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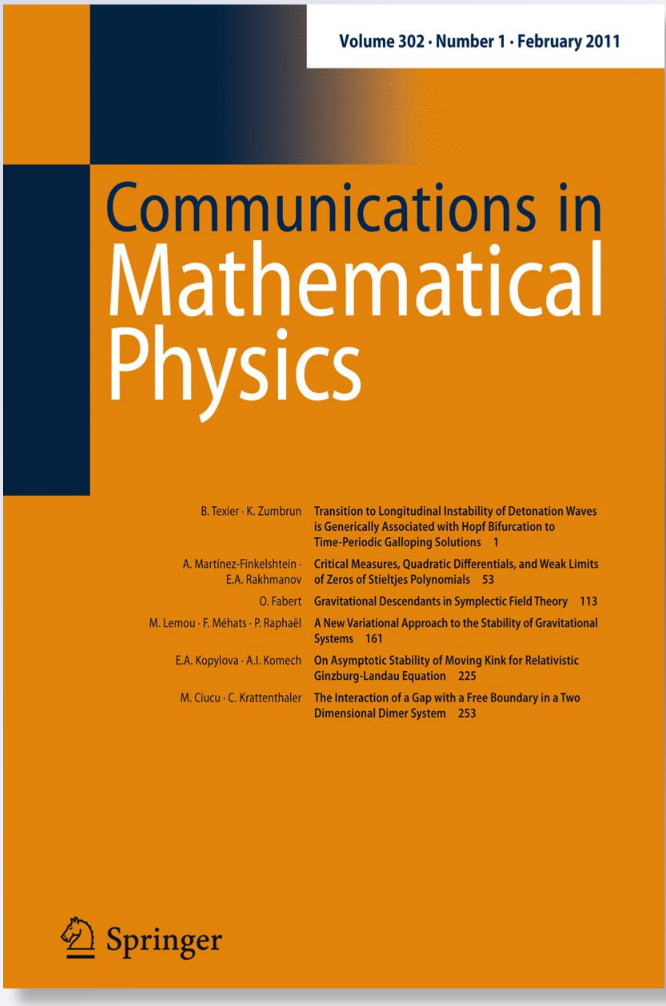

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On Asymptotic Stability of Moving Kink for Relativistic Ginzburg-Landau Equation

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Abstract: We prove the asymptotic stability of the moving kinks for the nonlinear relativistic wave equations in one space dimension with a Ginzburg-Landau potential: starting in a small neighborhood of the kink, the solution, asymptotically in time, is the sum of a uniformly moving kink and dispersive part described by the free Klein-Gordon equation. The remainder decays in a global energy norm. Our recent results on the weighted energy decay for the Klein-Gordon equations play a crucial role in the proofs.

1. Introduction

There has been widespread interest in the dynamics of topological excitations of classical relativistic field theories [2,3]. These excitations are finite energy solutions which do not decay to one of the true ground states because of topological constraints, said differently, these excitations are separated by an infinitely high potential barrier from the ground state. In our contribution we will study in mathematical detail one of the simplest examples. The field, ψ, is real valued and defined on the line, $\psi : \mathbb{R} \to \mathbb{R}$. The Hamiltonian function reads

$$\mathcal{H}(\psi, \pi) = \int_{\mathbb{R}} \left[\frac{1}{2}|\pi(x)|^2 + \frac{1}{2}|\psi'(x)|^2 + U(\psi(x)) \right] dx \tag{1.1}$$

with π the momentum canonically conjugate to ψ and a smooth potential U. This leads to the equation of motion

$$\ddot{\psi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \quad x \in \mathbb{R}, \tag{1.2}$$

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where $F(\psi) = -U'(\psi)$. For an introduction, let us consider the Ginzburg-Landau quartic double well potential of the form $U(\psi) = (\psi^2 - a^2)^2/(4a^2)$. Then the topological excitations are defined through $\mathcal{H} < \infty$ and the boundary conditions

$$\lim_{x\to\pm\infty} \psi(x,t) \to \pm a \tag{1.3}$$

with a fixed $a > 0$. Amongst them there are soliton-like solutions which travel with constant velocity,

$$\psi(x,t) = a \tanh \gamma \frac{x - vt - q}{\sqrt{2}},$$

where $\gamma = 1/\sqrt{1-v^2}$ is the Lorentz contraction. The solitons are related by a Lorentz boost, since Eq. (1.2) is relativistically invariant. We will consider more general double well potentials for which

$$U(\pm a) = U'(\pm a) = 0, \qquad U''(\pm a) > 0, \tag{1.4}$$

and

$$U(\psi) > 0 \quad \text{for} \quad \psi \in (-a, a), \tag{1.5}$$

similarly to the quartic potential. In this case the soliton-like solutions also exist,

$$\psi(x,t) = s(\gamma(x - vt - q)), \qquad v, q \in \mathbb{R}, \quad |v| < 1, \tag{1.6}$$

where $s(\cdot)$ is a "kink" solution to the corresponding stationary equation

$$s''(x) - U'(s(x)) = 0, \qquad s(\pm\infty) = \pm a. \tag{1.7}$$

In general our goal is to clarify the special role of the soliton-like solutions (1.6) as long time asymptotics for any finite energy topological excitations satisfying (1.3). Namely, if one chooses some arbitrary finite energy initial state satisfying (1.3), one would expect that for $t \to \infty$ the solution separates into two pieces: one piece is a finite collection of travelling solitons of the form (1.6) and their negatives with some velocities $v_j \in (-1, 1)$ and the shifts q_j depending in a complicated way on the initial data, and the second radiative piece which is a dispersive solution to the free Klein-Gordon equation which propagates to infinity with the velocity 1. Our aim here is to elucidate this general picture by mathematical arguments for initial data sufficiently close to a soliton (1.6).

Let us discuss our choice of the smooth potentials U. The condition (1.5) is necessary and sufficient for the existence of a finite energy static solution $s(x)$ to (1.7) when (1.4) holds. Indeed, the condition is obviously sufficient. On the other hand, the "energy conservation"

$$(s'(x))^2/2 - U(s(x)) = E \tag{1.8}$$

and $s(\pm\infty) = \pm a$ imply that $E = 0$. Therefore, $U(\psi) > 0$ for $\psi \in (-a, a)$ since otherwise the boundary conditions $s(\pm\infty) = \pm a$ would fail. As a byproduct, our kink solution is monotone increasing, and

$$s'(x) > 0, \qquad x \in \mathbb{R}. \tag{1.9}$$

Let us note that only the behavior of U near the interval $[-a, a]$ is of importance since the solution is expected to be close to a soliton. However, we will assume additionally the potential to be bounded from below

$$\inf_{\psi\in\mathbb{R}} U(\psi) > -\infty \tag{1.10}$$

to have a well posed Cauchy problem for all finite energy initial states.

Summarising, we formulate our first basic condition on the potential, for technical reasons adding a flatness condition.

Condition U1. *The potential U is a real smooth function which satisfies* (1.4)*,* (1.5)*,* (1.10)*, and the following condition holds with some $m > 0$,*

$$U(\psi) = \frac{m^2}{2}(\psi \mp a)^2 + \mathcal{O}(|\psi \mp a|^{14}), \qquad \psi \to \pm a. \tag{1.11}$$

Let us comment on the condition (1.11) (see also Remark 4.10). First, the condition means that $U''(-a) = U''(a)$, though we do not need the potential to be reflection symmetric. We consider the solutions close to the kink, $\psi(x, t) = s(\gamma(x-vt-q))+\phi(x, t)$, with small perturbations $\phi(x, t)$. For such solution the condition (1.11) and the asymptotics (1.3) mean that Eq. (1.2) is almost linear Klein-Gordon equation for large $|x|$ which is helpful for application of the dispersive properties. Finally, we expect that the degree 14 in (1.11) is technical, and a smaller degree should be sufficent. Let us note that a similar condition has been introduced in [4,5] in the context of the Schrödinger equation.

Further we need some assumptions on the spectrum of the linearised equation. Let us rewrite Eq. (1.2) in the vector form,

$$\left\{\begin{array}{l} \dot\psi(x,t) = \pi(x,t) \\ \dot\pi(x,t) = \psi''(x,t) + F(\psi(x,t)) \end{array}\right| \quad x \in \mathbb{R}. \tag{1.12}$$

Now the soliton-like solutions (1.6) become

$$Y_{q,v}(t) = (\psi_v(x - vt - q), \pi_v(x - vt - q)) \tag{1.13}$$

for $q, v \in \mathbb{R}$ with $|v| < 1$, where

$$\psi_v(x) = s(\gamma x), \quad \pi_v(x) = -v\psi_v'(x). \tag{1.14}$$

The states $S_{q,v} := Y_{q,v}(0)$ form the solitary manifold

$$\mathcal{S} := \{S_{q,v} : q, v \in \mathbb{R}, |v| < 1\}. \tag{1.15}$$

The linearized operator near the soliton solution $Y_{q,v}(t)$ is (see Sect. 4, formula (4.20))

$$A_v = \begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 - V_v(y) & v\nabla \end{pmatrix}, \quad \nabla = \frac{d}{dx}, \quad \Delta = \frac{d^2}{dx^2},$$

where

$$V_v(x) = -F'(\psi_v(x)) - m^2 = U''(\psi_v(x)) - m^2. \tag{1.16}$$

By (1.7) and condition **U1**, we have

$$V_v(x) \sim C(s(\gamma x) \mp a)^{12} \sim Ce^{-12m\gamma|x|}, \quad x \to \pm\infty, \tag{1.17}$$

since

$$s(x) \mp a \sim Ce^{-m|x|}, \quad x \to \pm\infty. \tag{1.18}$$

In Sect. 4 we show that the spectral properties of the operator A_v are determined by the corresponding properties of its determinant, which is the Schrödinger operator

$$H_v = -(1-v^2)\Delta + m^2 + V_v. \tag{1.19}$$

The spectral properties of H_v are identical for all $v \in (-1, 1)$ since the relation $V_v(x) = V_0(\gamma x)$ implies

$$H_v = T_v^{-1} H_0 T_v, \quad \text{where } \ T_v : \psi(x) \mapsto \psi(x/\gamma). \tag{1.20}$$

This equivalence manifests the relativistic invariance of Eq. (1.12). The continuous spectrum of the operator H_v coincides with $[m^2, \infty)$. The point 0 belongs to the discrete spectrum with corresponding eigenfunction ψ_v'. By (1.14) and (1.9) we have $\psi_v'(x) = \gamma s'(\gamma x) > 0$ for $x \in \mathbb{R}$. Hence, ψ_v' is the groundstate, and all remaining discrete spectrum is contained in $(0, m^2]$.

For $\alpha \in \mathbb{R}$, $p \geq 1$, and $l = 0, 1, 2, \ldots$ let us denote by $W_\alpha^{l,p}$, the weighted Sobolev space of the functions with the finite norm

$$\|\psi\|_{W_\alpha^{l,p}} = \sum_{k=0}^{l} \|(1+|x|)^\alpha \psi^{(k)}\|_{L^p} < \infty.$$

Denote $H_\alpha^l := W_\alpha^{l,2}$, so $L_\alpha^2 := H_\alpha^0$ are the Agmon's weighted spaces.

Definition 1.1 (cf. [9,16]). *A nonzero solution* $\psi \in L^2_{-1/2-0}(\mathbb{R}) \setminus L^2(\mathbb{R})$ *to* $H_v\psi = m^2\psi$ *is called a resonance.*

Now we can formulate our second basic condition on the potential.

Condition U2. *For any* $v \in (-1, 1)$,

i) 0 *is only eigenvalue of* H_v.
ii) m^2 *is not a resonance of* H_v.

We show that Condition **U2** implies the boundedness of the resolvent of the operator A_v in the corresponding weighted Agmon spaces at the edge points $\pm im/\gamma$ of its continuous spectrum.

Both conditions **U1**, **U2** can be satisfied though it is non-obvious. Let us note that the quartic Ginzburg-Landau potential does not satisfy (1.11) and condition **U2**. We will prove elsewhere that the corresponding examples of potentials satisfying both **U1** and **U2** can be constructed as smoothened piece-wise quadratic potentials.

We now can formulate the main result of our paper. Namely, we will prove the following asymptotics:

$$(\psi(x,t), \pi(x,t)) \sim (\psi_{v_\pm}(x - v_\pm t - q_\pm), \pi_{v_\pm}(x - v_\pm t - q_\pm)) + W_0(t)\Phi_\pm, \quad t \to \pm\infty \tag{1.21}$$

for solutions to (1.12) with initial states close to a soliton-like solution (1.13). Here $W_0(t)$ is the dynamical group of the free Klein-Gordon equation, $\Phi_\pm$ are the corresponding asymptotic states, and the remainder converges to zero $\sim t^{-1/2}$ in the global energy norm of the Sobolev space $H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$.

Let us comment on previous results in this field.

- *Orbital stability of the kinks.* For 1D relativistic nonlinear Ginzburg-Landau equations (1.2) the orbital stability of the kinks has been proved in [10].
- *The Schrödinger equation.* The asymptotics of type (1.21) were established for the first time by Soffer and Weinstein [23,24] (see also [19]) for nonlinear $U(1)$-invariant Schrödinger equation with a potential for small initial states and sufficiently small nonlinear coupling constant.

The results have been extended by Buslaev and Perelman [4] to the translation invariant 1D nonlinear $U(1)$-invariant Schrödinger equation. The novel techniques [4] are based on the "separation of variables" along the solitary manifold and in the transversal directions. The symplectic projection allows to exclude the unstable directions corresponding to the zero discrete spectrum of the linearized dynamics. Similar techniques were developed by Miller, Pego and Weinstein for the 1D modified KdV and RLW equations, [17,18].The extensions to higher dimensions were obtained in [6,12,22,27].

- *Nonrelativistic Klein-Gordon equations.* The asymptotics of type (1.21) were extended to the nonlinear 3D Klein-Gordon equations with a potential [25], and for translation invariant system of the 3D Klein-Gordon equation coupled to a particle [11].
- *Wave front of 3D Ginzburg-Landau equation.* The asymptotic stability of wave front was proved for 3D relativistic Ginzburg-Landau equation with initial data which differ from the wave front on a compact set [7]. The wave front is the solution which depends on one space variable only, so it is not a soliton. The equation differs from the 1D equation (1.2) by the additional 2D Laplacian which improves the dispersive decay for the corresponding linearized Klein-Gordon equation in the continuous spectral space.

The proving of the asymptotic stability of the solitons and kinks for relativistic equations remained an open problem till now. The investigation crucially depends on the spectral properties for the linearized equation which are completely unknown for higher dimensions. For the 1D case the main obstacle was the slow decay $\sim t^{-1/2}$ for the free 1D Klein-Gordon equation (see the discussion in [7, Introduction]).

Let us comment on our approach. We follow general strategy of [4–7,11,25]: symplectic projection onto the solitary manifold, modulation equations, linearization of the transversal equations and further Taylor expansion of the nonlinearity, etc. We develop for relativistic equations a general scheme which is common in almost all papers in this area: dispersive estimates for the solutions to the linearized equation, virial and $L^1 - L^\infty$ estimates and the method of majorants. However, the corresponding statements and their proofs in the context of relativistic equations are completely new.

Let us comment on our novel techniques.

I. The decay $\sim t^{-3/2}$ from Theorem 4.7 for the linearized transversal dynamics relies on our novel approach [13,14] to the 1D Klein-Gordon equation.

II. The novel "virial type" estimate (4.42) is the relativistic version of the bound [5, (1.2.5)] used in [5] in the context of the nonlinear Schrödinger equation (see Remark 4.10).

III. We establish an appropriate relativistic version (4.31) of $L^1 \to L^\infty$ estimates. Both estimates (4.42) and (4.31) play a crucial role in obtaining the bounds for the majorants.

IV. Finally, we give the complete proof of the soliton asymptotics (1.21). In the context of the Schrödinger equation, the proof of the corresponding asymptotics were sketched in [5].

Our paper is organized as follows. In Sect. 2 we formulate the main theorem. In Sect. 3 we introduce the symplectic projection onto the solitary manifold. The linearized equation is defined in Sect. 4. In Sect. 5 we split the dynamics in two components: along the solitary manifold and in the transversal directions. In Sect. 6 the modulation equations for the parameters of the soliton are displayed. The time decay of the transversal component is established in Sects. 7-11. Finally, in Sect. 12 we obtain the soliton asymptotics (1.21).

2. Main Results

2.1. Existence of dynamics. We consider the Cauchy problem for the Hamilton system (1.12) which we write as

$$\dot{Y}(t) = \mathcal{F}(Y(t)), \quad t \in \mathbb{R}: \quad Y(0) = Y_0. \tag{2.1}$$

Here $Y(t) = (\psi(t), \pi(t))$, $Y_0 = (\psi_0, \pi_0)$, and all derivatives are understood in the sense of distributions. To formulate our results precisely, let us introduce a suitable phase space for the Cauchy problem (2.1).

Definition 2.1. i) *$E_\alpha := H^1_\alpha \oplus L^2_\alpha$ is the space of the states $Y = (\psi, \pi)$ with finite norm*

$$\| Y \|_{E_\alpha} = \|\psi\|_{H^1_\alpha} + \|\pi\|_{L^2_\alpha} < \infty. \tag{2.2}$$

ii) *The phase space $\mathcal{E} := \mathcal{S} + E$, where $E = E_0$ and $\mathcal{S}$ is defined in* (1.15)*. The metric in $\mathcal{E}$ is defined as*

$$\rho_{\mathcal{E}}(Y_1, Y_2) = \|Y_1 - Y_2\|_E, \quad Y_1, Y_2 \in \mathcal{E}. \tag{2.3}$$

iii) *$W := W^{2,1}_0 \oplus W^{1,1}_0$ is the space of the states $Y = (\psi, \pi)$ with the finite norm*

$$\| Y \|_W = \|\psi\|_{W^{2,1}_0} + \|\pi\|_{W^{1,1}_0} < \infty. \tag{2.4}$$

Obviously, the Hamilton functional (1.1) is continuous on the phase space $\mathcal{E}$. The existence and uniqueness of the solutions to the Cauchy problem (2.1) follows by methods [15,20,26]:

Proposition 2.2. (i) *For any initial data $Y_0 \in \mathcal{E}$ there exists the unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ to the problem* (2.1).
(ii) *For every $t \in \mathbb{R}$, the map $U(t) : Y_0 \mapsto Y(t)$ is continuous in $\mathcal{E}$.*
(iii) *The energy is conserved, i.e.*

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y_0), \quad t \in \mathbb{R}. \tag{2.5}$$

2.2. Solitary manifold and main result. Let us consider the solitons (1.14). The substitution to (1.12) gives the following stationary equations:

$$\left.\begin{aligned} -v\psi_v'(y) &= \pi_v(y), \\ -v\pi_v'(y) &= \psi_v''(y) + F(\psi_v(y)). \end{aligned}\right| \tag{2.6}$$

Definition 2.3. *A soliton state is $S(\sigma) := (\psi_v(x-b), \pi_v(x-b))$, where $\sigma := (b, v)$ with $b \in \mathbb{R}$ and $v \in (-1, 1)$.*

Obviously, the soliton solution (1.13) admits the representation $S(\sigma(t))$, where

$$\sigma(t) = (b(t), v(t)) = (vt + q, v). \tag{2.7}$$

Definition 2.4. *A solitary manifold is the set $\mathcal{S} := \{S(\sigma) : \sigma \in \Sigma := \mathbb{R} \times (-1, 1)\}$.*

The main result of our paper is the following theorem

Theorem 2.5. *Let the conditions* **U1** *and* **U2** *hold, and $Y(t)$ be the solution to the Cauchy problem* (2.1) *with an initial state $Y_0 \in \mathcal{E}$ which is close to a kink $S(\sigma_0) = S_{q_0,v_0}$:*

$$Y_0 = S(\sigma_0) + X_0, \quad d_0 := \|X_0\|_{E_\beta \cap W} \ll 1, \tag{2.8}$$

where $\beta > 5/2$. Then for d_0 sufficiently small the solution admits the asymptotics:

$$Y(x,t) = (\psi_{v_\pm}(x - v_\pm t - q_\pm), \pi_{v_\pm}(x - v_\pm t - q_\pm)) + W_0(t)\Phi_\pm r_\pm(x,t), \quad t \to \pm\infty, \tag{2.9}$$

where $v_\pm$ and $q_\pm$ are constants, $\Phi_\pm \in E$, and $W_0(t)$ is the dynamical group of the free Klein-Gordon equation, while

$$\|r_\pm(t)\|_E = \mathcal{O}(|t|^{-1/2}). \tag{2.10}$$

It suffices to prove the asymptotics (2.9) for $t \to +\infty$ since the system (1.12) is time reversible.

3. Symplectic Projection

3.1. Symplectic structure and hamiltonian form. The system (2.1) reads as the Hamilton system

$$\dot{Y} = J\mathcal{D}\mathcal{H}(Y), \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = (\psi, \pi) \in \mathcal{E}, \tag{3.1}$$

where $\mathcal{D}\mathcal{H}$ is the Fréchet derivative of the Hamilton functional (1.1). Let us identify the tangent space of $\mathcal{E}$, at every point, with the space E. Consider the symplectic form Ω on E defined by

$$\Omega(Y_1, Y_2) = \langle Y_1, JY_2 \rangle, \quad Y_1, Y_2 \in E, \tag{3.2}$$

where

$$\langle Y_1, Y_2 \rangle := \langle \psi_1, \psi_2 \rangle + \langle \pi_1, \pi_2 \rangle$$

and $\langle \psi_1, \psi_2 \rangle = \int \psi_1(x)\psi_2(x)dx$, etc. It is clear that the form Ω is non-degenerate, i.e.

$$\Omega(Y_1, Y_2) = 0 \text{ for every } Y_2 \in E \implies Y_1 = 0.$$

Definition 3.1. i) *The symbol $Y_1 \nmid Y_2$ means that $Y_1 \in E$, $Y_2 \in E$, and Y_1 is symplectic orthogonal to Y_2, i.e. $\Omega(Y_1, Y_2) = 0$.*

ii) *A projection operator* $\mathbf{P} : E \to E$ *is said to be symplectic orthogonal if $Y_1 \nmid Y_2$ for $Y_1 \in \operatorname{Ker} \mathbf{P}$ and $Y_2 \in \operatorname{Range} \mathbf{P}$.*

3.2. Symplectic projection onto solitary manifold. Let us consider the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ of the manifold $\mathcal{S}$ at a point $S(\sigma)$. The vectors

$$\begin{aligned} \tau_1 = \tau_1(v) := \partial_b S(\sigma) = (-\psi_v'(y),\ -\pi_v'(y)),\\ \tau_2 = \tau_2(v) := \partial_v S(\sigma) = (\partial_v\psi_v(y),\ \partial_v\pi_v(y)) \end{aligned} \tag{3.3}$$

form a basis in $\mathcal{T}_{S(\sigma)}\mathcal{S}$. Here $y := x - b$ is the "moving frame coordinate". Let us stress that the functions τ_j are always regarded as functions of y rather than those of x. Formula (1.14) implies that

$$\tau_j(v) \in E_\alpha, \quad v \in (-1, 1), \quad j = 1, 2, \quad \forall\alpha \in \mathbb{R}. \tag{3.4}$$

Lemma 3.2. *The symplectic form Ω is nondegenerate on the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$, i.e. $\mathcal{T}_{S(\sigma)}\mathcal{S}$ is a symplectic subspace.*

Proof. Let us compute the vectors τ_1 and τ_2. Recall that $\psi_v(y) = s(\gamma y)$ and $\pi_v = -v\psi_v'(y) = -v\gamma s'(\gamma y)$ with $\gamma = 1/\sqrt{1-v^2}$. Then

$$\tau_1 = (\tau_1^1, \tau_1^2) = \Big(-\gamma s'(\gamma y),\ v\gamma^2 s''(\gamma y)\Big),$$
$$\tau_2 = (\tau_2^1, \tau_2^2) = \Big(vy\gamma^3 s'(\gamma y),\ -\gamma^3 s'(\gamma y) - v^2 y\gamma^4 s''(\gamma y)\Big).$$

Therefore

$$\Omega(\tau_1, \tau_2) = \langle\tau_1^1, \tau_2^2\rangle - \langle\tau_1^2, \tau_2^1\rangle = \gamma^4\langle s'(\gamma y), s'(\gamma y)\rangle > 0. \tag{3.5}$$

□

Now we show that in a small neighborhood of the soliton manifold $\mathcal{S}$ a "symplectic orthogonal projection" onto $\mathcal{S}$ is well-defined. Let us introduce the translations $T_q : (\psi(x), \pi(x)) \mapsto (\psi(x-q), \pi(x-q))$, $q \in \mathbb{R}$. Note that the manifold $\mathcal{S}$ is invariant with respect to the translations.

Definition 3.3. *For any $\overline{v} < 1$ denote by $\Sigma(\overline{v}) = \{\sigma = (b, v) : b \in \mathbb{R}, |v| \le \overline{v}\}$.*

Let us note that $\mathcal{S} \subset E_\alpha$ with $\alpha < -1/2$.

Lemma 3.4. *Let $\alpha < -1/2$ and $\overline{v} < 1$. Then*

i) *there exists a neighborhood $\mathcal{O}_\alpha(\mathcal{S})$ of $\mathcal{S}$ in E_α and a mapping $\mathbf{\Pi} : \mathcal{O}_\alpha(\mathcal{S}) \to \mathcal{S}$ such that $\mathbf{\Pi}$ is uniformly continuous on $\mathcal{O}_\alpha(\mathcal{S})$ in the metric of E_α,*

$$\mathbf{\Pi}Y = Y \ \textit{for}\ Y \in \mathcal{S}, \quad \textit{and} \quad Y - S \nmid \mathcal{T}_S\mathcal{S}, \ \textit{where}\ S = \mathbf{\Pi}Y. \tag{3.6}$$

ii) *$\mathcal{O}_\alpha(\mathcal{S})$ is invariant with respect to the translations T_q, and*

$$\mathbf{\Pi}T_qY = T_q\mathbf{\Pi}Y, \quad \textit{for}\ Y \in \mathcal{O}_\alpha(\mathcal{S}) \ \textit{and}\ q \in \mathbb{R}. \tag{3.7}$$

iii) *For any $\overline{v} < 1$ there exists an $r_\alpha(\overline{v}) > 0$ s.t. $S(\sigma) + X \in \mathcal{O}_\alpha(\mathcal{S})$ if $\sigma \in \Sigma(\overline{v})$ and $\|X\|_{E_\alpha} < r_\alpha(\overline{v})$.*

Proof. We have to find $\sigma = \sigma(Y)$ such that $S(\sigma) = \mathbf{\Pi} Y$ and

$$\Omega(Y - S(\sigma), \partial_{\sigma_j} S(\sigma)) = 0, \qquad j = 1, 2. \tag{3.8}$$

Let us fix an arbitrary $\sigma^0 \in \Sigma$ and note that the system (3.8) involves two smooth scalar functions of Y. Then for Y close to $S(\sigma^0)$, the existence of σ follows by the standard finite dimensional implicit function theorem if we show that the 2×2 Jacobian matrix with elements $M_{lj}(Y) = \partial_{\sigma_l} \Omega(Y - S(\sigma^0), \partial_{\sigma_j} S(\sigma^0))$ is non-degenerate at $Y = S(\sigma^0)$. First note that all the derivatives exist by (3.4). The non-degeneracy holds by Lemma 3.2 and the definition (3.3) since $M_{lj}(S(\sigma^0)) = -\Omega(\partial_{\sigma_l} S(\sigma^0), \partial_{\sigma_j} S(\sigma^0))$. Thus, there exists some neighborhood $\mathcal{O}_\alpha(S(\sigma^0))$ of $S(\sigma^0)$, where $\mathbf{\Pi}$ is well defined and satisfies (3.6), and the same is true in the union $\mathcal{O}'_\alpha(\mathcal{S}) = \cup_{\sigma^0 \in \Sigma} \mathcal{O}_\alpha(S(\sigma^0))$. The identity (3.7) holds for $Y, T_q Y \in \mathcal{O}'_\alpha(\mathcal{S})$, since the form Ω and the manifold $\mathcal{S}$ are invariant with respect to the translations. It remains to modify $\mathcal{O}'_\alpha(\mathcal{S})$ by the translations: we set $\mathcal{O}_\alpha(\mathcal{S}) = \cup_{b \in \mathbb{R}} T_b \mathcal{O}'_\alpha(\mathcal{S})$. Then the second statement obviously holds.

The last two statements and the uniform continuity in the first statement follow by translation invariance and the compactness arguments. □

We refer to $\mathbf{\Pi}$ as the symplectic orthogonal projection onto $\mathcal{S}$.

4. Linearization on the Solitary Manifold

Let us consider a solution to the system (1.12), and split it as the sum

$$Y(t) = S(\sigma(t)) + X(t), \tag{4.1}$$

where $\sigma(t) = (b(t), v(t)) \in \Sigma$ is an arbitrary smooth function of $t \in \mathbb{R}$. In detail, denote $Y = (\psi, \pi)$ and $X = (\Psi, \Pi)$. Then (4.1) means that

$$\left.\begin{aligned} \psi(x,t) &= \psi_{v(t)}(x - b(t)) + \Psi(x - b(t), t),\\ \pi(x,t) &= \pi_{v(t)}(x - b(t)) + \Pi(x - b(t), t). \end{aligned}\right| \tag{4.2}$$

Let us substitute (4.2) to (1.12), and linearize the equations in X. Setting $y = x - b(t)$ which is the "moving frame coordinate", we obtain that

$$\left.\begin{aligned} \dot\psi &= \dot v \partial_v \psi_v(y) - \dot b \psi'_v(y) + \dot\Psi(y,t) - \dot b \Psi'(y,t) = \pi_v(y) + \Pi(y,t),\\ \dot\pi &= \dot v \partial_v \pi_v(y) - \dot b \pi'_v(y) + \dot\Pi(y,t) - \dot b \Pi'(y,t) = \psi''_v(y) + \Psi''(y,t) + F(\psi_v(y) + \Psi(y,t)). \end{aligned}\right| \tag{4.3}$$

Using Eq. (2.6), we obtain from (4.3) the following equations for the components of the vector $X(t)$:

$$\left.\begin{aligned} \dot\Psi(y,t) &= \Pi(y,t) + \dot b \Psi'(y,t) + (\dot b - v)\psi'_v(y) - \dot v \partial_v \psi_v(y),\\ \dot\Pi(y,t) &= \Psi''(y,t) + \dot b \Pi'(y,t) + (\dot b - v)\pi'_v(y) - \dot v \partial_v \pi_v(y) + F(\psi_v(y) + \Psi(y,t)) - F(\psi_v(y)). \end{aligned}\right| \tag{4.4}$$

We can write Eq. (4.4) as

$$\dot X(t) = A(t) X(t) + T(t) + \mathcal{N}(t), \quad t \in \mathbb{R}, \tag{4.5}$$

where $T(t)$ is the sum of terms which do not depend on X, and $\mathcal{N}(t)$ is at least quadratic in X. The linear operator $A(t) = A_{v,w}$ depends on two parameters, $v = v(t)$, and $w = \dot{b}(t)$ and can be written in the form

$$A_{v,w}\begin{pmatrix}\Psi\\ \Pi\end{pmatrix} := \begin{pmatrix} w\nabla & 1\\ \Delta + F'(\psi_v) & w\nabla\end{pmatrix}\begin{pmatrix}\Psi\\ \Pi\end{pmatrix} = \begin{pmatrix} w\nabla & 1\\ \Delta - m^2 - V_v(y) & w\nabla\end{pmatrix}\begin{pmatrix}\Psi\\ \Pi\end{pmatrix}, \tag{4.6}$$

where

$$V_v(y) = -F'(\psi_v) - m^2. \tag{4.7}$$

Furthermore, $T(t)$ and $\mathcal{N}(t) = \mathcal{N}(\sigma, X)$ are given by

$$T = \begin{pmatrix}(w-v)\psi_v' - \dot{v}\partial_v\psi_v\\ (w-v)\pi_v' - \dot{v}\partial_v\pi_v\end{pmatrix}, \quad \mathcal{N}(\sigma, X) = \begin{pmatrix}0\\ N(v,\Psi)\end{pmatrix}, \tag{4.8}$$

where $v = v(t)$, $w = w(t)$, $\sigma = \sigma(t) = (b(t), v(t))$, $X = X(t)$, and

$$N(v,\Psi) = F(\psi_v + \Psi) - F(\psi_v) - F'(\psi_v)\Psi, \tag{4.9}$$

Remark 4.1. Formulas (3.3) and (4.8) imply:

$$T(t) = -(w-v)\tau_1 - \dot{v}\tau_2, \tag{4.10}$$

and hence $T(t) \in \mathcal{T}_{S(\sigma(t))}\mathcal{S}$, $t \in \mathbb{R}$. This fact suggests an unstable character of the nonlinear dynamics *along the solitary manifold.*

4.1. Linearized equation. Here we collect some Hamiltonian and spectral properties of the operator $A_{v,w}$. First, let us consider the linear equation

$$\dot{X}(t) = A_{v,w}X(t), \qquad t \in \mathbb{R} \tag{4.11}$$

with arbitrary fixed $v \in (-1, 1)$ and $w \in \mathbb{R}$. Let us define the space $E^+ := H^2(\mathbb{R}) \oplus H^1(\mathbb{R})$.

Lemma 4.2. i) *For any* $v \in (-1, 1)$ *and* $w \in \mathbb{R}$, *Eq.* (4.11) *can be represented as the Hamiltonian system,*

$$\dot{X}(t) = JD\mathcal{H}_{v,w}(X(t)), \qquad t \in \mathbb{R}, \tag{4.12}$$

where $D\mathcal{H}_{v,w}$ *is the Fréchet derivative of the Hamiltonian functional*

$$\mathcal{H}_{v,w}(X) = \frac{1}{2}\int\left[|\Pi|^2 + |\Psi'|^2 + (m^2 + V_v)|\Psi|^2\right]dy + \int \Pi w\Psi' dy. \tag{4.13}$$

ii) *The energy conservation law holds for the solutions* $X(t) \in C^1(\mathbb{R}, E^+)$,

$$\mathcal{H}_{v,w}(X(t)) = \text{const}, \qquad t \in \mathbb{R}. \tag{4.14}$$

iii) *The skew-symmetry relation holds*:

$$\Omega(A_{v,w}X_1, X_2) = -\Omega(X_1, A_{v,w}X_2), \qquad X_1, X_2 \in E. \tag{4.15}$$

Proof. i) The equation (4.11) reads as follows:

$$\frac{d}{dt}\begin{pmatrix}\Psi\\ \Pi\end{pmatrix} = \begin{pmatrix}\Pi + w\Psi'\\ \Psi'' - (m^2 + V_v)\Psi + w\Pi'\end{pmatrix}. \tag{4.16}$$

The equations correspond to the Hamilton form since

$$\Pi + w\Psi' = D_\Pi \mathcal{H}_{v,w}, \quad \Psi'' - (m^2 + V_v)\Psi + w\Pi' = -D_\Psi \mathcal{H}_{v,w}.$$

ii) The energy conservation law follows by (4.12) and the chain rule for the Fréchet derivatives:

$$\frac{d}{dt}\mathcal{H}_{v,w}(X(t)) = \langle D\mathcal{H}_{v,w}(X(t)), \dot{X}(t)\rangle = \langle D\mathcal{H}_{v,w}(X(t)), JD\mathcal{H}_{v,w}(X(t))\rangle = 0, \quad t \in \mathbb{R}, \tag{4.17}$$

since the operator J is skew-symmetric by (3.1), and $D\mathcal{H}_{v,w}(X(t)) \in E$ for $X(t) \in E^+$.

iii) The skew-symmetry holds since $A_{v,w}X = JD\mathcal{H}_{v,w}(X)$, and the linear operator $X \mapsto D\mathcal{H}_{v,w}(X)$ is symmetric as the Fréchet derivative of a real quadratic form. □

Lemma 4.3. *The operator $A_{v,w}$ acts on the tangent vectors $\tau = \tau_j(v)$ to the solitary manifold as follows*:

$$A_{v,w}[\tau_1] = (v - w)\tau_1', \quad A_{v,w}[\tau_2] = (w - v)\tau_2' + \tau_1. \tag{4.18}$$

Proof. In detail, we have to show that

$$A_{v,w}\begin{pmatrix}-\psi_v'\\ -\pi_v'\end{pmatrix} = \begin{pmatrix}(v - w)\psi_v''\\ (v - w)\pi_v''\end{pmatrix}, \quad A_{v,w}\begin{pmatrix}\partial_v\psi_v\\ \partial_v\pi_v\end{pmatrix} = \begin{pmatrix}(w - v)\partial_v\psi_v'\\ (w - v)\partial_v\pi_v'\end{pmatrix} + \begin{pmatrix}-\psi_v'\\ -\pi_v'\end{pmatrix}.$$

Indeed, differentiate Eqs. (2.6) in b and v, and obtain that the derivatives of the soliton state in parameters satisfy the following equations:

$$\begin{aligned} -v\psi_v'' &= \pi_v', \quad -v\pi_v'' = \psi_v''' + F'(\psi_v)\psi_v', \\ -\psi_v' - v\partial_v\psi_v' &= \partial_v\pi_v, \quad -\pi_v' - v\partial_v\pi_v' = \partial_v\psi_v'' + F'(\psi_v)\partial_v\psi_v. \end{aligned} \tag{4.19}$$

Then (4.18) follows from (4.19) by definition of $A_{v,w}$ in (4.6) □

Now we consider the operator $A_v = A_{v,v}$ corresponding to $w = v$:

$$A_v := \begin{pmatrix} v\nabla & 1\\ \Delta - m^2 - V_v & v\nabla\end{pmatrix}. \tag{4.20}$$

In that case the linearized equation has the following additional specific features. The continuous spectrum of the operator A_v coincides with

$$\Gamma := (-i\infty, -im/\gamma\,] \cup [im/\gamma,\ i\infty). \tag{4.21}$$

From (4.18) it follows that the tangent vector $\tau_1(v)$ is the zero eigenvector, and $\tau_2(v)$ is the corresponding root vector of the operator A_v, i.e.

$$A_v[\tau_1(v)] = 0, \quad A_v[\tau_2(v)] = \tau_1(v). \tag{4.22}$$

Lemma 4.4. *Zero root space of operator A_v is two-dimensional for any $v \in (-1, 1)$.*

Proof. It suffices to check that the equation $A_v u = \tau_2(v)$ has no solution in $L^2 \oplus L^2$. Indeed, the equation reads

$$\begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 - V_v & v\nabla \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} v\gamma^2 y\psi_v' \\ -\gamma^2\psi_v' - v^2\gamma^2 y\psi_v'' \end{pmatrix}. \tag{4.23}$$

From the first equation we get $u_2 = v\gamma^2 y\psi_v' - vu_1'$. Then the second equation implies that

$$H_v u_1 = \gamma^2(1 + v^2)\psi_v' + 2v^2\gamma^2 y\psi_v'', \tag{4.24}$$

where H_v is the Schrödinger operator defined in (1.19). Setting $u_1 = -\frac{1}{2}v^2\gamma^4 y^2\psi_v' + \tilde{u}_1$, we reduce the equation to

$$H_v\tilde{u}_1 = -\gamma^2\psi_v', \tag{4.25}$$

since $\psi_v''' = \gamma^2(m^2 + V_v)\psi_v'$ by the first line of (4.19). Hence, $\tilde{u}_1$ is the root function of the operator H_v since ψ_v' is an eigenfunction. However, this is impossible since H_v is a selfadjoint operator. □

Lemma 4.5. *The operator A_v has only eigenvalue $\lambda = 0$.*

Proof. Let us consider the eigenvalues problem for operator A_v:

$$\begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 - V_v & v\nabla \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

From the first equation we have $u_2 = -(v\nabla - \lambda)u_1$. Then the second equation implies that

$$(H_v + \lambda^2 - 2v\lambda\nabla)u_1 = 0. \tag{4.26}$$

Hence, for $v = 0$ the operator A_0 has only eigenvalue $\lambda = 0$ by Condition **U2** i).

Further, let us consider the case $v \neq 0$. Taking the scalar product with u_1, we obtain

$$\langle H_v u_1, u_1\rangle + \lambda^2\langle u_1, u_1\rangle = 0.$$

Hence, λ^2 is real since the operator H_v is selfadjoint. The nonzero eigenvalues can bifurcate either from the point $\lambda = 0$ or from the edge points $\pm im/\gamma$ of the continuous spectrum of the operator A_v. Let us consider each case separately.

i) The point $\lambda = 0$ cannot bifurcate since it is isolated, and the zero root space is two dimensional by Lemma 4.4.
ii) The bifurcation from the edge points also is impossible. Indeed, the bifurcated eigenvalue $\lambda \in (-im/\gamma, im/\gamma)$ is pure imaginary because λ^2 is real. Hence, (4.26) is equivalent to

$$\left(H_v + \gamma^2\lambda^2\right) p = 0, \tag{4.27}$$

where $p(x) = e^{\gamma^2 v\lambda x}u_1(x) \in L^2$ that is forbidden by Condition **U2** i) since $-\gamma^2\lambda^2 \in (0, m^2)$. □

4.2. Decay for the linearized dynamics. Let us consider the linearized equation

$$\dot{X}(t) = A_v X(t), \quad t \in \mathbb{R}, \tag{4.28}$$

where $A_v = A_{v,v}$ is given in (4.20) with V_v is defined in (4.7).

Definition 4.6. *For $|v| < 1$, denote by $\mathbf{P}_v^d$ the symplectic orthogonal projection of E onto the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$, and $\mathbf{P}_v^c = \mathbf{I} - \mathbf{P}_v^d$.*

Note that by the linearity,

$$\mathbf{P}_v^d X = \sum p_{jl}(v)\tau_j(v)\Omega(\tau_l(v), X), \quad X \in E \tag{4.29}$$

with some smooth coefficients $p_{jl}(v)$. Hence, the projector $\mathbf{P}_v^d$, in the variable $y = x - b$, does not depend on b.

Next decay estimates will play the key role in our proofs. The first estimate follows from our assumption **U2** by Theorem 3.15 of [14] since the condition of type [14, (1.3)] holds in our case (see also [13]).

Theorem 4.7. *Let the condition* **U2** *hold, and $\beta > 5/2$. Then for any $X \in E_\beta$, the weighted energy decay holds*:

$$\|e^{A_v t}\mathbf{P}_v^c X\|_{E_{-\beta}} \le C(v)(1+t)^{-3/2}\|X\|_{E_\beta}, \quad t \in \mathbb{R}, \tag{4.30}$$

Corollary 4.8. *For $\beta > 5/2$ and for $X \in E_\beta \cap W$,*

$$\|(e^{A_v t}\mathbf{P}_v^c X)_1\|_{L^\infty} \le C(v)(1+t)^{-1/2}(\|X\|_W + \|X\|_{E_\beta}), \quad t \in \mathbb{R}. \tag{4.31}$$

Here $(\cdot)_1$ stands for the first component of the vector function.

Proof. Let us apply the projector $\mathbf{P}_v^c$ to both sides of (4.28):

$$\mathbf{P}_v^c \dot{X} = A_v \mathbf{P}_v^c X = A_v^0 \mathbf{P}_v^c X + \mathbf{V}_v \mathbf{P}_v^c X, \tag{4.32}$$

where

$$A_v^0 = \begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 & v\nabla \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} 0 & 0 \\ -V_v & 0 \end{pmatrix}.$$

Hence, the Duhamel representation gives,

$$e^{A_v t}Y = e^{A_v^0 t}Y + \int_0^t e^{A_v^0(t-\tau)}\mathbf{V}e^{A_v\tau}Y d\tau, \quad Y = \mathbf{P}_v^c X, \quad t > 0. \tag{4.33}$$

Let us note that $e^{A_v^0 t}Z = e^{A_0^0 t}T_{vt}Z$, where $T_{vt}Z(x,t) = Z(x+vt,t)$. Then (4.33) reads

$$e^{A_v t}Y = e^{A_0^0 t}T_{vt}Y + \int_0^t e^{A_0^0(t-\tau)}T_{vt}[\mathbf{V}e^{A_v\tau}Y]d\tau, \quad t > 0. \tag{4.34}$$

Applying estimate (265) from [21], the Hölder inequality and Theorem 4.7 we obtain

$$\begin{aligned}
\|(e^{A_v t}Y)_1\|_{L^\infty} &\le C(1+t)^{-1/2}\|T_{vt}Y\|_W + C\int_0^t (1+t-\tau)^{-1/2}\|T_{vt}[V(e^{A_v\tau}Y)_1]\|_{W_0^{1,1}}\,d\tau\\
&= C(1+t)^{-1/2}\|Y\|_W + C\int_0^t (1+t-\tau)^{-1/2}\|V(e^{A_v\tau}Y)_1\|_{W_0^{1,1}}\,d\tau\\
&\le C(1+t)^{-1/2}\|X\|_W + C\int_0^t (1+t-\tau)^{-1/2}\|e^{A_v\tau}\mathbf{P}_v^c X\|_{E_{-\beta}}\,d\tau\\
&\le C(1+t)^{-1/2}\|X\|_W + C\int_0^t (1+t-\tau)^{-1/2}(1+\tau)^{-3/2}\|X\|_{E_\beta}\,d\tau\\
&\le C(1+t)^{-1/2}(\|X\|_W + \|X\|_{E_\beta}).
\end{aligned}$$

□

4.3. Taylor expansion for nonlinear term. Now let us expand $N(v,\Psi)$ from (4.9) in the Taylor series

$$\begin{aligned}
N(v,\Psi) &= N_2(v,\Psi) + N_3(v,\Psi) + \cdots + N_{12}(v,\Psi) + N_R(v,\Psi)\\
&= N_I(v,\Psi) + N_R(v,\Psi),
\end{aligned} \tag{4.35}$$

where

$$N_j(v,\Psi) = \frac{F^{(j)}(\psi_v)}{j!}\Psi^j, \quad j = 2,\dots,12 \tag{4.36}$$

and N_R is the remainder. By condition **U1** we have

$$F(\psi) = -m^2(\psi \mp a) + \mathcal{O}(|\psi \mp a|^{13}), \quad \psi \to \pm a.$$

Hence, the functions $F^{(j)}(\psi_v(y))$, $2 \le j \le 12$ decrease exponentially as $|y| \to \infty$ by (1.18) and (1.14). Therefore,

$$\|N_I\|_{L^2_\beta \cap W_0^{1,1}} = \mathcal{R}(\|\Psi\|_{L^\infty})\|\Psi\|_{L^\infty}\|\Psi\|_{H^1_{-\beta}} = \mathcal{R}(\|\Psi\|_{L^\infty})\|\Psi\|_{L^\infty}\|X\|_{E_{-\beta}}. \tag{4.37}$$

For the remainder N_R we have

$$|N_R| = \mathcal{R}(\|\Psi\|_{L^\infty})|\Psi|^{13}, \tag{4.38}$$

where $\mathcal{R}(A)$ is a general notation for a positive function which remains bounded as A is sufficiently small.

Lemma 4.9. *The bounds hold*:

$$\|N_R\|_{W_0^{1,1}} = \mathcal{R}(\|\Psi\|_{L^\infty})\|\Psi\|^{11}_{L^\infty}, \tag{4.39}$$

$$\|N_R\|_{L^2_{5/2+\nu}} = \mathcal{R}(\|\Psi\|_{L^\infty})(1+t)^{4+\nu}\|\Psi\|^{12}_{L^\infty}, \quad 0 < \nu < 1/2. \tag{4.40}$$

Proof. Step i) By the Cauchy formula,

$$N_R(x,t) = \frac{\Psi^{13}(x,t)}{(13)!}\int_0^1 (1-\rho)^{12} F^{(13)}(\psi_v + \rho\Psi(x,t))d\rho, \tag{4.41}$$

Therefore,

$$\begin{aligned}\|N_R\|_{L^1} &= \mathcal{R}(\|\Psi\|_{L^\infty})\int |\Psi|^{13}dx = \mathcal{R}(\|\Psi\|_{L^\infty})\|\Psi\|_{L^\infty}^{11}\|\Psi\|_2^2 \\ &= \mathcal{R}(\|\Psi\|_{L^\infty})\|)\|\Psi\|_{L^\infty}^{11},\end{aligned}$$

since $\|\Psi\|_{L^2} \le C(d_0)$ by the results of [10]. Differentiating (4.41) in x, we obtain

$$\begin{aligned}N_R' = &\frac{\Psi^{13}}{(13)!}\int_0^1 (1-\rho)^{12}(\psi_v' + \rho\Psi')F^{(14)}(\psi_v + \rho\Psi)d\rho \\ &+\frac{\Psi^{12}\Psi'}{(12)!}\int_0^1 (1-\rho)^{12}F^{(13)}(\psi_v + \rho\Psi)d\rho,\end{aligned}$$

Hence,

$$\begin{aligned}\|N_R'\|_{L^1} &= \mathcal{R}(\|\Psi\|_{L^\infty})\left[\|\Psi\|_{L^\infty}^{13} + \|\Psi\|_{L^\infty}^{11}\int |\Psi(x)\Psi'(x)|dx\right] \\ &\le \mathcal{R}(\|\Psi\|_{L^\infty})\|\Psi\|_{L^\infty}^{11},\end{aligned}$$

since $\int |\Psi(x)\Psi'(x)|dx \le \|\Psi\|_{L^2}\|\Psi'\|_{L^2} \le C(d_0)$. Then (4.39) follows.

Step ii) The bound (4.38) implies

$$\|N_R\|_{L^2_{5/2+\nu}} = \mathcal{R}(\|\Psi\|_{L^\infty})\|\Psi\|_{L^\infty}^{12}\|\Psi\|_{L^2_{5/2+\nu}}.$$

We will prove in Appendix B that

$$\|\Psi(t)\|_{L^2_{5/2+\nu}} \le C(d_0)(1+t)^{4+\nu}. \tag{4.42}$$

Then (4.40) follows. □

Remark 4.10. Our choice of the degree 14 in the condition (1.11) is due to the competition between the factors in the estimate (4.40) for the remainder. Namely, the factor $(1+t)^{4+\nu}$ with $\nu < 1/2$ comes from the virial type estimate (4.42) describing the expansion of the support for the perturbation of the kink. On the other hand, $\|\Psi\|_{L^\infty}^{12} \sim t^{-6}$ by the crucial decay estimate (7.1). Hence, the right-hand side (4.40) decays like $\sim t^{-2+\nu}$, where $-2+\nu < -3/2$ which is sufficient for the method of majorants (in integral inequalities (9.2) and (9.3)).

5. Symplectic Decomposition of the Dynamics

Here we decompose the dynamics in two components: along the manifold $\mathcal{S}$ and in transversal directions. Equation (4.5) is obtained without any assumption on $\sigma(t)$ in (4.1). We are going to choose $S(\sigma(t)) := \mathbf{\Pi} Y(t)$, but then we need to know that

$$Y(t) \in \mathcal{O}_\alpha(\mathcal{S}), \qquad t \in \mathbb{R} \tag{5.1}$$

with some $\mathcal{O}_\alpha(\mathcal{S})$ defined in Lemma 3.4. It is true for $t = 0$ by our main assumption (2.8) with sufficiently small $d_0 > 0$. Then $S(\sigma(0)) = \mathbf{\Pi} Y(0)$ and $X(0) = Y(0) - S(\sigma(0))$ are well defined. We will prove below that (5.1) holds with $\alpha = -\beta$ if d_0 is sufficiently small. First, we choose $\overline{v} < 1$ such that

$$|v(0)| \le \overline{v}. \tag{5.2}$$

Denote by $r_{-\beta}(\overline{v})$ the positive number from Lemma 3.4 iii) which corresponds to $\alpha = -\beta$. Then $S(\sigma) + X \in \mathcal{O}_{-\beta}(\mathcal{S})$ if $\sigma = (b, v)$ with $|v| < \overline{v}$ and $\|X\|_{E_{-\beta}} < r_{-\beta}(\overline{v})$. Therefore, $S(\sigma(t)) = \mathbf{\Pi} Y(t)$ and $X(t) = Y(t) - S(\sigma(t))$ are well defined for $t \ge 0$ so small that $\|X(t)\|_{E_{-\beta}} < r_{-\beta}(\overline{v})$. This is formalized by the standard definition of the "exit time". First, we introduce the "majorants"

$$m_1(t) := \sup_{s\in[0,t]} (1+s)^{3/2} \|X(s)\|_{E_{-\beta}}, \qquad m_2(t) := \sup_{s\in[0,t]} (1+s)^{1/2} \|\Psi(s)\|_{L^\infty}. \tag{5.3}$$

Here $X = (X_1, X_2) = (\Psi, \Pi)$. Let us denote by $\varepsilon \in (0, r_{-\beta}(\overline{v}))$ a fixed number which we will specify below.

Definition 5.1. *t_* is the exit time*

$$t_* = \sup\{t \ge 0 : m_j(s) < \varepsilon, \quad j = 1, 2, \quad 0 \le s \le t\}. \tag{5.4}$$

Let us note that $m_j(0) < \varepsilon$ for sufficiently small d_0. One of our main goals is to prove that $t_* = \infty$ if d_0 is sufficiently small. This would follow if we show that

$$m_j(t) < \varepsilon/2, \qquad 0 \le t < t_*. \tag{5.5}$$

6. Modulation Equations

In this section we present the modulation equations which allow to construct the solutions $Y(t)$ of Eq. (2.1) close at each time t to a kink, i.e. to one of the functions described in Definition 2.3 with time varying ("modulating") parameters $(b, v) = (b(t), v(t))$. We look for a solution to (2.1) in the form $Y(t) = S(\sigma(t)) + X(t)$ by setting $S(\sigma(t)) = \mathbf{\Pi} Y(t)$ which is equivalent to the symplectic orthogonality condition of type (3.7),

$$X(t) \nmid \mathcal{T}_{S(\sigma(t))}\mathcal{S}, \quad t < t_*, \tag{6.1}$$

The projection $\mathbf{\Pi} Y(t)$ is well defined for $t < t_*$ by Lemma 3.4 iii). Now we derive the "modulation equations" for the parameters $\sigma(t) = (b(t), v(t))$. For this purpose, let us write (6.1) in the form

$$\Omega(X(t), \tau_j(t)) = 0, \quad j = 1, 2, \tag{6.2}$$

where the vectors $\tau_j(t) = \tau_j(\sigma(t))$ span the tangent space $\mathcal{T}_{S(\sigma(t))}\mathcal{S}$. It would be convenient for us to use some other parameters (c, v) instead of $\sigma = (b, v)$, where $c(t) = b(t) - \int_0^t v(\tau) d\tau$ and

$$\dot{c}(t) = \dot{b}(t) - v(t) = w(t) - v(t) \tag{6.3}$$

Lemma 6.1. *Let $Y(t)$ be a solution to the Cauchy problem (2.1), and (6.2) hold. Then the parameters $c(t)$ and $v(t)$ satisfy the equations*

$$\dot{c} = \frac{\Omega(\tau_1, \tau_2)\Omega(\mathcal{N}, \tau_2) + \Omega(X, \partial_v \tau_1)\Omega(\mathcal{N}, \tau_2) - \Omega(X, \partial_v \tau_2)\Omega(\mathcal{N}, \tau_1)}{D} \quad (6.4)$$

$$\dot{v} = \frac{-\Omega(\tau_1, \tau_2)\Omega(\mathcal{N}, \tau_1) - \Omega(X, \tau_2')\Omega(\mathcal{N}, \tau_1) - \Omega(X, \tau_1')\Omega(\mathcal{N}, \tau_2)}{D}, \quad (6.5)$$

where

$$D = \Omega^2(\tau_1, \tau_2) + \mathcal{O}(\|X\|_{E_{-\beta}}).$$

Proof. Differentiating the orthogonality conditions (6.2) in t we obtain

$$0 = \Omega(\dot{X}, \tau_j) + \Omega(X, \dot{\tau}_j) = \Omega(A_{v,w}X + T + \mathcal{N}, \tau_j) + \Omega(X, \dot{\tau}_j), \quad j = 1, 2. \quad (6.6)$$

First, let us compute the principal (i.e. non-vanishing at $X = 0$) term $\Omega(T, \tau_j)$. By (4.10),

$$\Omega(T, \tau_1) = -\dot{v}\Omega(\tau_2, \tau_1) = \dot{v}\Omega(\tau_1, \tau_2); \quad \Omega(T, \tau_2) = -\dot{c}\Omega(\tau_1, \tau_2). \quad (6.7)$$

Second, let us compute $\Omega(A_{v,w}X, \tau_j)$. The skew-symmetry (4.15) implies that $\Omega(A_{v,w}X, \tau_j) = -\Omega(X, A_{v,w}\tau_j)$. Then by (4.18) we have

$$\Omega(A_{v,w}X, \tau_1) = \Omega(X, \dot{c}\tau_1'), \quad (6.8)$$

$$\Omega(A_{v,w}X, \tau_2) = -\Omega(X, \dot{c}\tau_2' + \tau_1) = -\Omega(X, \dot{c}\tau_2'), \quad (6.9)$$

since $\Omega(X, \tau_1) = 0$.

Finally, let us compute the last term $\Omega(X, \dot{\tau}_j)$ in (6.6). For $j = 1, 2$ one has $\dot{\tau}_j = \dot{b}\partial_b\tau_j + \dot{v}\partial_v\tau_j = \dot{v}\partial_v\tau_j$ since the vectors τ_j do not depend on b according to (3.3). Hence,

$$\Omega(X, \dot{\tau}_j) = \Omega(X, \dot{v}\partial_v\tau_j). \quad (6.10)$$

As the result, by (6.7)–(6.10), Eq. (6.6) becomes

$$0 = \dot{c}\Omega(X, \tau_1') + \dot{v}\,(\Omega(\tau_1, \tau_2) + \Omega(X, \partial_v\tau_1)) + \Omega(\mathcal{N}, \tau_1),$$
$$0 = -\dot{c}\left(\Omega(X, \tau_2') + (\Omega(\tau_1, \tau_2)\right) + \dot{v}\Omega(X, \partial_v\tau_2) + \Omega(\mathcal{N}, \tau_2).$$

Since $\Omega(\tau_1, \tau_2) \neq 0$ by (3.5) then the determinant D of the system does not vanish for small $\|X\|_{E_{-\beta}}$ and we obtain (6.4)–(6.5). □

Corollary 6.2. *Formulas (6.4)–(6.5) imply*

$$|\dot{c}(t)|, \ |\dot{v}(t)| \le C(\overline{v})\|\Psi(t)\|^2_{L^2_{-\beta}} \le C(\overline{v})\|X(t)\|^2_{E_{-\beta}}, \qquad 0 \le t < t_*. \quad (6.11)$$

7. Decay for the Transversal Dynamics

In Sect. 12 we will show that our main Theorem 2.5 can be derived from the following time decay of the transversal component $X(t)$:

Proposition 7.1. *Let all conditions of Theorem 2.5 hold. Then $t_* = \infty$, and*

$$\|X(t)\|_{E_{-\beta}} \le \frac{C(\overline{v}, d_0)}{(1+|t|)^{3/2}}, \quad \|\Psi(t)\|_{L^\infty} \le \frac{C(\overline{v}, d_0)}{(1+|t|)^{1/2}}, \qquad t \ge 0. \tag{7.1}$$

We will derive (7.1) in Sects. 11 from our Eq. (4.5) for the transversal component $X(t)$. This equation can be specified using Corollary 6.2. Indeed, (4.10) implies that

$$\|T(t)\|_{E_\beta \cap W} \le C(\overline{v})\|X\|^2_{E_{-\beta}}, \qquad 0 \le t < t_* \tag{7.2}$$

by (6.11) since $w - v = \dot{c}$. Thus (4.5) becomes the equation

$$\dot{X}(t) = A(t)X(t) + T(t) + \mathcal{N}_I(t) + \mathcal{N}_R(t), \qquad 0 \le t < t_*, \tag{7.3}$$

where $A(t) = A_{v(t),w(t)}$, $T(t)$ satisfies (7.2), and

$$\left.\begin{aligned} \|\mathcal{N}_I(t)\|_{E_\beta \cap W} &\le C(\overline{v})\|\Psi\|_{L^\infty}\|X\|_{E_{-\beta}}, \\ \|\mathcal{N}_R\|_{E_{5/2+\nu}} &\le C(\overline{v})(1+t)^{4+\nu}\|\Psi\|^{12}_{L^\infty}, \quad 0 < \nu < 1/2, \\ \|\mathcal{N}_R\|_W &\le C(\overline{v})\|\Psi\|^{11}_{L^\infty} \end{aligned}\right| \quad 0 \le t < t_*, \tag{7.4}$$

by (4.37), (4.39–(4.40)). In remaining part of our paper we will analyze mainly Eq. (7.3) to establish the decay (7.1). We are going to derive the decay using the bounds (7.2) and (7.4), and the orthogonality condition (6.1).

Let us comment on two main difficulties in proving (7.1). The difficulties are common for the problems studied in [4]. First, the linear part of the equation is non-autonomous, hence we cannot apply directly the methods of scattering theory. Similarly to the approach of [4], we reduce the problem to the analysis of the *frozen* linear equation,

$$\dot{X}(t) = A_1 X(t), \quad t \in \mathbb{R}, \tag{7.5}$$

where A_1 is the operator A_{v_1} defined by (4.6) with $v_1 = v(t_1)$ for a fixed $t_1 \in [0, t_*)$. Then we estimate the error by the method of majorants.

Second, even for the frozen equation (7.5), the decay of type (7.1) for all solutions does not hold without the orthogonality condition of type (6.1). Namely, by (4.22) Eq. (7.5) admits the *secular solutions*

$$X(t) = C_1\tau_1(v) + C_2[\tau_1(v)t + \tau_2(v)] \tag{7.6}$$

which arise also by differentiation of the soliton (1.13) in the parameters q and v in the moving coordinate $y = x - v_1 t$. Hence, we have to take into account the orthogonality condition (6.1) in order to avoid the secular solutions. For this purpose we will apply the corresponding symplectic orthogonal projection which kills the "runaway solutions" (7.6).

Remark 7.2. The solution (7.6) lies in the tangent space $\mathcal{T}_{S(\sigma_1)}\mathcal{S}$ with $\sigma_1 = (b_1, v_1)$ (for an arbitrary $b_1 \in \mathbb{R}$) that suggests an unstable character of the nonlinear dynamics *along the solitary manifold* (cf. Remark 4.1 ii)).

Definition 7.3. *Denote by* $\mathcal{X}_v = \mathbf{P}_v^c E$ *the space symplectic orthogonal to* $\mathcal{T}_{S(\sigma)}\mathcal{S}$ *with* $\sigma = (b, v)$ (*for an arbitrary* $b \in \mathbb{R}$).

Now we have the symplectic orthogonal decomposition

$$E = \mathcal{T}_{S(\sigma)}\mathcal{S} + \mathcal{X}_v, \qquad \sigma = (b, v) \tag{7.7}$$

and the symplectic orthogonality (6.1) can be written in the following equivalent forms:

$$\mathbf{P}^d_{v(t)} X(t) = 0, \quad \mathbf{P}^c_{v(t)} X(t) = X(t), \qquad 0 \le t < t_*. \tag{7.8}$$

Remark 7.4. The tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ is invariant under the operator A_v by (4.22), hence the space $\mathcal{X}_v$ is also invariant by (4.15): $A_v X \in \mathcal{X}_v$ on a dense domain of $X \in \mathcal{X}_v$.

8. Frozen Form of Transversal Dynamics

Now let us fix an arbitrary $t_1 \in [0, t_*)$, and rewrite Eq. (7.3) in a "frozen form"

$$\dot X(t) = A_1 X(t) + (A(t) - A_1)X(t) + T(t) + \mathcal{N}_I(t) + \mathcal{N}_R(t), \quad 0 \le t < t_*, \tag{8.1}$$

where $A_1 = A_{v(t_1),v(t_1)}$ and

$$A(t) - A_1 = \begin{pmatrix} (w(t) - v(t_1))\nabla & 0 \\ 0 & (w(t) - v(t_1))\nabla \end{pmatrix}.$$

The next trick is important since it allows us to kill the "bad terms" $(w(t) - v(t_1))\nabla$ in the operator $A(t) - A_1$.

Let us change the variables $(y, t) \mapsto (y_1, t) = (y + d_1(t), t)$, where

$$d_1(t) := \int_{t_1}^t (w(s) - v(t_1))ds, \quad 0 \le t \le t_1. \tag{8.2}$$

Next define

$$\tilde X(t) = (\Psi(y_1 - d_1(t), t), \Pi(y_1 - d_1(t), t)). \tag{8.3}$$

Then we obtain the final form of the "frozen equation" for the transversal dynamics

$$\dot{\tilde X}(t) = A_1 \tilde X(t) + \tilde T(t) + \tilde{\mathcal{N}}_I(t) + \tilde{\mathcal{N}}_R(t), \quad 0 \le t \le t_1, \tag{8.4}$$

where $\tilde T(t)$, $\tilde{\mathcal{N}}_I(t)$ and $\tilde{\mathcal{N}}_R(t)$ are $T(t)$, $\mathcal{N}_I(t)$ and $\mathcal{N}_R(t)$ expressed in terms of $y_1 = y + d_1(t)$. Now we derive appropriate bounds for the "remainder terms" in (8.4). Let us recall the following well-known inequality: for any $\alpha \in \mathbb{R}$,

$$(1 + |y + x|)^\alpha \le (1 + |y|)^\alpha (1 + |x|)^{|\alpha|}, \qquad x, y \in \mathbb{R}. \tag{8.5}$$

Lemma 8.1. *For* $f \in L^2_\alpha$ *with any* $\alpha \in \mathbb{R}$ *the following bound holds*:

$$\|f(y_1 - d_1)\|_{L^2_\alpha} \le \|f\|_{L^2_\alpha} (1 + |d_1|)^{|\alpha|}, \quad d_1 \in \mathbb{R}. \tag{8.6}$$

Proof. The bound (8.6) follows from (8.5) since

$$\|f(y_1-d_1)\|^2_{L^2_\alpha}=\int|f(y_1-d_1)|^2(1+|y_1|)^{2\alpha}dy_1=\int|f(y)|^2(1+|y+d_1|)^{2\alpha}dy$$
$$\le\int|f(y)|^2(1+|y|)^{2\alpha}(1+|d_1|)^{2|\alpha|}dy\le(1+|d_1|)^{2|\alpha|}\|f\|^2_{L^2_\alpha}.$$

□

Corollary 8.2. *The following bounds hold for* $0\le t\le t_1$ *by* (7.2) *and* (7.4)*:*

$$\left.\begin{aligned}\|\tilde T(t)\|_{E_\beta}&\le C(\overline{v})(1+|d_1(t)|)^\beta\|X\|^2_{E_{-\beta}},\quad \|\tilde T(t)\|_W\le C(\overline{v})\|X\|^2_{E_{-\beta}},\\ \|\tilde{\mathcal N}_I(t)\|_{E_\beta}&\le C(\overline{v})(1+|d_1(t)|)^\beta\|\Psi\|_{L^\infty}\|X\|_{E_{-\beta}},\quad \|\tilde{\mathcal N}_I(t)\|_W\le C(\overline{v})\|\Psi\|_{L^\infty}\|X\|_{E_{-\beta}},\\ \|\tilde{\mathcal N}_R\|_{E_{5/2+\nu}}&\le C(\overline{v})(1+|d_1(t)|)^{5/2+\nu}(1+t)^{4+\nu}\|\Psi\|^{12}_{L^\infty},\quad 0<\nu<1/2,\\ \|\tilde{\mathcal N}_R\|_W&\le C(\overline{v})\|\Psi\|^{11}_{L^\infty}.\end{aligned}\right| \tag{8.7}$$

9. Integral Inequality

Equation (8.4) can be written in the integral form:

$$\tilde X(t)=e^{A_1t}\tilde X(0)+\int_0^t e^{A_1(t-s)}[\tilde T(s)+\tilde{\mathcal N}_I(s)+\tilde{\mathcal N}_R(s)]ds,\quad 0\le t\le t_1. \tag{9.1}$$

We apply the symplectic orthogonal projection $\mathbf{P}^c_1:=\mathbf{P}^c_{v(t_1)}$ to both sides, and get

$$\mathbf{P}^c_1\tilde X(t)=e^{A_1t}\mathbf{P}^c_1\tilde X(0)+\int_0^t e^{A_1(t-s)}\mathbf{P}^c_1\,[\tilde T(s)+\tilde{\mathcal N}_I(s)+\tilde{\mathcal N}_R(s)]\,ds.$$

We have used here that $\mathbf{P}^c_1$ commutes with the group e^{A_1t} since the space $\mathcal X_1:=\mathbf{P}^c_1E$ is invariant with respect to e^{A_1t} by Remark 7.4. Applying (4.30) we obtain that

$$\|\mathbf{P}^c_1\tilde X(t)\|_{E_{-\beta}}\le\frac{C\|\tilde X(0)\|_{E_\beta}}{(1+t)^{3/2}}+C\int_0^t\frac{\|\tilde T(s)+\tilde{\mathcal N}_I(s)+\tilde{\mathcal N}_R(s)\|_{E_\beta}}{(1+|t-s|)^{3/2}}ds.$$

Then for $5/2<\beta<3$ and $0\le t\le t_1$ the bounds (8.7) imply

$$\|\mathbf{P}^c_1\tilde X(t)\|_{E_{-\beta}}\le\frac{C(\overline{d}_1(0))}{(1+t)^{3/2}}\|X(0)\|_{E_\beta}$$
$$+C(\overline{d}_1(t))\int_0^t\frac{\|X(s)\|^2_{E_{-\beta}}+\|\Psi(s)\|_{L^\infty}\|X(s)\|_{E_{-\beta}}+(1+s)^{3/2+\beta}\|\Psi(s)\|^{12}_{L^\infty}}{(1+|t-s|)^{3/2}}ds, \tag{9.2}$$

where $\overline{d}_1(t):=\sup_{0\le s\le t}|d_1(s)|$. Similarly, (4.31) and (8.7) imply

$$\|(\mathbf{P}^c_1\tilde X(t))_1\|_{L^\infty}\le\frac{C\|\tilde X(0)\|_{E_\beta\cap W}}{(1+t)^{1/2}}+C\int_0^t\frac{\|\tilde T(s)+\tilde{\mathcal N}_I(s)+\tilde{\mathcal N}_R(s)\|_{E_\beta\cap W}}{(1+|t-s|)^{1/2}}ds$$
$$\le\frac{C(\overline{d}_1(0))}{(1+t)^{1/2}}\|X(0)\|_{E_\beta\cap W}+C(\overline{d}_1(t))$$
$$\times\int_0^t\frac{\|X(s)\|^2_{E_{-\beta}}+\|\Psi(s)\|_{L^\infty}\|X(s)\|_{E_{-\beta}}+(1+s)^{3/2+\beta}\|\Psi(s)\|^{12}_{L^\infty}+\|\Psi(s)\|^{11}_{L^\infty}}{(1+|t-s|)^{1/2}}ds. \tag{9.3}$$

Lemma 9.1. *For $t_1 < t_*$ we have*

$$|d_1(t)| \le C\varepsilon^2, \quad 0 \le t \le t_1. \tag{9.4}$$

Proof. To estimate $d_1(t)$, we note that

$$w(s) - v(t_1) = w(s) - v(s) + v(s) - v(t_1) = \dot{c}(s) + \int_s^{t_1} \dot{v}(\tau)d\tau \tag{9.5}$$

by (6.3). Hence, the definitions (8.2), (5.3), and Corollary 6.2 imply that

$$\begin{aligned} |d_1(t)| &= |\int_{t_1}^t (w(s) - v(t_1))ds| \le \int_t^{t_1} \left(|\dot{c}(s)| + \int_s^{t_1} |\dot{v}(\tau)|d\tau \right) ds \\ &\le Cm_1^2(t_1) \int_t^{t_1} \left(\frac{1}{(1+s)^3} + \int_s^{t_1} \frac{d\tau}{(1+\tau)^3} \right) ds \le Cm_1^2(t_1) \le C\varepsilon^2, \quad 0 \le t \le t_1. \end{aligned} \tag{9.6}$$

□

Now (9.2) and (9.3) imply that for $t_1 < t_*$ and $0 \le t \le t_1$,

$$\|\mathbf{P}_1^c \tilde{X}(t)\|_{E_{-\beta}} \le \frac{C\|X(0)\|_{E_\beta}}{(1+t)^{3/2}} + C \int_0^t \frac{\|X(s)\|_{E_{-\beta}}^2 + \|\Psi(s)\|_{L^\infty} \|X(s)\|_{E_{-\beta}} + (1+s)^{3/2+\beta} \|\Psi(s)\|_{L^\infty}^{12}}{(1+|t-s|)^{3/2}} ds, \tag{9.7}$$

$$\|(\mathbf{P}_1^c \tilde{X}(t))_1\|_{L^\infty} \le \frac{C\|X(0)\|_{E_\beta \cap W}}{(1+t)^{1/2}} + C \int_0^t \frac{\|X(s)\|_{E_{-\beta}}^2 + \|\Psi(s)\|_{L^\infty} \|X(s)\|_{E_{-\beta}} + (1+s)^{3/2+\beta} \|\Psi(s)\|_{L^\infty}^{12} + \|\Psi(s)\|_{L^\infty}^{11}}{(1+|t-s|)^{1/2}} ds. \tag{9.8}$$

10. Symplectic Orthogonality

Finally, we are going to change $\mathbf{P}_1^c \tilde{X}(t)$ by $X(t)$ in the left-hand side of (9.7) and (9.8). We will prove that it is possible since $d_0 \ll 1$ in (2.8).

Lemma 10.1. *For sufficiently small $\varepsilon > 0$, we have for $t_1 < t_*$,*

$$\|X(t)\|_{E_{-\beta}} \le C\|\mathbf{P}_1^c \tilde{X}(t)\|_{E_{-\beta}}, \quad 0 \le t \le t_1,$$
$$\|\Psi(t)\|_{L^\infty} \le 2\|(\mathbf{P}_1^c \tilde{X}(t))_1\|_{L^\infty}, \quad 0 \le t \le t_1,$$

where the constant C does not depend on t_1.

Proof. The proof is based on the symplectic orthogonality (7.8), i.e.

$$\mathbf{P}_{v(t)}^d X(t) = 0, \quad t \in [0, t_1] \tag{10.1}$$

and on the fact that all the spaces $\mathcal{X}(t) := \mathbf{P}_{v(t)}^c E$ are almost parallel for all t.

Namely, we first note that $\|\Psi(t)\|_{L^\infty} = \|\tilde\Psi(t)\|_{L^\infty}$, and $\|X(t)\|_{E_{-\beta}} \le C\|\tilde X(t)\|_{E_{-\beta}}$ by Lemma 8.1, since $|d_1(t)| \le \text{const}$ for $t \le t_1 < t_*$ by (9.4). Therefore, it suffices to prove that

$$\|\tilde\Psi(t)\|_{L^\infty} \le 2\|(\mathbf{P}_1^c\tilde X(t))_1\|_{L^\infty}, \quad \|\tilde X(t)\|_{E_{-\beta}} \le 2\|\mathbf{P}_1^c\tilde X(t)\|_{E_{-\beta}}, \quad 0\le t\le t_1. \tag{10.2}$$

This estimate will follow from

$$\|(\mathbf{P}_1^d\tilde X(t))_1\|_{L^\infty} \le \frac{1}{2}\|\tilde\Psi(t)\|_{L^\infty}, \quad \|\mathbf{P}_1^d\tilde X(t)\|_{E_{-\beta}} \le \frac{1}{2}\|\tilde X(t)\|_{E_{-\beta}}, \;\; 0\le t\le t_1. \tag{10.3}$$

since $\mathbf{P}_1^c\tilde X(t) = \tilde X(t) - \mathbf{P}_1^d\tilde X(t)$. To prove (10.3), we write (10.1) as,

$$\tilde{\mathbf{P}}^d_{v(t)}\tilde X(t) = 0, \quad t\in[0,t_1] \tag{10.4}$$

where $\tilde{\mathbf{P}}^d_{v(t)}\tilde X(t)$ is $\mathbf{P}^d_{v(t)}X(t)$ expressed in terms of the variable $y_1 = y + d_1(t)$. Hence, (10.3) follows from (10.4) if the difference $\mathbf{P}_1^d - \tilde{\mathbf{P}}^d_{v(t)}$ is small uniformly in t, i.e.

$$\|\mathbf{P}_1^d - \tilde{\mathbf{P}}^d_{v(t)}\| < 1/2, \qquad 0\le t\le t_1. \tag{10.5}$$

It remains to justify (10.5) for small enough $\varepsilon > 0$. In order to prove the bound (10.5), we will need the formula (4.29) and the following relation which follows from (4.29):

$$\tilde{\mathbf{P}}^d_{v(t)}\tilde X(t) = \sum p_{jl}(v(t))\tilde\tau_j(v(t))\Omega(\tilde\tau_l(v(t)), \tilde X(t)), \tag{10.6}$$

where $\tilde\tau_j(v(t))$ are the vectors $\tau_j(v(t))$ expressed in the variables y_1. In detail (cf. (3.3)),

$$\begin{aligned}\tilde\tau_1(v) &:= (-\psi_v'(y_1-d_1(t)), -\pi_v'(y_1-d_1(t))),\\ \tilde\tau_2(v) &:= (\partial_v\psi_v(y_1-d_1(t)), \partial_v\pi_v(y_1-d_1(t))),\end{aligned} \tag{10.7}$$

where $v = v(t)$. Since τ_j' are smooth and rapidly decaying at infinity functions, then Lemma 9.1 implies

$$\|\tilde\tau_j(v(t)) - \tau_j(v(t))\|_{E_\beta} \le C\varepsilon^2, \quad 0\le t\le t_1, \quad j=1,2. \tag{10.8}$$

Furthermore,

$$\tau_j(v(t)) - \tau_j(v(t_1)) = \int_t^{t_1}\dot v(s)\partial_v\tau_j(v(s))ds,$$

and therefore

$$\|\tau_j(v(t)) - \tau_j(v(t_1))\|_{E_\beta} \le C\int_t^{t_1}|\dot v(s)|ds, \quad 0\le t\le t_1, \tag{10.9}$$

$$|p_{jl}(v(t)) - p_{jl}(v(t_1))| = |\int_t^{t_1}\dot v(s)\partial_v p_{jl}(v(s))ds| \le C\int_t^{t_1}|\dot v(s)|ds, \quad 0\le t\le t_1, \tag{10.10}$$

since $|\partial_v p_{jl}(v(s))|$ is uniformly bounded by (5.2). Further,

$$\int_t^{t_1}|\dot v(s)|ds \le Cm_1^2(t_1)\int_t^{t_1}\frac{ds}{(1+s)^3} \le C\varepsilon^2, \quad 0\le t\le t_1. \tag{10.11}$$

Hence, the bounds (10.5) will follow from (4.29), (10.6) and (10.8)–(10.10) if we choose $\varepsilon > 0$ small enough. The proof is completed. □

11. Decay of Transversal Component

Here we prove Proposition 7.1.

Step i) We fix $\varepsilon > 0$ and $t_* = t_*(\varepsilon)$ for which Lemma 10.1 holds. Then the bounds of type (9.7) and (9.8) holds with $\|\mathbf{P}_1^c \tilde{X}(t)\|_{E_{-\beta}}$ and $\|(\mathbf{P}_1^c \tilde{X}(t))_1\|_{L^\infty}$ in the left-hand sides replaced by $\|X(t)\|_{E_{-\beta}}$ and $\|\Psi(t)\|_{L^\infty}$:

$$\|X(t)\|_{-\beta} \le \frac{C\|X(0)\|_{E_\beta}}{(1+t)^{3/2}} + C\int_0^t \frac{\|X(s)\|^2_{E_{-\beta}} + \|\Psi(s)\|_{L^\infty}\|X(s)\|_{E_{-\beta}} + (1+s)^{3/2+\beta}\|\Psi(s)\|^{12}_{L^\infty}}{(1+|t-s|)^{3/2}} ds, \tag{11.1}$$

$$\|\Psi(t)\|_{L^\infty} \le \frac{C\|X(0)\|_{E_\beta\cap W}}{(1+t)^{1/2}} + C\int_0^t \frac{\|X(s)\|^2_{E_{-\beta}} + \|\Psi(s)\|_{L^\infty}\|X(s)\|_{E_{-\beta}} + (1+s)^{3/2+\beta}\|\Psi(s)\|^{12}_{L^\infty} + \|\Psi(s)\|^{11}_{L^\infty}}{(1+|t-s|)^{1/2}} ds \tag{11.2}$$

for $0 \le t \le t_1$ and $t_1 < t_*$. This implies an integral inequality for the majorants m_1 and m_2. Namely, multiplying both sides of (11.1) by $(1+t)^{3/2}$, and taking the supremum in $t \in [0, t_1]$, we obtain

$$m_1(t_1) \le C\|X(0)\|_{E_\beta} + C \sup_{t\in[0,t_1]} \int_0^t \frac{(1+t)^{3/2} ds}{(1+|t-s|)^{3/2}} \times \left[\frac{m_1^2(s)}{(1+s)^3} + \frac{m_1(s)m_2(s)}{(1+s)^2} + \frac{m_2^{12}(s)(1+s)^{3/2+\beta}}{(1+s)^6}\right]$$

for $t_1 < t_*$. Taking into account that $m(t)$ is a monotone increasing function, we get

$$m_1(t_1) \le C\|X(0)\|_{E_\beta} + C[m_1^2(t_1) + m_1(t_1)m_2(t_1) + m_2^{12}(t_1)]I_1(t_1), \quad t_1 < t_*, \tag{11.3}$$

where

$$I_1(t_1) = \sup_{t\in[0,t_1]} \int_0^t \frac{(1+t)^{3/2}}{(1+|t-s|)^{3/2}} \frac{ds}{(1+s)^{9/2-\beta}} \le \overline{I}_1 < \infty, \quad t_1 \ge 0, \quad 5/2 < \beta < 3.$$

Therefore, (11.3) becomes

$$m_1(t_1) \le C\|X(0)\|_{E_\beta} + C\overline{I}_1[m_1^2(t) + m_1(t_1)m_2(t_1) + m_2^{12}(t_1)], \quad t_1 < t_*. \tag{11.4}$$

Similarly, multiplying both sides of (11.2) by $(1+t)^{1/2}$, and taking the supremum in $t \in [0, t_1]$, we get

$$m_2(t_1) \le C\|X(0)\|_{E_\beta\cap W} + C[m_1^2(t_1) + m_1(t_1)m_2(t_1) + m_2^{12}(t_1) + m_2^{11}(t_1)]I_2(t_1), \quad t_1 < t_*, \tag{11.5}$$

where

$$I_2(t_1) = \sup_{t\in[0,t_1]} \int_0^t \frac{(1+t)^{1/2}}{(1+|t-s|)^{1/2}} \frac{ds}{(1+s)^{9/2-\beta}} \le \overline{I}_2 < \infty, \quad t_1 \ge 0, \quad 5/2 < \beta < 3.$$

Therefore, (11.5) becomes

$$m_2(t_1) \le C\|X(0)\|_{E_\beta\cap W} + C\overline{I}_2[m_1^2(t_1) + m_1(t_1)m_2(t_1) + m_2^{12}(t_1) + m_2^{11}(t_1)] \quad t_1 < t_*, \tag{11.6}$$

Inequalities (11.4) and (11.6) imply that $m_1(t_1)$ and $m_2(t_1)$ are bounded for $t_1 < t_*$, and moreover,

$$m_1(t_1),\ m_2(t_1) \le C\|X(0)\|_{E_\beta\cap W}, \quad t_1 < t_* \tag{11.7}$$

since $m_1(0) = \|X(0)\|_{E_{-\beta}}$ and $m_2(0) = \|\Psi(0)\|_{L^\infty}$ are sufficiently small by (2.8).

Step ii) The constant C in the estimate (11.7) does not depend on t_* by Lemma 10.1. We choose d_0 in (2.8) so small that $\|X(0)\|_{E_\beta\cap W} < \varepsilon/(2C)$. It is possible due to (2.8). Finally, this implies that $t_* = \infty$, and (11.7) holds for all $t_1 > 0$ if d_0 is small enough.

12. Soliton Asymptotics

Here we prove our main Theorem 2.5 using the decay (7.1). The estimates (6.11) and (7.1) imply that

$$|\dot c(t)| + |\dot v(t)| \le \frac{C_1(\overline{v}, d_0)}{(1+t)^3}, \qquad t \ge 0. \tag{12.1}$$

Therefore, $c(t) = c_+ + \mathcal{O}(t^{-2})$ and $v(t) = v_+ + \mathcal{O}(t^{-2})$, $t \to \infty$. Similarly,

$$b(t) = c(t) + \int_0^t v(s)ds = v_+t + q_+ + \alpha(t), \quad \alpha(t) = \mathcal{O}(t^{-1}). \tag{12.2}$$

We have obtained the solution $Y(x,t) = (\psi(x,t), \pi(x,t))$ to (1.12) in the form

$$Y(x,t) = Y_{v(t)}(x - b(t), t) + X(x - b(t), t), \tag{12.3}$$

where we define now $v(t) = \dot b(t) = v_+ + \dot\alpha(t)$. Since

$$\|Y_{v(t)}(x - b(t), t) - Y_{v_+}(x - v_+t - q_+, t)\|_E = \mathcal{O}(t^{-1}),$$

it remains to extract the dispersive wave $W_0(t)\Phi_+$ from the term $X(x - b(t), t)$. Substituting (12.3) into (1.12) we obtain by (2.6) the inhomogeneous Klein-Gordon equation for the $X(x - b(t), t)$:

$$\dot X(y,t) = A_v^0 X(y,t) + R(y,t), \quad 0 \le t \le \infty, \tag{12.4}$$

where $y = x - b(t)$, and

$$A_v^0 = \begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 & v\nabla \end{pmatrix}, \quad R(t) = \begin{pmatrix} \dot v\partial_v\psi_v \\ \dot v\partial_v\pi_v + F(\Psi + \psi_v) - F(\psi_v) + m^2\Psi \end{pmatrix},$$

Now we change the variable $y \mapsto y_1 = y + \alpha(t) + q_+$. Then we obtain the "frozen" equation

$$\dot{\tilde X}(t) = A_+\tilde X(t) + \tilde R(t), \quad 0 \le t \le \infty, \tag{12.5}$$

where $\tilde{X}(t)$ and $\tilde{R}(t)$ are $X(t)$ and $R(t)$ of $y = y_1 - \alpha(t) - q_+$, and

$$A_+ = \begin{pmatrix} v_+\nabla & 1 \\ \Delta - m^2 & v_+\nabla \end{pmatrix}, \tag{12.6}$$

Equation (12.5) implies

$$\tilde{X}(t) = W_+(t)\tilde{X}(0) + \int_0^t W_+(t-s)\tilde{R}(s)ds, \tag{12.7}$$

where $W_+(t) = e^{A_+t}$ is the integral operator with integral kernel

$$W_+(y_1 - z, t) = W_0(y_1 - z + v_+t, t) = W_0(x - z, t),$$

since by (12.2)

$$y_1 + v_+t = y + \alpha(t) + q_+ + v_+t = x - b(t) + \alpha(t) + q_+ + v_+t = x.$$

Hence, Eq. (12.7) implies

$$X(x - b(t), t) = W_0(t)\tilde{X}(0) + \int_0^t W_0(t-s)\tilde{R}(s)ds. \tag{12.8}$$

Let us rewrite (12.8) as

$$\begin{aligned} X(x - b(t), t) &= W_0(t)\left(\tilde{X}(0) + \int_0^\infty W_0(-s)\tilde{R}(s)ds\right) - \int_t^\infty W_0(t-s)\tilde{R}(s)ds \\ &= W_0(t)\Phi_+ + r_+(t). \end{aligned}$$

To establish the asymptotics (2.9), it suffices to prove that

$$\Phi_+ = \tilde{X}(0) + \int_0^\infty W_0(-s)\tilde{R}(s)ds \in E \quad \text{and} \quad \|r_+(t)\|_E = \mathcal{O}(t^{-1/2}). \tag{12.9}$$

Assumption (2.8) implies that $\tilde{X}(0) \in E$. Let us split $\tilde{R}(s)$ as the sum

$$\tilde{R}(s) = \begin{pmatrix} \dot{v}\partial_v\tilde{\psi}_v \\ \dot{v}\partial_v\tilde{\pi}_v \end{pmatrix} + \begin{pmatrix} 0 \\ F(\tilde{\Psi} + \tilde{\psi}_v) - F(\tilde{\psi}_v) + m^2\tilde{\Psi} \end{pmatrix} = \tilde{R}'(s) + \tilde{R}''(s).$$

By (12.1), we obtain

$$\|\tilde{R}'(s)\|_E = \mathcal{O}(s^{-3}). \tag{12.10}$$

Let us consider $\tilde{R}'' = (0, \tilde{R}_2'')$. We have

$$\tilde{R}_2'' = F(\tilde{\Psi}+\tilde{\psi}_v) - F(\tilde{\psi}_v) + m^2\tilde{\Psi} = (F'(\tilde{\psi}_v) + m^2)\tilde{\Psi} + \tilde{N}(v, \tilde{\Psi}) = -\tilde{V}_v\tilde{\Psi} + \tilde{N}(v, \tilde{\Psi}),$$

By (1.17) and (7.1), we obtain

$$\|\tilde{V}_v\tilde{\Psi}(s)\|_{L^2} \le C\|\tilde{\Psi}(s)\|_{L^2_{-\beta}} \le C(\overline{v}, d_0)(1 + |s|)^{-3/2}, \tag{12.11}$$

since $|q_+ + \alpha(s)| \le C$. Finally, (7.1), (7.4), and (8.6) imply

$$\|\tilde{N}(v, \tilde{\Psi}(s))\|_{L^2} \le C(\overline{v}, d_0)(1 + |s|)^{-3/2}. \tag{12.12}$$

Hence, (12.11)–(12.12) imply

$$\|\tilde{R}''(s)\|_E = \mathcal{O}(s^{-3/2}), \tag{12.13}$$

and (12.9) follows by (12.10) and (12.13).

A. Virial Type Estimates

Here we prove the weighted estimate (4.42). Let us recall that we split the solution $Y(t) = (\psi(\cdot,t), \pi(\cdot,t)) = S(\sigma(t)) + X(t)$, and denote $X(t) = (\Psi(t), \Pi(t))$, $(\Psi_0, \Pi_0) := (\Psi(0), \Pi(0))$. Our basic condition (2.8) implies that for some $\nu > 0$,

$$\|X_0\|_{E_{5/2+\nu}} \le d_0 < \infty. \tag{A.1}$$

Proposition A.1. *Let the potential U satisfy conditions* **U1***, and X_0 satisfy* (A.1)*. Then the bounds hold*

$$\|\Psi(t)\|_{L^2_{5/2+\nu}} \le C(\overline{v}, d_0)(1+t)^{4+\nu}, \qquad t > 0. \tag{A.2}$$

We will deduce the proposition from the following two lemmas. The first lemma is well known. Denote

$$e(x,t) = \frac{|\pi(x,t)|^2}{2} + \frac{|\psi'(x,t)|^2}{2} + U(\psi(x,t)).$$

Lemma A.2. *For the solution $\psi(x,t)$ of Klein-Gordon equation* (1.2) *the local energy estimate holds*

$$\int_{a_1}^{a_2} e(x,t)\,dx \le \int_{a_1-t}^{a_2+t} e(x,0)\,dx, \quad a_1 < a_2, \quad t > 0. \tag{A.3}$$

Proof. The estimate follows by standard arguments: multiplication of Eq. (1.2) by $\dot\psi(x,t)$ and integration over the trapezium $ABCD$, where $A = (a_1 - t, 0)$, $B = (a_1, t)$, $C = (a_2, t)$, $D = (a_2 + t, 0)$. Then (A.3) is obtained after partial integration using that $U(\psi) \ge 0$. □

Lemma A.3. *For any $\sigma \ge 0$ and $b \in \mathbb{R}$,*

$$\int (1 + |x-b|^\sigma) e(x,t) dx \le C(\sigma)(1+t+|b|)^{\sigma+1} \int (1+|x|^\sigma) e(x,0) dx. \tag{A.4}$$

Proof. By (A.3)

$$\int (1+|y|^\sigma) \left(\int_{y+b-1}^{y+b} e(x,t)dx \right) dy \le \int (1+|y|^\sigma) \left(\int_{y+b-1-t}^{y+b+t} e(x,0)dx \right) dy.$$

Hence,

$$\int e(x,t) \left(\int_{x-b}^{x-b+1} (1+|y|^\sigma) dy \right) dx \le \int e(x,0) \left(\int_{x-b-t}^{x-b+1+t} (1+|y|^\sigma) dy \right) dx. \tag{A.5}$$

Obviously,

$$\int_{x-b}^{x-b+1} (1+|y|^\sigma) dy \ge c(\sigma)(1+|x-b|^\sigma) \tag{A.6}$$

with some $c(\sigma) > 0$. On the other hand,

$$\int_{x-b-t}^{x-b+1+t} (1+|y|^\sigma) dy \le (2t+1)(1+t+|b|+|x|)^\sigma \le C(1+t+|b|)^{\sigma+1}(1+|x|^\sigma), \tag{A.7}$$

since $\sigma \ge 0$. Finally, (A.5)–(A.7) imply (A.4). □

Proof of Proposition A.1. First, we verify that

$$U_0 = \int (1 + |x|^{5+2\nu}) U(\psi_0(x)) dx < C(d_0), \quad \psi_0(x) = \psi(x, 0). \tag{A.8}$$

Indeed, $\psi_0(x) = \psi_{v_0}(x - q_0) + \Psi_0(x)$ is bounded since $\Psi_0 \in H^1(\mathbb{R})$. Hence **U1** implies that

$$|U(\psi_0(x))| \leq C(d_0)(\psi_0(x) \pm a)^2 \leq C(d_0)\Big((\psi_{v_0}(x - q_0) \pm a)^2 + \Psi_0^2(x)\Big)$$

and then (A.8) follows by (1.14), (1.18) and (A.1). Further, we have

$$\begin{aligned} \|\Psi(t)\|^2_{L^2_{5/2+\nu}} &= \int (1 + |y|^{5+2\nu}) \left(\int_0^t \dot\Psi(y, s) ds - \Psi_0(y)\right)^2 dy \\ &\leq 2d_0^2 + 2t \int (1 + |y|^{5+2\nu}) dy \int_0^t \dot\Psi^2(y, s) ds. \end{aligned} \tag{A.9}$$

Due to (4.2) and (12.1)–(12.2) we have

$$\begin{aligned} \dot\Psi^2(y, s) &= \left[\dot b(s)\psi'(y + b(s), s) + \pi(y + b(s), s) - \dot v \partial_v \psi_v(y)\right]^2 \\ &\leq C(\overline{v}, d_0)\Big((\psi'(y + b(s), s))^2 + \pi^2(y + b(s), s) + (\partial_v \psi_v(y))^2\Big) \\ &\leq C(\overline{v}, d_0)\Big(e(y + b(s), s) + (\partial_v \psi_v(y))^2\Big). \end{aligned} \tag{A.10}$$

Substituting (A.10) into (A.9) and changing variables we obtain by (A.4) and (A.8) that

$$\begin{aligned} \|\Psi(t)\|^2_{L^2_{5/2+\nu}} &\leq 2d_0^2 + C(\overline{v}, d_0)t \int_0^t \left(\int (1 + |x - b(s)|^{5+2\nu}) e(x, s) dx + C(\overline{v})\right) ds \\ &\leq 2d_0^2 + C(\overline{v}, d_0)t^2 + C(\overline{v}, d_0)t \int (1 + |x|^{5+2\nu}) e(x, 0) dx \\ &\quad \times \int_0^t (1 + s + |b(s)|)^{6+2\nu} ds \\ &\leq 2d_0^2 + C(\overline{v}, d_0)t^2 + C(\overline{v}, d_0)(1 + t)^{8+2\nu}\left[\|X_0\|^2_{E_{5/2+\nu}} + U_0\right] \\ &\leq C(\overline{v}, d_0)(1 + t)^{8+2\nu}. \end{aligned}$$

□

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