i) The set Z_ρ is discrete and finite;

(ii) $\sigma(\omega)\ne 0$ for all $\omega\in(-m,m)\cup M\cup Z_\rho$, except perhaps at one point which will be denoted ω_σ.

Let us mention that $\sigma(\omega)$ is well defined at the points $(-m,m)\cup M\cup Z_\rho$. Indeed, the regularization in the integral (3.4) makes it convergent for all $\omega\ne\pm m$. The finiteness of $\sigma(\omega)$ for $\omega\in M$ follows from the vanishing of $\hat\rho_\pm$ specified in Definition 8.1, as it could be seen from the expression for $\sigma(\omega)$ in terms of $\hat\rho_\pm(\xi)$,

$$\sigma(\omega) = \int_{\mathbb{R}^n}\left(\frac{|\hat\rho_+(\xi)|^2}{\omega-\sqrt{\xi^2+m^2}} + \frac{|\hat\rho_-(\xi)|^2}{\omega+\sqrt{\xi^2+m^2}}\right)\frac{d^n\xi}{(2\pi)^n}.$$

Theorem 3.3 remains true if Assumption B is substituted by Assumption C.

THEOREM 8.3. *Let $n \geq 1$. Assume that the nonlinearity $F(z)$ satisfies Assumption* A *and that the coupling function $\rho(x)$ satisfies Assumption* C. *Then for any $\psi_0 \in \mathscr{X}$ the corresponding solution $\psi(t) \in C(\mathbb{R}, \mathscr{X})$ to the Cauchy problem* (2.1) *converges to $\mathbf{S}$ in the space $\mathscr{Y}$:*

$$\lim_{t\to\pm\infty} \operatorname{dist}_{\mathscr{Y}}(\psi(t), \mathbf{S}) = 0, \tag{8.1}$$

where $\operatorname{dist}_{\mathscr{Y}}(\psi, \mathbf{S}) := \inf_{\boldsymbol{s}\in\mathbf{S}} \|\psi - \boldsymbol{s}\|_{\mathscr{Y}}$. *See Figure* 1.

Here are the modifications of propositions and lemmas which one needs when proving Theorem 3.3 under Assumption C instead of Assumption B. Let us note that the proofs require only slight modifications.

PROPOSITION 6.1′. $\tilde f \in L^1_{loc}(\mathbb{R}\backslash([-m,m]\cup Z_\rho))$.

That is, now $\tilde f$ is locally integrable if $I \subset \mathbb{R}\backslash[-m,m]$ has no points of Z_ρ. Indeed, by Lemma A.1 from Appendix A, at these points one has $\varSigma(\cdot,\omega) \in \mathscr{X}$; hence the inequality (6.3) breaks down at Z_ρ.

Proposition 6.1′ leads to the following version of Proposition 7.1.

PROPOSITION 7.1′. $\operatorname{supp}\tilde g \subset [-m,m]\cup Z_\rho$.

The statement of Proposition 7.2 remains unchanged.

PROPOSITION 7.2′. $\operatorname{supp}\tilde g \subset \{\omega_\sigma\} \cup \operatorname{supp}\langle\tilde{\mathfrak{z}}(\cdot,\omega),\rho\rangle$.

Indeed, by the spectral relation (7.6) and Assumption C that $\sigma(\omega)$ does not vanish on $(-m,m)\cup Z_\rho$ except at most at one point ω_σ, we see that Lemma 7.3 remains valid.

LEMMA 7.3′. $\operatorname{supp}\tilde g \subset \{\pm m\} \cup \operatorname{supp}\langle\tilde{\mathfrak{z}}(\cdot,\omega),\rho\rangle \cup \omega_\sigma$.

There is a small modification in the proof of Lemma 7.4, since now one could have $\varSigma(\cdot,\omega) \notin \mathscr{X}$ for $\omega \in \{\pm m\}$.

LEMMA 7.4′. *If $\omega_0 \in \{\pm m\}\backslash\omega_\sigma$ belongs to* $\operatorname{supp}\tilde g$, *then also* $\omega_0 \in \operatorname{supp}\langle\mathfrak{z},\rho\rangle$.

Proof. If $\omega_0 \in M\backslash\omega_\sigma$ is not an isolated point in the support of $\tilde g$, then $\omega_0 \in \operatorname{supp}\langle\mathfrak{z},\rho\rangle$ by Lemma 7.3′. If it is an isolated point in $\operatorname{supp}\tilde g$, then $\omega_0 \in \operatorname{supp}\langle\mathfrak{z},\rho\rangle$ by the same arguments as in Lemma 7.4. Finally, if $\omega_0 \in \{\pm m\}\backslash M$, then we could conclude that $\varSigma(\cdot,\omega_0) \in \mathscr{X}$ by (7.9) and (7.10); on the other hand, this would contradict the inclusion $\omega_0 \in \{\pm m\}\backslash M$ (see Lemma A.1 from Appendix A). ∎

Thus, $\operatorname{supp}\tilde g$ could be larger than $\operatorname{supp}\langle\tilde{\mathfrak{z}}(\cdot,\omega),\rho\rangle$ by at most one point, ω_σ. As in Proposition 7.5, the Titchmarsh theorem allows us to conclude that this can only happen when the support of $\langle\tilde{\mathfrak{z}}(\cdot,\omega),\rho\rangle$ consists of at most one point $\omega_+ \in M \cup Z_\rho$, which implies the representation (4.5). Hence, Theorem 8.3 follows.

Appendix A. Solitary waves. Let us describe the solitary wave solutions to (1.11).

LEMMA A.1. *For $\omega \in \mathbb{R}$, the function $\varSigma(x,\omega) \in \mathscr{X}$ if and only if $\omega \in (-m,m)\cup M \cup Z_\rho$.*

Recall that the sets M and Z_ρ are introduced in Definitions 8.1 and 8.2.

Proof. By (6.5), $\|\varSigma(\cdot,\omega+i0)\|_{\mathscr{X}}$ is finite at a point $\omega > m$ (respectively, at $\omega < -m$) if and only if $\hat\rho_+(\xi)$ (respectively, $\hat\rho_-(\xi)$) vanishes identically on the sphere $|\xi| = \sqrt{\omega^2 - m^2}$. By Definition 8.2, this is precisely the condition for $\omega \in Z_\rho$.

For $\omega = \pm m$, the integral in the right-hand side of (6.5) is convergent if and only if $\hat\rho_\pm$ vanishes at $\xi = 0$ of the order specified in Definition 8.1 ; hence $\varSigma(\cdot,\pm m+i0) \in \mathscr{X}$ if and only if $\pm m \in M$. ∎

Remark A.2. Let us mention that the regularization $\varSigma(\cdot,\omega) = \varSigma(\cdot,\omega+i0)$ for $\omega \in \mathbb{R}$, which we introduced in (5.2), is not necessary for $\omega \in (-m,m)\cup M \cup Z_\rho$.

The following proposition shows that, given $\rho \in \mathscr{S}(\mathbb{R}^n)$, one can always choose a nonlinearity $F(z) = a(|z|^2)z$ so that there are nontrivial solitary wave solutions.

PROPOSITION A.3. *Assume that* $F(z) = a(|z|^2)z$ *satisfies Assumption* A *and that* $\rho \in \mathscr{S}(\mathbb{R}^n, \mathbb{C}^N)$, $\rho \not\equiv 0$. *Let*

$$\Omega = \{\omega \in (-m, m) \cup M \cup Z_\rho\colon\ \sigma(\omega)a(s) = 1 \quad \textit{for some } s > 0\},$$

where $\sigma(\omega)$ *is defined in* (3.4). *Then for each* $\omega \in \Omega$ *there is a nonzero solitary wave solution* $\phi(x)e^{-i\omega t}$ *to* (2.1).

Proof. Substituting the ansatz $\phi(x)e^{-i\omega t}$ into (1.11), we get the following equation on ϕ:

$$\omega\phi(x) = (-i\boldsymbol{\alpha}\cdot\nabla + \beta m)\phi(x) + \rho(x)F(\langle\phi, \rho\rangle), \qquad x \in \mathbb{R}^n. \tag{A.1}$$

Therefore, ϕ satisfies the relation $(\omega - \boldsymbol{\alpha}\cdot\xi - \beta m)\hat{\phi}(\xi) = \hat{\rho}(\xi)F(\langle\phi, \rho\rangle)$, which implies that $\phi(x) = C\Sigma(x, \omega)$, with some $C \in \mathbb{C}$. By Lemma A.1, $\Sigma(\cdot, \omega) \in \mathscr{X}$ if and only if $\omega \in (-m, m) \cup M \cup Z_\rho$. Substituting $\phi(x) = C\Sigma(x, \omega)$ into (A.1) and using (2.3), we obtain the following condition on C:

$$\sigma(\omega)a(|C|^2|\sigma(\omega)|^2) = 1. \tag{A.2}$$

Thus, for $\omega \in \mathbb{R}$, there is a nonzero solitary wave if and only if $\Sigma(x, \omega) \in \mathscr{X}$ and (A.2) has a solution $C \in \mathbb{C}\backslash 0$; that is, if and only if $\omega \in \Omega$. ∎

Remark A.4. For some ω, (A.2) could have roots C which not only differ by a unitary factor but also have different magnitude. Each such root corresponds to a nonzero solitary wave.

Remark A.5. By Proposition A.3, for any $\rho \in \mathscr{S}(\mathbb{R}^n)$ one can choose a polynomial $a(s)$ so that there are nonzero solitary wave solutions to (1.11).

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