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On transitions to stationary states in Hamiltonian nonlinear wave equations

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Abstract

We consider the convergence to stationary states of all finite energy solutions to nonlinear wave equations without dissipation in the long-time limits $t \to \pm\infty$. The investigation is inspired by Bohr's postulate on the transitions between stationary states, by de Broglie's wave-particle duality, and by radiative damping in classical electrodynamics. © 1998 Published by Elsevier Science B.V.

1. Introduction. Stabilization in conservative wave equations

It is very surprising that the stationary states play an outstanding role in many phenomena described by reversible conservative wave equations. The permanent reproduction of the stationary states suggests the convergence

$$Y(t) \to S^{\pm}, \quad \text{as } t \to \pm\infty \qquad (1.1)$$

of all finite energy solutions $Y(t)$ to some stationary states $S^{\pm}$, depending on the solution $Y(t)$. We call such a long-time behavior *stabilization*. It means that the set $\mathcal{S}$ of all stationary states is a point attractor of the corresponding wave equation,

$$Y(t) \to \mathcal{S}, \quad \text{as } t \to \pm\infty. \qquad (1.2)$$

On the other hand, such a convergence seems to be very paradoxical and non-compatible with the reversibility and the conservativeness of the dynamics. The problems of such type that have been known for a long time are, for instance, the radiative damping in classical electrodynamics [15,24], Bohr's transitions between stationary states [6], de Broglie's wave-particle duality [7] in quantum theory, the stability of the shock waves in gas and fluid dynamics, and so on. Moreover, Schrödinger [45] relates the paradox of the reproduction of genes to the "Heitler-London theory," which is the theory of Bohr's transitions of molecules to the stationary states.

We establish the stabilization (1.1) and (1.2) for 1D nonlinear wave equations on the real line $\mathbb{R}$ with a space-localized nonlinear term, for the scalar field coupled to a particle in three-dimensional space $\mathbb{R}^3$, and for the Maxwell-Lorentz system of the Maxwell field coupled to a particle in $\mathbb{R}^3$. The results for the field-particle systems lead, for instance, to the solution of the problem of the radiative damping for one particle.

Let us note that the convergence of type (1.1) and (1.2) to stationary states is one of the main results

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of the theory of attractors for the dissipative nonlinear partial derivative equations [2,21,51]. The convergence holds then in the global energy norm (for $t \to +\infty$ only).

We consider Hamiltonian wave equations without dissipation and restrict our attention to the finite energy solutions. Then the convergence (1.1) and (1.2) in the global energy norm is in general impossible due to the energy conservation, and the vital question is to find the topology in the phase space in which the convergence holds. This is one of the main results of our investigation (very surprising and quite natural at the same time) that an appropriate topology is the Fréchet topology defined by local energy seminorms; this topology seems to be optimal.

The linearization followed by the perturbation adjustments is a powerful method for the analysis of nonlinear problems. However, the problems of the long-time behavior cannot be solved in this way if the trajectory is not close to a known solution; in general, the latter is the case in the cases of the problems of Bohr's transitions, of the wave–particle duality, and of the radiative damping. Then a nonperturbative approach to the problems is necessary [23]. The key role here is played by the scattering of the energy to infinity, discovered initially in linear and nonlinear scattering theory [9,18,19,22,25,35,36,40,42,46,49,50,52,53]. The scattering plays the role of dissipation and provides the convergence (1.1) and (1.2). The convergence fails in general for the problems in a bounded region due to the reflections of the waves from the boundary.

We do not consider the solutions with infinite energy. Their long-time asymptotics essentially depend on the behavior of the initial data at infinity. Let us note that space-periodic ingoing electromagnetic waves result in time-periodic radiations in the photoeffect and in the Compton effect.

In the next section we discuss the physical problems which have inspired our investigation, and list previously known results. In the last section we state our recent results on the stabilization.

2. Related physical problems and known results

2.1. Bohr's transitions between stationary states

Bohr's postulate [6] says that the quantum system is almost always in a stationary state and sometimes makes "jumps" between two such states,

$$S^- \mapsto S^+. \tag{2.1}$$

The postulate suggests the asymptotics of type (1.1) with a short time of relaxation, and the definition of quantum stationary states as long-time asymptotics of the solutions to the dynamical equations. Let us note that the quantum stationary states in the Schrödinger–Dirac theory are the "eigenfunctions," i.e. the solutions of type $(\phi(x), \exp(i\omega t)\psi(x))$ to the coupled Maxwell–Dirac system, [11,44] (the existence of the solutions is proved in Ref. [13]). Hence, more precisely, the transitions (2.1) suggest the long-time asymptotics similar to (1.1),

$$(\phi(x,t), \psi(x,t)) - (\phi^{\pm}(x), \exp(i\omega^{\pm}t)\psi^{\pm}(x)) \to 0, \quad \text{as } t \to \pm\infty. \tag{2.2}$$

Such asymptotics seem to be related to the global gauge group $U(1)$ of the Maxwell–Dirac system (they have not been proved yet for any actual quantum system).

The Yang–Mills system of the strong interaction is invariant with respect to the symmetry group $SU(2)$ [57] or to $SU(3)$ [17], the system of the electro-weak interaction with respect to the group $SU(2) \times U(1)$ (see Ref. [54] and others). Generally, for such systems with symmetry there exist solitary waves [20], i.e. solutions of the form $\exp(i\Omega t)\Psi(x)$, where Ω is an element of the Lie algebra of the corresponding symmetry group. For instance, the eigenfunction $(\phi(x), \exp(i\omega t)\psi(x))$ is such a solitary wave, and the corresponding matrix Ω has the eigenvalues ω and zero. For the solitary wave $\exp(i\Omega t)\Psi(x)$ the asymptotics (2.2) do not hold if the matrix Ω has distinct nonzero eigenvalues $\omega_1 \neq \omega_2$. The solitary waves probably play the role of the Schrödinger eigenfunctions in the asymptotics (2.2) for the solutions to the systems with the symmetry groups. This is in accordance with the experimentally known relations between the elementary particles and the Lie algebras [17].

Our investigation is inspired by these relations of the quantum problems to the long-time asymptotics. There are still no results on the asymptotics of type (2.2) for the equations with nontrivial symmetry groups, even with the Abelian group $U(1)$. Our results on the stabilization (1.1) correspond to the case of the trivial symmetry group $G = \{e\}$.

Previously known results on the stabilization (1.1) and (1.2) in the Fréchet topology concern relativistic-invariant nonlinear wave equations

$$\ddot{u}(x,t) = \Delta u(x,t) - m^2 u(x,t) + f(u(x,t)), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}. \tag{2.3}$$

The nonlinear term $f(u)$ satisfies $f(0) = f'(0) = 0$ and the "stability" assumption

$$u \cdot f(u) \leqslant 0 \quad \text{for } u \in \mathbb{R}. \tag{2.4}$$

This assumption and some power estimates on $|f(u)|$ provide the local energy decay

$$\|(u(\cdot,t),\dot{u}(\cdot,t))\|_R^2 = \int_{|x|\leqslant R} (|\dot{u}(x,t)|^2 + |\nabla u(x,t)|^2 + |u(x,t)|^2)\,\mathrm{d}^3x \to 0, \tag{2.5}$$
$$\text{as } t \to \pm\infty, \quad \forall R > 0.$$

This justifies the stabilization (1.1) and (1.2), since $\mathcal{S} = \{0\}$ due to (2.4). The decay (2.5) was established at first in the scattering theory for linear problems, [35,36,52,53] and it was extended in Refs. [9,18,19,40,42,46,49,50] to the nonlinear equations (2.3) satisfying the assumption (2.4). The decay is established in Refs. [22,25] for small initial data without the assumption (2.4) on nonlinear term.

The long-time asymptotics of all finite energy solutions to nonlinear wave equations without the assumption (2.4) or with $\mathcal{S} \neq \{0\}$ were not considered previously. However, the absence of the local energy decay for some solutions to the equations without the assumption (2.4) was observed in Refs. [10,43]. We prove the asymptotics (1.1) and (1.2) for the wave equations and systems with the set $\mathcal{S}$ containing arbitrary finite or infinite number of stationary points or containing continuous components.

2.2. De Broglie's wave-particle duality

Wave-particle duality [7] and the Davisson–Germer effect suggest soliton-like asymptotics,

$$u(x,t) \sim \sum_1^N s_k^\pm(x - v_k^\pm t), \quad \text{as } t \to \pm\infty, \tag{2.6}$$

of all finite energy solutions $u(x,t)$ to the translation-invariant wave equations. For the relativistic-invariant equations the soliton becomes a stationary state in the moving frame. The soliton-like asymptotics would then follow automatically from the stabilization in these equations, giving a better description of the long-time behavior of solutions. Such asymptotics are still not proved for any relativistic-invariant nonlinear wave equation with nontrivial solitons.

Soliton-like asymptotics similar to (2.6) are proved in Ref. [39] for some translation-invariant 1D completely integrable equations and in Ref. [16] for translation-invariant and $U(1)$-invariant 1D nonlinear reaction systems. The asymptotics of type (2.6) with $N = 1$ are established for the solutions close to a soliton to a 1D nonlinear Schrödinger equation [8] and for 2D and 3D nonlinear Schrödinger equations [47], [48]. We prove the soliton-like asymptotics of type (2.6) with $N = 1$ for the scalar field coupled to a single particle; this system is translation invariant, but not relativistic invariant.

2.3. Radiative damping

Soliton-like asymptotics also arise in the translation-invariant coupled field–particle systems, as a result of nonlinear interaction. If the particle moves with an acceleration, the energy is transferred to the wave field, and its part is eventually transported to infinity. Thus, the particle feels a sort of friction, and one expects the relaxation of the acceleration,

$$\ddot{q}(t) \to 0, \quad \text{as } t \to \pm\infty, \tag{2.7}$$

and the convergence of the field around the particle to a comoving Coulombian form, [23]. The velocity converges to zero if the trajectory is bounded. We establish such a long-time behavior for one classical particle coupled to a scalar field or to the Maxwell field according to Abraham's model, when there is a given form of the "charge density" associated to the particle. For a scalar field we prove that the limits of the velocity exist if the external force vanishes,

$$\dot{q}(t) \to v^{\pm}, \quad \text{as } t \to \pm\infty. \tag{2.8}$$

This implies the soliton-like asymptotics of type (2.6), with $N = 1$. The radiative damping is important because it is observable in crucial physical phenomena, X-ray emission, synchrotron radiation, and so on. It has been widely considered in physics letters, see Refs. [1,4,12,14,15,24,38,55] and the references thereafter. However, an exact mathematical statement of the phenomenon is missing up to now for different reasons; one of them is treating particles as having the "point" charge density [4,12,14,15,55]. On the other hand, Abraham and Lorentz [1,38] consider "distributed" charge density for the particle; let us note that all known elementary particles are considered as distributed, with nonzero cross-sections.

The main difficulty in the theoretical description of the radiative damping is the essential nonlinearity of the field–particle interaction, the particle generates the field, while the field acts on the particle. This "self-action" of a particle cannot be understood as the consecutive processes of such type, described by linear wave problem and finite-dimensional Newtonian equation, respectively. For instance, for a point particle, the proper energy and the energy of the interaction are infinite and the self-action does not allow the correct description, while the both processes do. This implies in particular that the perturbation theory is not applicable in this situation. This is why Abraham smoothened out the coupling by the charge density $\rho(x)$, providing the finiteness of the proper energy [1]. The energy of the interaction is then also finite. The corresponding dynamics was analyzed by Lorentz [38]. To take into account the self-action of the "distributed" particle, Lorentz expands the particle trajectory $q(t)$ into the Taylor series. This leads to the infinite order differential equation for $q(t)$ [15,38] that has never been investigated rigorously. The cut-off of the Taylor series at the term with $\dddot{q}\,(t)$ leads to a non-reversible equation, which allowed to explain the energy radiation for periodic particle motion [15,24]. However, this did not give an exact result and led to some new questions on "runaway" solutions [4,12,15,24]. We give a solution to the problem of the radiative damping for the case of one particle coupled to a scalar or to the Maxwell field.

For the classical Maxwell–Lorentz system the problem of the long-time behavior is discussed in Refs. [3,4,12,14,38,55].

Below we state our three recent results on the stabilization for the nonlinear wave equations and systems. The results for the systems are obtained in accordance with Spohn and Künze [33] and with Spohn [31].

3. Results on the stabilization

3.1. One-dimensional wave equations

We consider the long-time asymptotics of the solutions to the Cauchy problem

$$\ddot{u}(x,t) = u''(x,t) + f(x,u(x,t)), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{3.1}$$

$$u|_{t=0} = u_0(x), \quad \dot{u}|_{t=0} = v_0(x). \tag{3.2}$$

The solutions $u(x,t)$ take the values in $\mathbb{R}^d$ with $d \geqslant 1$. All the derivatives in (3.1) and everywhere below are understood in the sense of distributions. Physically, Eq. (3.1) describes small crosswise oscillations of a string interacting with an elastic nonlinear medium. We assume $f(x,u) = 0$ for $|x| \geqslant a$ with some $a > 0$, and

$$f(x,u) = \chi(x)F(u), \quad F \in C^1(\mathbb{R}^d,\mathbb{R}^d), \quad \chi(x) \in C(\mathbb{R}), \tag{3.3}$$

$$F(u) = -\nabla V(u) \quad \text{and } V(u) \to +\infty, \quad \text{as } |u| \to \infty, \tag{3.4}$$

$$\chi(x) \geqslant 0, \quad \chi(x) \not\equiv 0 \quad \text{and } \chi(x) = 0, \quad \text{for } |x| \geqslant a. \tag{3.5}$$

We introduce the "configuration space" $\mathcal{Q}$ and the phase space $\mathcal{E}$ of finite energy states for the system (3.1). We denote by L^2 the Hilbert space $L^2(\mathbb{R},\mathbb{R}^d)$ with the norm $|\cdot|$, and we denote by $|\cdot|_R$ the norm in $L^2(-R,R;\mathbb{R}^d)$ for $R > 0$.

Definition 3.1. $\mathcal{Q}$ is the Hilbert space $\{u(x) \in C(\mathbb{R},\mathbb{R}^d) : u'(x) \in L^2\}$ with the norm $\|u\|_{\mathcal{Q}} = |u'| + |u(0)|$. $\mathcal{E} = \mathcal{Q} \oplus L^2$ is the Hilbert space of the pairs $(u(x),v(x))$, with the norm $\|(u,v)\|_{\mathcal{E}} =$

$\|u\|_Q+|v|$. $\mathcal{E}_F$ is the space $\mathcal{E}$ endowed with the Fréchet topology defined by the seminorms $\|(u,v)\|_R \equiv |u'|_R + |u(0)| + |v|_R$, $R > 0$.

Note that both the space $\mathcal{E}$ and $\mathcal{E}_F$ are metrizable and that $\mathcal{E}_F$ is not complete.

We denote by $V(x,u) = \chi(x)V(u)$ the potential of the nonlinear force. With the assumptions (3.3)-(3.5), Eq. (3.1) is formally a Hamiltonian system with the phase space $\mathcal{E}$ and the Hamiltonian functional

$$\mathcal{H}(u,v) = \int (\tfrac{1}{2}|v(x)|^2 + \tfrac{1}{2}|u'(x)|^2 + V(x,u(x)))\,\mathrm{d}x \tag{3.6}$$

for $(u,v) \in \mathcal{E}$. We consider solutions $u(x,t)$ such that $Y(t) = (u(\cdot,t), \dot{u}(\cdot,t)) \in C(\mathbb{R},\mathcal{E})$ and we write the Cauchy problem (3.1)-(3.2) in the form

$$\dot{Y}(t) = \mathcal{V}(Y(t)), \quad \text{for } t \in \mathbb{R}, \quad Y(0) = Y_0, \tag{3.7}$$

where $Y_0 = (u_0, v_0)$.

Proposition 3.2. Let $d \geqslant 1$ and let the assumptions (3.3)-(3.5) be fulfilled. Then for every $Y^0 \in \mathcal{E}$ the Cauchy problem (3.7) has the unique solution $Y(t) \in C(\mathbb{R},\mathcal{E})$. The mapping $W_t : Y^0 \mapsto Y(t)$ is continuous in $\mathcal{E}$ and $\mathcal{E}_F$ for all $t \in \mathbb{R}$. The energy is conserved,

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y_0), \quad \text{for } t \in \mathbb{R}. \tag{3.8}$$

We denote by $\mathcal{S}$ the set of all stationary states $S = ((s(x),0) \in \mathcal{E}$ for the system (3.7). We establish the long-time convergence in the Fréchet topology

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}, \quad \text{as } t \to \pm\infty \tag{3.9}$$

for finite energy solutions $Y(t)$. The convergence means by definition that for every neighborhood $\mathcal{O}(\mathcal{S})$ of $\mathcal{S}$ in $\mathcal{E}_{\mathcal{F}}$ there exists a $T > 0$ such that $Y(t) \in \mathcal{O}(\mathcal{S})$ for $|t| > T$. Thus, the set $\mathcal{S}$ is the point attractor of the system (3.7) in the Fréchet topology of the space $\mathcal{E}_F$.

Theorem 3.3. Let all the assumptions of proposition 3.2hold and the initial state Y^0 have a bounded norm in $\mathcal{E}$. Then

(i) the convergence (3.9) holds for the solution $Y(t) \in C(\mathbb{R},\mathcal{E})$ to the Cauchy problem (3.7).

(ii) Let, moreover, $d = 1$ and let the function $F(u)$ be real analytic on $\mathbb{R}$. Then there exist some stationary states $S_\pm \in \mathcal{S}$ depending on the solution $Y(t)$ such that

$$Y(t) \xrightarrow{\mathcal{E}_F} S_\pm, \quad \text{as t} \to \pm\infty. \tag{3.10}$$

Remarks.

(i) The convergence (3.10) and (3.3)-(3.6) imply, by the Fatoux theorem,

$$\mathcal{H}(S_\pm) \leqslant \mathcal{H}(Y(t)) \equiv \mathcal{H}(Y^0), \quad t \in \mathbb{R}, \tag{3.11}$$

similarly to the well-known property of weak convergence in Hilbert and Banach spaces.

(ii) Similar results are established in Refs. [27,28] for $f(x,u) = \delta(x)F(u)$ and in Ref. [29] for $f(x,u) = \sum_1^N \delta(x-x_k)F_k(u)$. We assume $f(x,u) = \chi(x)F(u)$ for the simplicity of our exposition. All the results can easily be extended to $f(x,u)$ satisfying the conditions from Ref. [29] generalized in a suitable way.

As a trivial example let us consider $f(x,u) \equiv 0$. Then Eq. (3.1) becomes the d'Alembert equation and the assumptions (3.4)-(3.5) fail. For the solutions $Y(t) \in C(\mathbb{R},\mathcal{E})$ the convergence (3.10) generally does not hold. Meanwhile, in that case the convergence (3.9) holds for every solution $Y(t) \in C(\mathbb{R},\mathcal{E})$ and the convergence (3.10) holds if $u_0(x) = C_\pm$ and $v_0(x) = 0$ for $|x| \geqslant$ const. This evidently follows from the d'Alembert formula for the solution to the Cauchy problem.

3.2. Scalar field coupled to a particle

We consider a scalar field $\phi(x)$ coupled to a particle with a position $q \in \mathbb{R}^3$. Let $\pi(x)$ be the canonically conjugate field to $\phi(x)$ and let p be the momentum of the particle. The Hamiltonian (energy functional) reads

$$\mathcal{H}(\phi,q,\pi,p) = (1+p^2)^{1/2} + V(q) + \int (\tfrac{1}{2}|\pi(x)|^2 + \tfrac{1}{2}|\nabla\phi(x)|^2 + \phi(x)\rho(x-q))\,\mathrm{d}^3x. \tag{3.12}$$

The mass of the particle and the propagation speed for ϕ have been set equal to 1. The relativistic kinetic energy is chosen to insure that $|\dot{q}| < 1$. The interaction term of type $\phi(q)$ would result in an energy which is not bounded from below; therefore we smoothen out the coupling by the function $\rho(x)$, which is assumed to be radial and having compact support. In analogy with the Maxwell-Lorentz equations we call $\rho(x)$ the "charge distribution." Taking formally the variational derivatives in Eq. (3.12), we derive the following equations for the coupled dynamics,

$$\begin{aligned}
&\dot{\phi}(x,t) = \pi(x,t),\\
&\dot{\pi}(x,t) = \Delta\phi(x,t) - \rho(x-q(t)),\\
&\dot{q}(t) = p(t)/(1+p^2(t))^{1/2},\\
&\dot{p}(t) = -\nabla V(q(t))\\
&\quad + \int \phi(x,t)\nabla\rho(x-q(t))\,\mathrm{d}^3x. \qquad (3.13)
\end{aligned}$$

We consider the long-time behavior of all finite energy solutions to the system (3.13). The appropriate phase space will be introduced below, but first we note that the Hamiltonian functional (3.12) is conserved along sufficiently smooth solution trajectories of Eq. (3.13). It is then natural to choose as a phase space the set of all finite energy states.

3.2.1. Notations and definitions

We have to state our assumptions on V,

$$(P_{\min}) \quad V(x) \in C^2(\mathbb{R}^3) \quad \text{and} \quad \inf_{x\in\mathbb{R}^3} V(x) > -\infty.$$

We will also use the following conditions,

$$(P_{\max}) \quad \nabla V(x), \nabla\nabla V(x) \in L^\infty(\mathbb{R}^3),$$

$$(P_\infty) \quad \lim_{|q|\to\infty} V(q) = \infty,$$

$$(Q) \quad q_0 = \sup_{t\in\mathbb{R}} |q(t)| < \infty.$$

The charge distribution ρ is smooth, radially symmetric, and compactly supported,

$$(C) \quad \rho \in C_0^\infty(\mathbb{R}^3), \quad \rho(x) = \rho_r(|x|),$$
$$\rho(x) = 0, \quad \text{for } |x| \geqslant R_\rho.$$

A further important assumption is the Wiener condition,

$$(W) \quad \hat{\rho}(k) = F\rho(k) = \int \mathrm{e}^{ikx}\rho(x)\,\mathrm{d}^3x \neq 0,$$
$$\text{for } k \in \mathbb{R}^3.$$

It insures that all modes of the field are coupled to the charge. In particular, the total charge $\overline{\rho} = \hat{\rho}(0)$ is different from zero. We have constructed in Ref. [33] some generic examples of the charge distributions satisfying both (W) and (C).

We investigate the long-time behavior of all finite energy solutions to Eq. (3.13). Clearly, the first step is to determine stationary solutions. For every $q \in \mathbb{R}^3$, we define the state $S_q = (\phi_q, q, \pi, p)$ by

$$\begin{aligned}
&S_q = (\phi_q, q, 0, 0),\\
&\phi_q(x) = -\int \frac{\mathrm{d}^3y}{4\pi|y-x|}\rho(y-q). \qquad (3.14)
\end{aligned}$$

Let $Z = \{q \in \mathbb{R}^3 : \nabla V(q) = 0\}$ be the set of points for which the external force vanishes. The set of stationary solutions is simply given by

$$\mathcal{S} = \{S_q : q \in Z\}. \qquad (3.15)$$

The second set of asymptotic solutions corresponds to the charge traveling with a uniform velocity, v, when $V(q) \equiv 0$. Up to a translation, they are of the form

$$T^v(t) = (\phi^v(x-vt), vt, \pi^v(x-vt), p^v), \qquad (3.16)$$

with an arbitrary velocity $v \in \mathbf{V} = \{v \in \mathbb{R}^3 : |v| < 1\}$, where

$$\begin{aligned}
&\phi^v(x) = -\int \frac{\mathrm{d}^3y}{4\pi|v(y-x)_\parallel + \lambda(y-x)_\perp|}\rho(y),\\
&\pi^v(x) = -v\phi^v(x),\\
&p^v = \frac{v}{\sqrt{1-v^2}}.
\end{aligned}$$

Here $\lambda = \sqrt{1-v^2}$ and we set $x = vx_\parallel + x_\perp$, where $x_\parallel \in \mathbb{R}$ and $v \perp x_\perp \in \mathbb{R}^3$ for $x \in \mathbb{R}^3$.

We need to introduce a suitable phase space and to establish existence and uniqueness of solutions to (3.13). Let L^2 be the real Hilbert space $L^2(\mathbb{R}^3)$ with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$, and let $\dot{H}^1$ be the completion of the real space $C_0^\infty(\mathbb{R}^3)$ with norm $\|\phi(x)\| = |\nabla\phi(x)|$. Equivalently, using the Sobolev embedding theorem, $\dot{H}^1 = \{\phi(x) \in L^6(\mathbb{R}^3) : |\nabla\phi(x)| \in L^2\}$ (see Ref. [37]). Let $|\phi|_R$ denote the norm in $L^2(B_R)$ for $R > 0$, where $B_R = \{x \in \mathbb{R}^3 : |x| \leqslant R\}$. Then the seminorms $\|\phi\|_R = |\nabla\phi|_R + |\phi|_R$ are continuous on $\dot{H}^1$.

Definition 3.4.

(i) $\mathcal{F}$ and $\mathcal{E}$ are the Hilbert spaces $\dot{H}^1 \oplus L^2$ and $\dot{H}^1 \oplus \mathbb{R}^3 \oplus L^2 \oplus \mathbb{R}^3$ with finite norms

$$\|(\phi,\pi)\|_{\mathcal{F}} = \|\phi\| + |\pi| \quad \text{and}$$
$$\|Y\|_{\mathcal{E}} = \|\phi\| + |q| + |\pi| + |p|, \quad \text{for} \quad Y = (\phi, q, \pi, p).$$

(ii) $\mathcal{E}^\sigma$ for $\sigma \geqslant 0$ is the set of states $Y = (\phi(x), q, \pi(x), p) \in \mathcal{E}$ such that for some $R^0 = R^0(Y) > 0$ the functions $\phi(x)$,$\pi(x)$ are C^2, C^1-differentiable respectively outside the ball B_{R^0} and

$$DY(x) = |\phi(x)| + |x|(|\nabla\phi(x)| + |\pi(x)|) + |x|^2(|\nabla\nabla\phi(x)| + |\nabla\pi(x)|) = \mathcal{O}(|x|^{-\sigma}), \quad \text{as } |x| \to \infty. \tag{3.17}$$

(iii) $\mathcal{F}_F$, $\mathcal{E}_F$ are the spaces $\mathcal{F}$, $\mathcal{E}$ endowed with the Fréchet topology defined by the local energy seminorms

$$\|(\phi,\pi)\|_R = \|\phi\|_R + |\pi|_R \quad \text{and}$$
$$\|Y\|_R = \|\phi\|_R + |q| + |\pi|_R + |p|, \quad \text{for} \quad Y = (\phi,q,\pi,p), \quad \forall R > 0. \tag{3.18}$$

Note that the spaces $\mathcal{F}_F$ and $\mathcal{E}_F$ are metrizable and not complete; $\dot{H}^1$ is not contained in L^2 and for instance $|\phi_q| = \infty$. On the other hand, $S_q \in \mathcal{E}$ and $S_q \in \mathcal{E}^\sigma$ for every $\sigma > 1/2$. For smooth $\phi(x)$ vanishing at infinity we have

$$-\frac{1}{8\pi}\int\int \frac{\rho(x)\rho(y)\mathrm{d}^3x\mathrm{d}^3y}{|x-y|} = \frac{1}{2}(\rho, \Delta^{-1}\rho) \leqslant \tfrac{1}{2}|\nabla\phi|^2 + (\phi(x), \rho(x-q)) \leqslant |\nabla\phi|^2 - \tfrac{1}{2}(\rho, \Delta^{-1}\rho). \tag{3.19}$$

Therefore, $\mathcal{E}$ is the space of finite energy states. The Hamiltonian functional $\mathcal{H}$ is continuous on the space $\mathcal{E}$ and the lower bound in Eq. (3.19) implies that the energy (3.12) is bounded from below. In the point charge limit this lower bound tends to $-\infty$.

Let us write the system (3.13) as a dynamical equation on functions in the space $\mathcal{E}$,

$$\dot{Y}(t) = \mathcal{V}(Y(t)) \quad \text{for } t \in \mathbb{R}, \tag{3.20}$$

where $Y(t) = (\phi(x,t), q(t), \pi(x,t), p(t)) \in \mathcal{E}$.

Definition 3.5. $\mathcal{S} = \{S \in \mathcal{E} : \mathcal{V}(S) = 0\}$ is the set of all finite energy stationary states of the system (3.13).

Proposition 3.6. Let ($P_{\min}$) and (C) hold and let $Y^0 = (\phi^0(x), q^0, \pi^0(x), p^0) \in \mathcal{E}$. Then the system (3.13) has a unique solution $Y(t) = (\phi(x,t), q(t), \pi(x,t), p(t)) \in C(\mathbb{R}, \mathcal{E})$ with $Y(0) = Y^0$. The energy is conserved, i.e. Eq. (3.8) holds. If, in addition, (P_∞) holds, then (Q) holds. The set $\mathcal{S}$ is given by Eq. (3.15). The following minimum value is attained,

$$\min_{\phi \in \dot{H}^1} \int (\tfrac{1}{2}|\nabla\phi(x)|^2 + \phi(x)\rho(x-q))\,\mathrm{d}^3x = \tfrac{1}{2}\langle\rho, \Delta^{-1}\rho\rangle. \tag{3.21}$$

3.2.2. Stabilization

From physical intuition one is tempted to conjecture that every solution $Y(t)$ of finite energy would converge to some stationary state S_q as $t \to \infty$. We do not achieve such a global result. First of all, the behavior of initial fields at infinity, Eq. (3.17), should not only be as required by finite energy, but also with some additional smoothness. Secondly, the set Z is not necessarily discrete; in this case $Y(t)$ may never settle to a definite S_q but could instead wander around, only approaching $\mathcal{S}$ as a set.

We denote by $\xrightarrow{\mathcal{F}_F}$ and $\xrightarrow{\mathcal{E}_F}$ the convergence in the Fréchet topology of the spaces $\mathcal{F}_F$ and $\mathcal{E}_F$, respectively. We establish three types of long-time asymptotics of the finite energy solutions in the Fréchet topology.

Theorem 3.7. [33] Let ($P_{\min}$), (C), and (W) hold. Let $Y(t) \in C(\mathbb{R}, \mathcal{E})$ be the solution to the system (3.13) with initial state $Y^0 \in \mathcal{E}^\sigma$ with some $\sigma > 1/2$. We have the following.

(i) If either ($P_{\max}$) or (Q) holds, then

$$\ddot{q}(t) \to 0, \quad \text{as } t \to \pm\infty, \tag{3.22}$$

$$(\phi(q(t)+x,t), \pi(q(t)+x,t)) - (\phi^{v(t)}(x), \pi^{v(t)}(x)) \xrightarrow{\mathcal{F}_F} 0, \quad \text{as } t \to \pm\infty. \tag{3.23}$$

(ii) If (Q) holds, then in addition to Eqs. (3.22) and (3.23),

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}, \quad \text{as } t \to \pm\infty. \tag{3.24}$$

(iii) If (Q) holds, and in addition Z is a discrete set in $\mathbb{R}^3$, then there exist stationary states $S_\pm \in \mathcal{S}$ depending on Y^0 such that

$$Y(t) \xrightarrow{\mathcal{E}_F} S_\pm, \quad \text{as } t \to \pm\infty. \tag{3.25}$$

Remarks.

(i) The convergence (3.25) and (3.12) imply Eq. (3.11) by the Fatoux theorem.

(ii) Let $V_\infty = \liminf_{|q|\to\infty} V(q)$. If $\mathcal{H}(Y^0)\langle 1 + V_\infty + \frac{1}{2} < \rho, \Delta^{-1}\rho\rangle$, then Eq. (3.21) implies (Q) by the conservation of energy.

(iii) The assumption (C) can be weakened to a finite differentiability and some decay of $\rho(x)$ at infinity.

(iv) Our method fails in the case of a bounded space-periodic potential V together with a sufficiently large initial energy and a discrete set Z. In this case we expect convergence to some definite S_q, but cannot prove that $q(t)$ remains bounded.

(v) If the Wiener condition (W) is violated, then on the linearized level one can construct periodic solutions, provided the coupling strength is adjusted to the zeros of $\hat\rho$. The global convergence to $\mathcal{S}$ fails if such a periodic solution would persist for the full nonlinear equations (3.13).

To prove theorem 3.7 we will estimate the energy scattering by decomposing ϕ into a near and far field. The energy is scattered into the far field. Since the energy is bounded from below, such a scattering cannot go on forever and a certain scattered energy functional has to be bounded. This functional can be written as a convolution. By the Tauberian theorem of Wiener, using (W), we conclude that $\lim_{t\to\infty} \ddot q(t) = 0$. Therefore also $\lim_{t\to\infty} \dot q(t) = 0$, due to (Q). This implies that $\mathcal{A} = \{S_q : |q| \leqslant q_0)\}$ is a compact attracting set for the trajectory $Y(t)$. Relaxation and compactness reduce $\mathcal{A}$ to $\mathcal{S}$ as a minimal attractor.

3.2.3. The rate of stabilization

To establish the rate of convergence to a stationary state S_q we have to assume that the point $q \in Z$ is stable, in the following sense,

Definition 3.8. A point $q \in Z$ is said to be stable if $d^2V(q) > 0$ as a quadratic form.

Even for a stable $q \in Z$, a slow decay of the initial fields in space will result in a slow convergence in time.

Theorem 3.9. [33] Let all assumptions of theorem 3.7 hold, $V \in C^3(\mathbb{R}^3)$, and

$$Y(t) \xrightarrow{\mathcal{E}_F} S_q, \quad \text{as } t \to +\infty, \tag{3.26}$$

with a stable point $q \in Z$. Then

(i) for every $R, \varepsilon > 0$,

$$\|Y(t) - S_q\|_R = \mathcal{O}(t^{-\sigma+\varepsilon}), \quad \text{as } t \to \infty. \tag{3.27}$$

(ii) Let additionally

$$DY^0(x) = \mathcal{O}(\mathrm{e}^{-\alpha|x|}), \quad \text{as } |x| \to \infty, \tag{3.28}$$

with some $\alpha > 0$. Then there exists a $\gamma^* = \gamma^*(q) > 0$ such that for every $R > 0$

$$\|Y(t) - S_q\|_R = \mathcal{O}(\mathrm{e}^{-\beta t}), \quad \text{as } t \to \infty, \tag{3.29}$$

with $\beta = \alpha$ if $\alpha < \gamma^*$ and with arbitrary $\beta < \gamma^*$ if $\alpha \geqslant \gamma^*$.

We prove theorem 3.9 by controlling the nonlinear part of Eq. (3.13) by the linearized equation. For the linearized equation exponential convergence can be established by Paley–Wiener techniques for complex Fourier transforms [26,41]. As a byproduct we also establish asymptotic stability of stationary states S_q with stable points $q \in Z$. Let us note that for general wave equations with local nonlinear terms a Liapunov type criterion of the asymptotic stability of the stationary states is proved in Ref. [34].

3.2.4. Soliton-like asymptotics

We establish in Ref. [31] the soliton-like asymptotics for the system (3.13) with zero external potential, $V(q) \equiv 0$. Then the system is translation-invariant and admits soliton-like solutions (3.16) with an arbitrary velocity $v \in \mathbf{V}$. On physical grounds one is tempted to conjecture that every solution $y(t)$ of finite energy will converge to some soliton-like solution as $t \to \infty$. We do not achieve such a global result in two respects. The $t = 0$ fields are not only required to decay at infinity so that they have a finite energy, but also some additional smoothness requirement (3.17) is imposed. More severely, we do not control the asymptotics for the position in the form $q(t) \sim vt + q$. We only prove that $\dot q(t)$ has a limit and the field around particle converges to the comoving Coulombian form.

Theorem 3.10. [31] Let $V(q) \equiv 0$, and let (C), (W) hold. Let $Y(t) \in C(\mathbb{R}, \mathcal{E})$ be the solution of the system (3.13) with initial state $Y^0 \in \mathcal{E}^\sigma$ with some $\sigma > 1/2$. Then there exists a velocity $v \in V$ such that for every $R > 0$

$$\|\phi(q(t) + \cdot, t) - \phi_v(\cdot)\|_R + |\pi(q(t) + \cdot, t) - \pi_v(\cdot)|_R + |\dot{q}(t) - v| \to 0, \quad \text{as } t \to \infty. \tag{3.30}$$

Remark. Since the Hamiltonian system (3.13) is invariant under the reversal of time, our results also hold for $t \to -\infty$.

3.3. Maxwell–Lorentz system

We consider a single charge, coupled to the Maxwell field and subject to prescribed time-independent external potentials. If $q(t) \in \mathbb{R}^3$ denotes the position of charge at time t, then the coupled Maxwell–Lorentz equations read

$$\begin{aligned}
&\operatorname{div} E(x,t) = \rho(x - q(t)),\\
&\operatorname{rot} E(x,t) = -\dot{B}(x,t), \quad \operatorname{div} B(x,t) = 0,\\
&\operatorname{rot} B(x,t) = \dot{E}(x,t) + \rho(x - q(t))\dot{q}(t),]\\
&\dot{q}(t) = \frac{p(t)}{\sqrt{1 + p^2(t)}},\\
&\dot{p}(t) = \int [E(x,t) + \overline{E}(x) + \dot{q}(t) \wedge (B(x,t) + \overline{B}(x))]\rho(x - q(t))\, d^3x.
\end{aligned} \tag{3.31}$$

The last line is the Lorentz force equation and the first two lines are the inhomogeneous Maxwell equations. $\overline{E}$ is the external electric field related to the electrostatic potential $\overline{\phi}$ by $\overline{E} = -\nabla\overline{\phi}$. Similarly, $\overline{B}$ is the external magnetic field with $\overline{B} = \operatorname{rot}\overline{A}$. $(E(x,t), B(x,t))$ is the Maxwell field and $(E + \overline{E}, B + \overline{B})$ is the total electromagnetic field. ρ is the charge distribution of the particle, on which we will comment below. We use units such that the velocity of light, ε_0, and the mass of the particle are equal to 1. Despite their physical importance, the coupled Maxwell–Lorentz equations have received only little mathematical attention. In Ref. [5] the existence and smoothness of the solutions were studied, and in Ref. [3] the orbital stability of the solitary waves. We consider all finite energy solutions to Eqs. (3.31). The appropriate phase space will be introduced below, but first we note that the energy integral

$$\mathcal{H}(E,B,q,p) = (1 + p^2)^{1/2} + \overline{V}(q) + \tfrac{1}{2}\int (|E(x)|^2 + |B(x)|^2)\, d^3x \tag{3.32}$$

is conserved along sufficiently smooth solution trajectories of Eq. (3.31). Here, we denote

$$\overline{V}(q) = \int \overline{\phi}(x)\rho(x - q)\, d^3x.$$

It is then natural to choose as a phase space the set of all finite energy states. In fact, Eq. (3.31) can be put into Hamiltonian form. In the canonical coordinates (depending on $\overline{B}$) the energy $\mathcal{H}$ is then the Hamiltonian of the system.

Next we have to state our assumptions on $\overline{\phi}$, $\overline{A}$, generalizing similar assumptions for the scalar field above,

$(P_{\min})$ $\quad \overline{\phi}(x), \overline{A}(x) \in L^\infty_{loc}(\mathbb{R}^3)$ and $\inf_{x\in\mathbb{R}^3} \overline{V}(x) > -\infty.$

We will also use the following conditions,

$(P_{\max})$ $\quad \overline{E}(x), \overline{B}(x), \nabla\overline{E}(x), \nabla\overline{B}(x) \in L^\infty(\mathbb{R}^3)$ or $\overline{V}(x), \overline{A}(x) \in L^\infty(\mathbb{R}^3)$.

(P_∞) $\quad \lim_{|q|\to\infty} \overline{V}(q) = \infty,$

(Q) $\quad q_0 = \sup_{t\in\mathbb{R}} |q(t)| < \infty.$

For ρ we assume the same properties (C) and (W) as for the scalar field above. The generic examples of densities ρ satisfying both assumptions (C) and (W) and, moreover, $\rho(x) \geqslant 0$, have been constructed in Ref. [33]. Then the conditions $(P_{\min})$, $(P_{\max})$ and (P_∞) would hold if we there replaced $\overline{V}$ by $\overline{\phi}$. In the case $\rho(x) \leqslant 0$ we could replace $\overline{V}$ by $-\overline{\phi}$.

We investigate the long-time behavior of all finite energy solutions to Eq. (3.31). Clearly, the first step is to determine the stationary solutions. For every $q \in \mathbb{R}^3$ we define the state $S_q = (E_q, B, q, p)$ by

$$S_q = (E_q, 0, q, 0), \quad E_q(x) = -\nabla\phi_q(x),$$

$$\phi_q(x) = \int \frac{d^3y}{4\pi|y-x|}\rho(y-q). \qquad (3.33)$$

Let $Z = \{q \in \mathbb{R}^3 : \nabla\overline{V}(q) = 0\}$ be the set of points for which the external electrostatic force vanishes. The set of stationary solutions is simply given by

$$\mathcal{S} = \{S_q : q \in Z\}. \qquad (3.34)$$

The second set of asymptotic solutions corresponds to the charge traveling with a uniform velocity v, in the case when $\overline{E} = 0 = \overline{B}$. Up to a translation, these solutions are of the form

$$T^v(t) = (E^v(x-vt), B^v(x-vt), vt, p^v), \qquad (3.35)$$

with an arbitrary velocity $v \in \mathbf{V} = \{v \in \mathbb{R}^3 : |v| < 1\}$, where

$$E^v(x) = -\nabla\phi^v(x),$$

$$\phi^v(x) = \int \frac{d^3y}{4\pi|v(y-x)_{\|} + \lambda(y-x)_{\perp}|}\rho(y),$$

$$B^v(x) = \operatorname{rot} A^v(x), \quad A^v(x) = v\phi^v(x),$$

$$p^v = \frac{v}{\sqrt{1-v^2}}.$$

Here $\lambda = \sqrt{1-v^2}$, and we set $x = vx_{\|} + x_{\perp}$, where $x_{\|} \in \mathbb{R}$ and $v \perp x_{\perp} \in \mathbb{R}^3$ for $x \in \mathbb{R}^3$.

We define a suitable phase space. Let L^2 denote the real Hilbert space $L^2(\mathbb{R}^3, \mathbb{R}^3)$ with the norm $|\cdot|$. We introduce the Hilbert spaces $\mathcal{F} = L^2 \oplus L^2$ and $\mathcal{L} = \mathcal{F} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with finite norms

$$\|(E(x), B(x))\|_{\mathcal{L}} = |E| + |B| \quad \text{and}$$

$$\|Y\|_{\mathcal{L}} = |E| + |B| + |q| + |p|, \qquad (3.36)$$

$$\text{for } Y = (E(x), B(x), q, p) \in \mathcal{L}.$$

$\mathcal{L}$ is the space of finite energy states, e.g. $S_q \in \mathcal{L}$. The energy functional $\mathcal{H}$ is continuous on the space $\mathcal{L}$. On $\mathcal{F}$ and $\mathcal{L}$ we define the local energy seminorms by

$$\|(E(x), B(x)\|_R = |E|_R + |B|_R \quad \text{and}$$

$$\|Y\|_R = |E|_R + |B|_R + |q| + |p|, \qquad (3.37)$$

$$\text{for } Y = (E(x), B(x), q, p)$$

for every $R > 0$, where $|\cdot|_R$ is the norm in $L^2(B_R)$, B_R the ball $\{x \in \mathbb{R}^3 : |x| < R\}$. Let us denote by $\mathcal{F}_F$, $\mathcal{L}_F$ the spaces $\mathcal{F}$, $\mathcal{L}$ equipped with the Fréchet topology induced by these seminorms. Note that the spaces $\mathcal{L}$ and $\mathcal{L}_F$, $\mathcal{F}_F$ are metrizable, but $\mathcal{L}_F$, $\mathcal{F}_F$ are not complete.

The system (3.31) is overdetermined. Therefore, its actual phase space is a nonlinear submanifold of the linear space $\mathcal{L}$.

Definition 3.11.

(i) The phase space $\mathcal{M}$ for Maxwell-Lorentz equations (3.31) is the metric space of states $(E(x), B(x), q, p) \in \mathcal{L}$ which satisfy the constraints

$$\operatorname{div} E(x) = \rho(x-q) \quad \text{and}$$

$$\operatorname{div} B(x) = 0, \quad \text{for } x \in \mathbb{R}^3. \qquad (3.38)$$

The metric on $\mathcal{M}$ is induced through the embedding $\mathcal{M} \subset \mathcal{L}$.

(ii) $\mathcal{M}^\sigma$ for $0 \leqslant \sigma \leqslant 1$ is the set of the states $(E(x), B(x), q, p) \in \mathcal{M}$ such that for some $R^0 = R^0(Y) > 0$ the fields $E(x)$,$B(x)$ are C^1-differentiable outside the ball B_{R^0} and

$$|E(x)| + |B(x)| + |x|(|\nabla E(x)| + |\nabla B(x)|)$$
$$= \mathcal{O}(|x|^{-1-\sigma}), \quad \text{as } |x| \to \infty. \qquad (3.39)$$

(iii) $\mathcal{M}_F$ denotes the space $\mathcal{M}$ endowed with the Fréchet topology induced through the embedding $\mathcal{M} \subset \mathcal{L}_F$.

Remarks.

(i) $\mathcal{M}$ is a complete metric space, a nonlinear submanifold of $\mathcal{L}$. The spaces $\mathcal{M}, \mathcal{M}_F$ are metrizable and $\mathcal{M}_F$ is not complete.

(ii) All stationary states S_q belong to $\mathcal{M}^1$, and the set $\mathcal{M}^1$ is dense in $\mathcal{M}$. On the other hand, since the total charge $\overline{\rho} = \hat{\rho}(0) \neq 0$, $\mathcal{M}^\sigma = \emptyset$ for $\sigma > 1$ because of the Gauss theorem. For the same reason supp$E(x)$ cannot be a compact set (in contrast to supp$B(x)$).

Let us write the system (3.31) as a dynamical equation on the functions in the space $\mathcal{M}$,

$$\dot{Y}(t) = \mathcal{V}(Y(t)), \quad \text{for } t \in \mathbb{R}, \qquad (3.40)$$

where $Y(t) = (E(x,t), B(x,t), q(t), p(t))) \in \mathcal{M}$.

Definition 3.12. $\mathcal{S} = \{S \in \mathcal{M} : \mathcal{V}(S) = 0\}$ is the set of all finite energy stationary states of the system (3.31).

Proposition 3.13. Let $(P_{\min})$ and (C) hold and $Y^0 = (E^0(x), B^0(x), q^0, p^0) \in \mathcal{M}$. Then the system (3.31) has the unique solution $Y(t) =$

$(E(x,t), B(x,t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{M})$ with $Y(0) = Y^0$. The energy is conserved, i.e. Eq. (3.8) holds. If, in addition, (P_∞) holds, then (Q) holds. The set $\mathcal{S}$ is given by Eq. (3.34).

We denote by $\xrightarrow{\mathcal{F}_F}$ and $\xrightarrow{\mathcal{M}_F}$ the convergence in the Fréchet topology of the spaces $\mathcal{F}_F$ and $\mathcal{M}_F$, respectively. We establish three types of long-time asymptotics of finite energy solutions in the Fréchet topology.

Theorem 3.14. [32] Let $(P_{\min})$, (C), and (W) hold. Let $Y(t) \in C(\mathbb{R}, \mathcal{M})$ be the solution to the Maxwell–Lorentz equations (3.31) with the initial state $Y^0 \in \mathcal{M}^\sigma$ with some $\sigma > 1/2$. We have

(i) If either $(P_{\max})$ or (Q) holds, then

$$\ddot{q}(t) \to 0, \quad \text{as } t \to \pm\infty, \tag{3.41}$$

$$(E(q(t)+x,t), B(q(t)+x,t)) - (E^{v(t)}(x), B^{v(t)}(x)) \xrightarrow{\mathcal{F}_F} 0, \tag{3.42}$$

as $t \to \pm\infty$.

(ii) If (Q) holds, in addition to (3.41) and (3.42), then

$$Y(t) \xrightarrow{\mathcal{M}_F} \mathcal{S}, \quad \text{as } t \to \pm\infty. \tag{3.43}$$

(iii) If, in addition to (Q), Z is a discrete set in $\mathbb{R}^3$, then there exist stationary states $S_\pm \in \mathcal{S}$ depending on Y^0 such that

$$Y(t) \xrightarrow{\mathcal{M}_F} S_\pm, \quad \text{as } t \to \pm\infty. \tag{3.44}$$

Remarks.

(i) The convergence (3.44) and (3.32) imply Eq. (3.11) by the Fatoux theorem.

(ii) Let $\overline{V}_\infty = \liminf_{|q|\to\infty} \overline{V}(q)$. If $\mathcal{H}(Y^0) < 1 + \overline{V}_\infty$, then (Q) holds by the conservation of energy.

(iii) Our method fails in the case of a constant magnetic field, $\overline{B}(x) = B_0$, $\overline{E}(x) = 0$. A similar example (only one with a discrete set Z) is a bounded space-periodic potential $\overline{\phi}$, together with a sufficiently large initial energy. In both cases we expect convergence to some definite S_q, but cannot prove that $|q(t)|$ remains bounded.

The general strategy of the proof here is parallel to the scalar field above, though the algebraic details are different. As an input of the proof, we establish a bound on the energy scattering to infinity and the Liénard–Wiechert asymptotics of the Maxwell field along the light cone. Then we derive Eq. (3.41) by the Wiener Tauberian theorem. This is a crucial point of the proof. We deduce Eqs. (3.42)–(3.44) from Eq. (3.41).

We suggest that theorems 3.9 and 3.10 can be generalized for the Maxwell–Lorentz equations (3.31); however, the generalizations are open problems. Similarly, we suggest a generalization of theorem 3.9 for translation-invariant systems (3.13) and (3.31) with zero external fields.

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