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On convergence to equilibrium distribution for wave equation in even dimensions

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Abstract. Consider a wave equation (WE) with constant coefficients in $\mathbb{R}^n$ for even $n \geq 2$ and with variable coefficients for even $n \geq 4$. We study the distribution μ_t of the *random solution* at time $t \in \mathbb{R}$. The initial probability measure μ_0 has a translation-invariant covariance, zero mean and finite mean density for the energy. It also satisfies a Rosenblatt- or Ibragimov–Linnik-type mixing condition. The main result is the convergence of μ_t to a Gaussian probability measure as $t \to \infty$ which gives a Central Limit Theorem (CLT) for the WE. The proof for the case of constant coefficients is based on stationary phase asymptotics of the solution in the Fourier representation and Bernstein's 'room–corridor' argument. The case of variable coefficients is reduced to that of constant ones by a version of the scattering theory, based on Vainberg's results on local energy decay.

1. *Introduction*

This paper studies the asymptotic behavior of statistical solutions of the wave equation (WE) for the case of an even number of space dimensions. Here we consider the linear WEs in $\mathbb{R}^n$, with an arbitrary even $n \geq 4$:

$$\begin{aligned} \ddot{u}(x,t) = Lu(x,t) &\equiv \sum_{j=1}^{n} (\partial_j - iA_j(x))^2 u(x,t), \\ u|_{t=0} = u_0(x), &\quad \dot{u}|_{t=0} = v_0(x), \end{aligned} \tag{1.1}$$

where

$$\partial_j \equiv \frac{\partial}{\partial x_j}, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

We consider complex-valued solutions $u(x,t)$ and assume that the potentials $A_j(x)$ are real and vanish outside a bounded domain.

Denote $Y(t) = (Y^0(t), Y^1(t)) \equiv (u(\cdot,t), \dot{u}(\cdot,t))$, $Y_0 = (Y_0^0, Y_0^1) \equiv (u_0, v_0)$. Then (1.1) becomes

$$\dot{Y}(t) = \mathcal{A}Y(t), \quad t \in \mathbb{R}; \quad Y(0) = Y_0. \tag{1.2}$$

Here we denote

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}, \quad \text{where } A = \sum_{j=1}^{n} (\partial_j - iA_j(x))^2.$$

We assume that the initial state $Y_0 = (u_0, v_0)$ is a random element of a complex functional space $\mathcal{H}$ of the states with a finite local energy, see Definition 2.1. The distribution of Y_0 is a probability measure μ_0 of mean zero satisfying some additional assumptions, see conditions (S2)–(S4).

We identify $\mathbb{C} \equiv \mathbb{R}^2$ and denote by $\otimes$ the tensor product of real vectors. The covariance functions (CFs) of the initial measure are supposed to be translation-invariant, i.e. for $i, j = 0, 1$,

$$Q_0^{ij}(x,y) := E(Y_0^i(x) \otimes Y_0^j(y)) = q_0^{ij}(x-y), \quad x, y \in \mathbb{R}^n. \tag{1.3}$$

Next, we assume that the initial mean energy density is finite:

$$e_0 := E(|v_0(x)|^2 + |\nabla u_0(x)|^2 + |u_0(x)|^2) = \mathrm{tr}(q_0^{11}(0) - \Delta q_0^{00}(0) + q_0^{00}(0)) < \infty. \tag{1.4}$$

Finally, we assume that μ_0 satisfies a mixing condition. Roughly speaking, this means that

$$Y_0(x) \text{ and } Y_0(y) \text{ are asymptotically independent as } |x-y| \to \infty.$$

Denote by μ_t, $t \in \mathbb{R}$, the measures on $\mathcal{H}$ that give the distribution of the random solution $Y(t)$ to problem (1.2).

Our main result gives the (weak) convergence

$$\mu_t \rightharpoonup \mu_\infty, \quad t \to \infty \tag{1.5}$$

to a limiting measure μ_∞ which is a stationary Gaussian probability measure (GPM) on $\mathcal{H}$.

Results of this kind were obtained in [**2**] for the WE in odd dimension $n \geq 3$ and in [**1**] for the Klein–Gordon equation (KGE) in $\mathbb{R}^n$ with $n \geq 1$. The main difference from these previous results lies in the more complicated long-time asymptotic behavior of the corresponding Green function as $t \to \infty$, which we illustrate later.

We prove convergence (1.5) by using the following general strategy [**1–3**]. At first, we prove convergence (1.5) for equations with constant coefficients (i.e. $A_j(x) \equiv 0$) for any even $n \geq 2$ in three steps.

(I) The family of measures μ_t, $t \geq 0$, is compact in an appropriate Fréchet space.

(II) The CFs converge to a limit: for $i, j = 0, 1$,

$$Q_t^{ij}(x,y) = \int Y^i(x) \otimes Y^j(y) \mu_t(dY) \to Q_\infty^{ij}(x,y), \quad t \to \infty. \tag{1.6}$$

(III) The characteristic functionals converge to a Gaussian one:

$$\hat{\mu}_t(\Psi) = \int \exp\{i\langle Y, \Psi\rangle\}\mu_t(dY) \to \exp\left\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi, \Psi)\right\}, \quad t \to \infty, \tag{1.7}$$

where Ψ is an arbitrary element of a dual space and $\mathcal{Q}_\infty$ is the quadratic form with the integral kernel $(Q_\infty^{ij}(x, y))_{i,j=0,1}$. $\langle Y, \Psi\rangle$ denotes the scalar product in a real Hilbert space $L^2(\mathbb{R}^n) \otimes \mathbb{R}^n$.

Property (I) follows from the Prokhorov Compactness Theorem by using the methods in [**14**]. With the help of the Fourier transform (FT), we prove a uniform bound for the mean local energy in measure μ_t. The conditions of Prokhorov's theorem then follow from Sobolev's Embedding Theorem. Property (II) is deduced from an analysis of oscillatory integrals arising in the FT using the mixing condition.

To prove property (III), we use Bernstein's 'room–corridor' method. This method is based on the dispersive mechanism of the WE, exploited in [**2**] for the case of odd $n \geq 3$. We illustrate it first for the case $n = 3$ and $u_0 = 0$. Kirchhoff's formula holds:

$$u(x, t) = \frac{1}{4\pi t}\int_{S_t(x)} v_0(y)\, dS(y), \quad x \in \mathbb{R}^3, \tag{1.8}$$

where $dS(y)$ is the Lebesgue measure on the sphere $S_t(x) : |y - x| = t$. Divide the sphere of integration into $N \sim t^2$ 'rooms' R_k of a fixed width $d \gg 1$, separated by 'corridors' C_k of a fixed width $\rho \ll d$. Denoting by r_k the integral over R_k, we rewrite (1.8) as

$$u(x, t) \sim \sum_1^N r_k \Big/ \sqrt{N}, \tag{1.9}$$

where the r_k are nearly independent owing to the mixing condition. Then (1.7) follows by the well-known Ibragimov–Linnik Central Limit Theorem (CLT) [**7**].

For odd $n > 3$, the Kirchhoff formula is replaced by the corresponding Herglotz–Petrovskii's formulas. However, the formulas contain higher-order derivatives of the initial function. Then the earlier Kirchhoff-type arguments require a modification (see [**2**]). Namely, we apply Bernstein's method to prove the Gaussian property for the average $a(t) := \langle Y(x, t), \Psi(x)\rangle$ as $t \to \infty$ where $\Psi \in D$ is a function with the support in a ball $|x| \leq \overline{r}$. An important role is played by the argument of Lemma 7.1 which states that the dual representation $a(t) = \langle Y_0(x), \Phi(x, t)\rangle$, where $\Phi(x, t)$ is a solution of a dual wave problem with initial data Ψ. The support of $\Phi(x, t)$ belongs to the 'inflated future cone' $t - \overline{r} \leq |x| \leq t + \overline{r}$ by the strong Huygen principle. Hence, $a(t) = \langle Y_0(x), \Phi(x, t)\rangle$ becomes an integral of the Kirchhoff-type (1.8) over the inflated sphere $S_t^{\overline{r}}(x) : t - \overline{r} \leq |x| \leq t + \overline{r}$. Finally, this integral admits a representation of type (1.9) since

$$\sup_{x\in\mathbb{R}^n} \Phi(x, t) = \mathcal{O}(t^{-(n-1)/2}). \tag{1.10}$$

The asymptotics follow by the stationary phase method applied to the oscillatory integral representation for $\Phi(x, t)$. In [**1**] we extended the result to the KGE in $\mathbb{R}^n$ with $n \geq 1$. In this case, the strong Huygen principle breaks down. Respectively $a(t) :=$

$\langle Y(x,t), \Psi(x)\rangle$ becomes an integral over the ball $B_t(x) : |x| \le t$. This integral also admits a Bernstein-type representation (1.9) since

$$\sup_{x\in\mathbb{R}^n} \Phi(x,t) = \mathcal{O}(t^{-n/2}). \qquad (1.11)$$

In this paper we construct a suitable extension of Bernstein's method to the WE with an arbitrary even $n \ge 2$. In this case the strong Huygen principle is missing, as for the KGE. However, the uniform decay is $t^{-(n-1)/2}$ which is worse than (1.11). Therefore, the previous Kirchhoff-type arguments are not applicable directly. To overcome this difficulty we find the following two main arguments.

(1) To obtain a good decay for $\Phi(x,t)$ we consider the initial functions Ψ satisfying the following, spectral condition: the FT $\hat\Psi(k) = 0$ for small $|k|$ (a 'regular wave packet' in the terminology of [**11**]). Let us note that the support of a non-zero $\Psi(x)$ is not compact: overwise $\hat\Psi(k)$ would be analytic in $k \in \mathbb{C}^n$ and identically vanish by the spectral condition. Therefore, $\Phi(x,t)$ is not supported by a cone $|x| < t + C$ in contrast to the WE in odd dimension and the KGE in all dimensions. Hence, we need the asymptotics substituting (1.10), (1.11) everywhere in the space $\mathbb{R}^{n+1}$, inside and outside the light cone. Namely, for the regular wave packet, we prove that

$$|\Phi(x,t)| \le C(N,\Psi)t^{-(n-1)/2}(|t-|x||+1)^{-N}, \quad (x,t) \in \mathbb{R}^{n+1} \qquad (1.12)$$

for any $N \in \mathbb{N}$. The asymptotics substitute for the strong Huygen principle in our proof since they mean that an integral $a(t) = \langle Y_0(x), \Phi(x,t)\rangle$ is concentrated near the cone $|x| = t$ similarly to the case of odd $n \ge 3$. Then the main part of $a(t) = \langle Y_0(x), \Phi(x,t)\rangle$ becomes an integral of the Kirchhoff-type (1.8) over the inflated sphere $S_t^C(x) : t - C \le |x| \le t + C$. This again allows us to apply the Bernstein-type technique [**2**] to prove the Gaussian property. Note that the asymptotics (1.12) generalize the bounds [**11**, Theorem XI.18]. In [**11**] the bound (1.10) has been established and also $|\Phi(x,t)| \le C(N,\Psi)(t + |x| + 1)^{-N}$ in regions $|x| \le t(1-\varepsilon)$ and $|x| \ge t(1+\varepsilon)$ which are not sufficient for our purposes. We need more detailed asymptotics (1.12) in the region $t(1-\varepsilon) \le |x| \le t(1+\varepsilon)$.

(2) Further we prove (1.7) for the case of general Ψ without the spectral condition. For this purpose we partition the Fourier transform $\hat\Psi(k)$ into two summands: one vanishes for small $|k|$, the other is concentrated near $|k| = 0$. The *covariance* of the contribution of the second summand is small due to the absolute continuity of the correlation function in the Fourier space which is provided by the mixing condition.

Then we prove the convergence in (1.5) for problem (1.1) with variable coefficients. In this case explicit formulas for the solution are unavailable. The case of zero magnetic field, curl $A \equiv 0$, is reduced to a constant coefficient by a gauge transformation. To prove (1.5) for the case of non-zero magnetic field we, roughly speaking, construct a scattering theory for solutions of infinite global energy (this strategy is similar to that in [**1, 2**]). This also allows us to reduce the proof to the case of constant coefficients. More precisely, we establish the 'dual' long-time asymptotics

$$U'(t)\Psi = U_0'(t)W\Psi + r(t)\Psi, \quad t > 0, \qquad (1.13)$$

where $U'(t)$ is a 'formal adjoint' to the dynamical group in equation (1.2), while $U_0'(t)$ corresponds to the 'free' equation, with $A_j(x) \equiv 0$ and W is a scattering operator. The remainder $r(t)$ is small in mean:

$$E|\langle Y_0, r(t)\Psi\rangle|^2 \to 0, \quad t \to \infty.$$

This version of scattering theory is essentially based on Vainberg's estimates [**13**] for the local energy decay. We prove (1.5) for the case of non-zero magnetic field in three steps (I)–(III) as for the equations with constant coefficients. Now property (III) stands:

$$\hat{\mu}_t(\Psi) = \int \exp\{i\langle Y, \Psi\rangle\}\mu_t(dY) \to \exp\left\{-\frac{1}{2}\mathcal{Q}_\infty(W\Psi, W\Psi)\right\}, \quad t \to \infty. \tag{1.14}$$

Scattering theory allows us to prove (1.14) for any Ψ from a set $\Pi = W^{-1}D$. Finally, we prove that W is an isomorphism; hence, set Π is dense in a dual space.

Note that the operators $U_0'(t)$ and $U'(t)$ are unitary with respect to two different norms. Scattering theory requires the equivalence of these norms (cf. [**11**, §XI.10]). To prove the equivalence for the case $n \ge 4$, we apply the Sobolev inequality

$$\mathring{H}^1(\mathbb{R}^n) \in L^q(\mathbb{R}^n), \quad q = \frac{2n}{n-2},$$

which breaks down for the case $n = 2$. Respectively, in the case $n = 2$ the problem for variable coefficients is open.

The paper is organized as follows. In §2 we formally state our main result. Sections 3–8 deal with the case of constant coefficients: main results are stated in §3, the compactness (property (I)) and the convergence (1.6) are proved in §4, and convergence (1.7) in §§5–8. In §9 we construct the scattering theory and in §10 we establish convergence (1.5) for variable coefficients for non-zero magnetic field curl $A(x) \not\equiv 0$. In Appendix A we have collected the FT-type calculations. In Appendix B we construct the 'rooms–corridors' partition in the case $n = 2$. Convergence (1.5) for zero magnetic field is proved in Appendix C.

2. *Main results*

2.1. *Notation.* We assume that the functions $A_j(x)$ in (1.1) satisfy the following conditions.

(E1) $A_j(x)$ are real C^∞-function.

(E2) $A_j(x) = 0$ for $|x| \ge R_0$, where $R_0 < \infty$.

We assume that the initial state Y_0 belongs to the phase space $\mathcal{H}$ defined as follows.

Definition 2.1. $\mathcal{H} \equiv H^1_{\mathrm{loc}}(\mathbb{R}^n) \oplus H^0_{\mathrm{loc}}(\mathbb{R}^n)$ is the Fréchet space of pairs $Y(x) \equiv (u(x), v(x))$ of complex functions $u(x)$, $v(x)$, endowed with local energy seminorms

$$\|Y\|_R^2 = \int_{|x|<R} (|u(x)|^2 + |\nabla u(x)|^2 + |v(x)|^2)\, dx < \infty, \quad \forall R > 0. \tag{2.1}$$

Proposition 2.1 follows from [**9**, Theorems V.3.1, V.3.2] as the speed of propagation for equation (1.1) is finite.

PROPOSITION 2.1.
(i) *For any $Y_0 \in \mathcal{H}$, there exists a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{H})$ to Cauchy problem (1.2).*
(ii) *For any $t \in \mathbb{R}$, the operator $U(t) : Y_0 \mapsto Y(t)$ is continuous in $\mathcal{H}$.*

Let us choose a function $\zeta(x) \in C_0^\infty(\mathbb{R}^n)$ with $\zeta(0) \neq 0$. Denote by $H^s_{\rm loc}(\mathbb{R}^n)$, $s \in \mathbb{R}$, the local Sobolev spaces, i.e. the Fréchet spaces of distributions $u \in D'(\mathbb{R}^n)$ with finite seminorms

$$\|u\|_{s,R} := \|\Lambda^s(\zeta(x/R)u)\|_{L^2(\mathbb{R}^n)},$$

where $\Lambda^s v := F^{-1}_{k\to x}(\langle k\rangle^s \hat{v}(k))$, $\langle k\rangle := \sqrt{|k|^2+1}$ and $\hat{v} := Fv$ is the FT of a tempered distribution v. For $\psi \in D$ define $F\psi(k) = \int e^{ik\cdot x}\psi(x)\,dx$.

Definition 2.2. For $s \in \mathbb{R}$ denote $\mathcal{H}^s \equiv H^{1+s}_{\rm loc}(\mathbb{R}^n) \oplus H^s_{\rm loc}(\mathbb{R}^n)$.

Using standard techniques of pseudodifferential operators and Sobolev's theorem (see, e.g., [**6**]), it is possible to prove that $\mathcal{H}^0 = \mathcal{H} \subset \mathcal{H}^{-\varepsilon}$ for every $\varepsilon > 0$, and the embedding is compact. We denote the scalar product by $\langle\cdot,\cdot\rangle$ in real Hilbert space $L^2(\mathbb{R}^n)$ or in $L^2(\mathbb{R}^n) \otimes \mathbb{R}^N$ or its various extensions.

2.2. *Random solution: convergence to equilibrium.* Let (Ω, Σ, P) be a probability space with expectation E and $\mathcal{B}(\mathcal{H})$ denote the Borel σ-algebra in $\mathcal{H}$. We assume that $Y_0 = Y_0(\omega, \cdot)$ in (1.2) is a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. In other words, $(\omega, x) \mapsto Y_0(\omega, x)$ is a measurable map $\Omega \times \mathbb{R}^n \to \mathbb{C}^2$ with respect to the (completed) σ-algebras $\Sigma \times \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{C}^2)$. Then $Y(t) = U(t)Y_0$ is also a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ owing to Proposition 2.1. We denote by $\mu_0(dY_0)$ a Borel probability measure in $\mathcal{H}$ giving the distribution of the Y_0. Without loss of generality, we assume $(\Omega, \Sigma, P) = (\mathcal{H}, \mathcal{B}(\mathcal{H}), \mu_0)$ and $Y_0(\omega, x) = \omega(x)$ for $\mu_0(d\omega) \times dx$-almost all $(\omega, x) \in \mathcal{H} \times \mathbb{R}^n$.

Definition 2.3. μ_t is a Borel probability measure in $\mathcal{H}$ which gives the distribution of $Y(t)$:

$$\mu_t(B) = \mu_0(U(-t)B), \quad \forall B \in \mathcal{B}(\mathcal{H}), \quad t \in \mathbb{R}.$$

Our main goal is to derive the weak convergence of the measures μ_t in the Fréchet spaces $\mathcal{H}^{-\varepsilon}$ for each $\varepsilon > 0$:

$$\mu_t \overset{\mathcal{H}^{-\varepsilon}}{\rightharpoonup} \mu_\infty, \quad t \to \infty, \tag{2.2}$$

where μ_∞ is a limiting measure on the space $\mathcal{H}$. This means the convergence

$$\int f(Y)\mu_t(dY) \to \int f(Y)\mu_\infty(dY) \quad \text{as } t \to \infty$$

for any bounded continuous functional f on space $\mathcal{H}^{-\varepsilon}$. Recall that we have identified $\mathbb{C} \equiv \mathbb{R}^2$ and that $\otimes$ stands for the tensor product of real vectors. Denote $M^2 = \mathbb{R}^2 \otimes \mathbb{R}^2$.

Definition 2.4. The CFs of measure μ_t are defined by

$$Q_t^{ij}(x, y) \equiv E(Y^i(x,t) \otimes Y^j(y,t)), \quad i, j = 0, 1 \text{ for almost all } x, y \in \mathbb{R}^n \times \mathbb{R}^n \tag{2.3}$$

assuming that the expectations in the right-hand side are finite.

We set $\mathcal{D} = D \oplus D$, and $\langle Y, \Psi \rangle = \langle Y^0, \Psi^0 \rangle + \langle Y^1, \Psi^1 \rangle$ for $Y = (Y^0, Y^1) \in \mathcal{H}$ and $\Psi = (\Psi^0, \Psi^1) \in \mathcal{D}$. For a probability measure μ on $\mathcal{H}$, denote by $\hat{\mu}$ the characteristic functional (FT)

$$\hat{\mu}(\Psi) \equiv \int \exp\{i \langle Y, \Psi \rangle\} \mu(dY), \quad \Psi \in \mathcal{D}.$$

A probability measure μ is called the GPM (of mean zero) if its characteristic functional has the form

$$\hat{\mu}(\Psi) = \exp\{-\tfrac{1}{2}\mathcal{Q}(\Psi, \Psi)\}, \quad \Psi \in \mathcal{D},$$

where $\mathcal{Q}$ is a real non-negative quadratic form in $\mathcal{D}$. A measure μ is called translation-invariant if

$$\mu(T_h B) = \mu(B), \quad \forall B \in \mathcal{B}(\mathcal{H}), \quad h \in \mathbb{R}^n,$$

where $T_h Y(x) = Y(x - h)$, $x \in \mathbb{R}^n$.

2.3. *Mixing condition.* Let $O(r)$ denote the set of all pairs of open bounded subsets $\mathcal{A}$, $\mathcal{B} \subset \mathbb{R}^n$ at distance $\mathrm{dist}(\mathcal{A}, \mathcal{B}) \geq r$ and $\sigma(\mathcal{A})$ be the σ-algebra of the subsets in $\mathcal{H}$ generated by all linear functionals $Y \mapsto \langle Y, \Psi \rangle$, where $\Psi \in \mathcal{D}$ with $\operatorname{supp} \Psi \subset \mathcal{A}$. We define the Ibragimov–Linnik mixing coefficient of a probability measure μ_0 on $\mathcal{H}$ by (cf. [**7**, Definition 17.2.2])

$$\varphi(r) \equiv \sup_{(\mathcal{A},\mathcal{B}) \in O(r)} \sup_{\substack{A \in \sigma(\mathcal{A}),\ B \in \sigma(\mathcal{B}), \\ \mu_0(B) > 0}} \frac{|\mu_0(A \cap B) - \mu_0(A)\mu_0(B)|}{\mu_0(B)}.$$

Definition 2.5. Measure μ_0 satisfies the strong uniform Ibragimov–Linnik mixing condition if

$$\varphi(r) \to 0, \quad r \to \infty.$$

Here, we specify the rate of the decay of φ (see condition (S3)).

2.4. *Main theorem.* We assume that measure μ_0 satisfies the following properties:

(S1) μ_0 has the zero expectation value,

$$E Y_0(x) \equiv 0, \quad x \in \mathbb{R}^n.$$

(S2) μ_0 has translation-invariant CFs, i.e. equation (1.3) holds for almost all $x, y \in \mathbb{R}^n$.

(S3) μ_0 has a finite mean energy density, i.e. equation (1.4) holds.

(S4) μ_0 satisfies the strong uniform Ibragimov–Linnik mixing condition, with

$$\begin{aligned} &\int_0^\infty r^{n-1} \varphi^{n/(2(n+2))}(r)\, dr < \infty, \quad n \geq 4, \\ &\int_0^\infty r |\log r| \varphi^{1/2}(r)\, dr < \infty, \quad n = 2. \end{aligned} \tag{2.4}$$

Denote

$$\mathcal{E}(x) = \begin{cases} C_n |x|^{2-n}, & n > 2, \\ \dfrac{1}{2\pi} \log|x|, & n = 2 \end{cases}$$

the fundamental solution of the Laplacian, i.e. $\Delta\mathcal{E}(x) = \delta(x)$ for $x \in \mathbb{R}^n$. Define, for almost all $x, y \in \mathbb{R}^n$, the matrix-valued function

$$Q_\infty(x, y) = (Q_\infty^{ij}(x, y))_{i,j=0,1} = (q_\infty^{ij}(x - y))_{i,j=0,1},$$

where

$$(q_\infty^{ij})_{i,j=0,1} = \frac{1}{2}\begin{pmatrix} q_0^{00} - \mathcal{E} * q_0^{11} & q_0^{01} - q_0^{10} \\ q_0^{10} - q_0^{01} & q_0^{11} - \Delta q_0^{00} \end{pmatrix}. \tag{2.5}$$

Conditions (S1), (S3) and (S4) imply by [**7**, Lemma 17.2.3] (see Lemma 6.2(1)) that the derivatives $\partial^\alpha q_0^{ij}$ are bounded by the mixing coefficient:

$$|\partial^\alpha q_0^{ij}(z)| \le Ce_0\varphi^{1/2}(|z|), \quad |\alpha| \le 2 - i - j, \quad i, j = 0, 1, \ z \in \mathbb{R}^n. \tag{2.6}$$

Hence, (2.4) implies the existence of the convolution $\mathcal{E} * q_0^{11}$ in (2.5). Note that (2.4) and (2.6) imply that $q_0^{ij} \in L^1(\mathbb{R}^n)$; hence, the FT $\hat{q}_0^{ij} \in C(\mathbb{R}^n) \otimes M^2$. For $n = 2$ we assume, in addition, the following condition:
(S5) $|k|^{-2}\hat{q}_0^{11}(k) \in L^1(\mathbb{R}^2) \otimes M^2$.
Let $\mathring{H}$ denote the space of complex-valued functions $\Psi = (\Psi^0, \Psi^1)$ which is a completion of $\mathcal{D}$ in the energy norm

$$\|\Psi\|^2_{\mathring{H}} = \int_{\mathbb{R}^n} (|\Psi^0(x)|^2 + |\nabla\Psi^1(x)|^2)\, dx.$$

Denote by $\mathcal{Q}_\infty$ a real quadratic form defined by

$$\mathcal{Q}_\infty(\Psi, \Psi) = \sum_{i,j=0,1} \int_{\mathbb{R}^n\times\mathbb{R}^n} (Q_\infty^{ij}(x, y)\Psi^i(x), \Psi^j(y))\, dx\, dy, \quad \Psi \in \mathcal{D}, \tag{2.7}$$

where $(\cdot, \cdot)$ stands for the real scalar product in $\mathbb{C}^2 \equiv \mathbb{R}^4$. Our main result is the following theorem.

THEOREM A. *Let $n \ge 4$ be even, and assume that (E1), (E2), (S1)–(S4) hold. Then the following hold.*

(1) *The convergence in (2.2) holds for any $\varepsilon > 0$.*
(2) *The limiting measure μ_∞ is a Gaussian measure on $\mathcal{H}$.*
(3) *The limiting characteristic functional has the form*

$$\hat{\mu}_\infty(\Psi) = \exp\{-\tfrac{1}{2}\mathcal{Q}_\infty(W\Psi, W\Psi)\}, \quad \Psi \in \Pi,$$

where Π is dense in $\mathring{H}$, $W : \mathring{H} \to \mathring{H}$ is an isomorphism, and $W\Pi \subset \mathcal{D}$.

2.5. *Examples of initial measures with mixing condition.* In this section we represent the Gaussian initial measure μ_0 satisfying (S1)–(S5) for $n = 2$ (for $n > 2$ similar measure satisfying (S1)–(S4) have been constructed in [**1**]). We set

$$f(z) = (1 - \cos z)\left(\frac{1 - \cos z}{z^2}\right)^2, \quad z \in \mathbb{R}$$

and $\hat{q}_0^{ij}(k) = \delta^{ij} f(k_1) f(k_2) \geq 0$. Then the function $q_0^{ij}(x) = F^{-1}_{k\to x}(\hat{q}_0^{ij}(k)) \in C^2(\mathbb{R}^n) \otimes M^2$ and it has a compact support: $q_0^{ij} = 0$, $|x| \geq 3\sqrt{2}$. Let μ_0 be a Gaussian measure in the space $\mathcal{H}$ with the characteristic functional

$$\hat{\mu}_0(\Psi) \equiv E \exp\{i\langle Y, \Psi\rangle\} = \exp\{-\tfrac{1}{2}\mathcal{Q}_0(\Psi, \Psi)\}, \quad \Psi \in \mathcal{D}.$$

Here $\mathcal{Q}_0$ is a real non-negative quadratic form with an integral kernel

$$Q_0^{ij}(x, y) \equiv q_0^{ij}(x - y), \quad i, j = 0, 1.$$

Then (S1)–(S5) are satisfied with $\varphi(r) = 0$, $r \geq 3\sqrt{2}$.

3. *Equations with constant coefficients*

In §§3–8 we consider the Cauchy problem (1.1) with constant coefficients, i.e.

$$\begin{aligned} \ddot{u}(x, t) &= \Delta u(x, t), \quad x \in \mathbb{R}^n, \\ u|_{t=0} &= u_0(x), \quad \dot{u}|_{t=0} = v_0(x). \end{aligned} \tag{3.1}$$

Rewrite (3.1) in a form similar to (1.2):

$$\dot{Y}(t) = \mathcal{A}_0 Y(t), \quad t \in \mathbb{R}; \quad Y(0) = Y_0. \tag{3.2}$$

Here we denote

$$\mathcal{A}_0 = \begin{pmatrix} 0 & 1 \\ A_0 & 0 \end{pmatrix}, \tag{3.3}$$

where $A_0 = \Delta$. Denote by $U_0(t)$, $t \in \mathbb{R}$, the dynamical group for problem (3.2), then $Y(t) = U_0(t)Y_0$. Set $\mu_t(B) = \mu_0(U_0(-t)B)$, $B \in \mathcal{B}(\mathcal{H})$, $t \in \mathbb{R}$. The main result for problem (3.2) is the following theorem.

THEOREM B. *Let $n \geq 2$ be even and conditions (S1)–(S4) for $n \geq 4$ or conditions (S1)–(S5) for $n = 2$ hold. Then the conclusions of Theorem A hold with $W = I$ and limiting measure μ_∞ is translation-invariant.*

Theorem B can be deduced from Propositions 3.1 and 3.2, by using the same arguments as in [**14**, Theorem XII.5.2].

PROPOSITION 3.1. *The family of measures $\{\mu_t, t \geq 0\}$ is weakly compact in $\mathcal{H}^{-\varepsilon}$ with any $\varepsilon > 0$, and the bounds hold:*

$$\sup_{t\geq 0} E\|U_0(t)Y_0\|_R^2 < \infty, \quad R > 0. \tag{3.4}$$

Let $\mathcal{S}$ be the Schwartz space of a smooth test function with rapid decay at infinity.

PROPOSITION 3.2. *For any $\Psi \in \mathcal{S}$,*

$$\hat{\mu}_t(\Psi) \equiv \int \exp\{i\langle Y, \Psi\rangle\}\mu_t(dY) \to \exp\left\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi, \Psi)\right\}, \quad t \to \infty. \tag{3.5}$$

Propositions 3.1 and 3.2 are proved in §§4 and 5–8, respectively. We repeatedly use the FTs (A.2) and (A.3) from Appendix A.

4. *Compactness of the measures family*

Here we prove Proposition 3.1 with the help of the FT.

4.1. *Mixing in terms of spectral density.*

The next proposition gives the mixing property in terms of the FT $\hat{q}_0^{ij}$ of the initial CFs q_0^{ij}: assumption (S3) implies that $q_0^{ij}(z)$ is a measurable bounded function. Therefore, it belongs to the Schwartz space of tempered distributions as well as its FT.

PROPOSITION 4.1. *Let the assumptions of Theorem B hold. Then $\hat{q}_0^{ij}(k) \in L^p(\mathbb{R}^n) \otimes M^2$ with $1 \le p \le \infty$, and*

$$\int |k|^l |\hat{q}_0^{ij}(k)|\, dk < \infty, \quad -i-j \le l \le 2-i-j. \tag{4.1}$$

Proof. The bounds (2.6) and (2.4) imply that $q_0^{ij}(x) \in L^1(\mathbb{R}^n)$; hence, $\hat{q}_0^{ij}(k) \in C(\mathbb{R}^n) \otimes M^2$. Further we consider $i = j = 1$ since, in all other cases, the proof is similar. The function $\hat{q}_0^{11}$ is non-negative by the Bohner theorem. Hence,

$$\int |\hat{q}_0^{11}(k)|\, dk = \int \hat{q}_0^{11}(k)\, dk = q_0^{11}(0) < \infty,$$

owing to (S3). Hence, $\hat{q}_0^{11}(k) \in L^p(\mathbb{R}^n) \otimes M^2$ with any $p \in [1, \infty]$ by interpolation. Since the singularity $|k|^{-2}$ is summable for $n > 2$ we obtain the bound (4.1) with $l = -2$. For $n = 2$, this bound follows from assumption (S5). □

COROLLARY 4.1. *Formula (2.5) implies that functions $\hat{q}_\infty^{ij}(k)$ belong to $L^1(\mathbb{R}^n) \otimes M^2$, $i, j = 0, 1$.*

4.2. *Proof of the compactness of measures family.*

We now prove bound (3.4). Proposition 3.1 then can be deduced with the help of the Prokhorov Theorem [**14**, Lemma II.3.1] as in [**14**, Theorem XII.5.2]. Formulas (A.2), (A.3) and Proposition 4.1 imply

$$\begin{aligned} E(u(x,t) \otimes u(y,t)) &:= q_t^{00}(x-y) \\ &= \frac{1}{(2\pi)^n} \int e^{-ik(x-y)} \Big[\frac{1+\cos 2|k|t}{2} \hat{q}_0^{00}(k) + \frac{\sin 2|k|t}{2|k|} (\hat{q}_0^{01}(k) + \hat{q}_0^{10}(k)) \\ &\quad + \frac{1-\cos 2|k|t}{2|k|^2} \hat{q}_0^{11}(k) \Big]\, dk, \end{aligned} \tag{4.2}$$

where the integral converges and defines a continuous function. Similar representations hold for q_t^{ij} with all $i, j = 0, 1$. Therefore, we have, as in (1.4),

$$e_t := \mathrm{tr}(q_t^{11}(0) - \Delta q_t^{00}(0) + q_t^{00}(0)) = \frac{1}{(2\pi)^n} \int \mathrm{tr}(\hat{q}_t^{11}(k) + |k|^2 \hat{q}_t^{00}(k) + \hat{q}_t^{00}(k))\, dk. \tag{4.3}$$

It remains to estimate the last integral. Equation (4.2) implies the following representation for $\hat{q}_t^{00}$:

$$\hat{q}_t^{00}(k) = \frac{1+\cos 2|k|t}{2} \hat{q}_0^{00}(k) + \frac{\sin 2|k|t}{2|k|} (\hat{q}_0^{01}(k) + \hat{q}_0^{10}(k)) + \frac{1-\cos 2|k|t}{2|k|^2} \hat{q}_0^{11}(k). \tag{4.4}$$

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Similarly, formulas (A.3), (A.2) imply

$$\hat{q}_t^{11}(k) = |k|^2 \frac{1-\cos 2|k|t}{2} \hat{q}_0^{00}(k) - |k| \frac{\sin 2|k|t}{2} (\hat{q}_0^{01}(k) + \hat{q}_0^{10}(k)) + \frac{1-\cos 2|k|t}{2} \hat{q}_0^{11}(k). \tag{4.5}$$

Therefore, (4.1) and (4.3) imply that $e_t \le C_1(\varphi)e_0$. Hence, taking expectation in (2.1), we get (3.4):

$$E\|U_0(t)Y_0\|_R^2 = e_t|B_R| \le C_1(\varphi)e_0|B_R|,$$

where B_R denotes the ball $\{x \in \mathbb{R}^n : |x| \le R\}$ and $|B_R|$ is its volume.

COROLLARY 4.2. *Bound (3.4) implies the convergence of the integrals in (2.3).*

4.3. *Convergence of covariance functions.* Here we prove the convergence of the CVs of measures μ_t. This convergence is used in §6.

LEMMA 4.1. *The following convergence holds as $t \to \infty$:*

$$q_t^{ij}(z) \to q_\infty^{ij}(z), \quad \forall z \in \mathbb{R}^n, \quad i, j = 0, 1. \tag{4.6}$$

Proof. (4.4) and (4.5) imply the convergence for $i = j$: the oscillatory terms there converge to zero as they are absolutely continuous and summable by Proposition 4.1. For $i \ne j$ the proof is similar. □

5. *Bernstein's argument for the wave equation*

In this and the subsequent section we develop a version of Bernstein's 'room–corridor' method. We use the standard integral representation for solutions, divide the domain of integration into 'rooms' and 'corridors' and evaluate their contribution. As a result, $\langle U_0(t)Y_0, \Psi\rangle$ is represented as the sum of weakly dependent random variables. We evaluate the variances of these random variables which will be important in next section.

First we prove Proposition 3.2 under an additional assumption on the function $\Psi \in \mathcal{S}$.

Spectral condition for an $\varepsilon > 0$,

$$\hat{\Psi}(k) = 0, \quad |k| \le \varepsilon. \tag{5.1}$$

We get rid of this restriction in §8.

5.1. *The dual dynamics.* First, we evaluate $\langle Y(t), \Psi\rangle$ in (3.5) by using the duality arguments. For $t \in \mathbb{R}$, introduce 'formal adjoint' operators $U'(t)$, $U_0'(t)$ from space $\mathcal{S}$ to a suitable space of distributions. For example,

$$\langle Y, U_0'(t)\Psi\rangle = \langle U_0(t)Y, \Psi\rangle, \quad \Psi \in \mathcal{S}, \ Y \in \mathcal{H}, \ t \in \mathbb{R}. \tag{5.2}$$

The adjoint groups admit a convenient description. Lemma 5.1 states that the action of groups $U_0'(t)$, $U'(t)$ coincides, respectively, with the action of $U_0(t)$, $U(t)$, up to the order of the components. In particular, $U_0'(t)$ is a continuous group in $\mathcal{S}$.

LEMMA 5.1. *For* $\Psi = (\Psi^0, \Psi^1) \in \mathcal{S}$

$$U_0'(t)\Psi = (\dot{\phi}(\cdot, t), \phi(\cdot, t)), \quad U'(t)\Psi = (\dot{\psi}(\cdot, t), \psi(\cdot, t)), \tag{5.3}$$

where $\phi(x, t)$ *is the solution of equation (3.1) with the initial state* $(u_0, v_0) = (\Psi^1, \Psi^0)$ *and* $\psi(x, t)$ *is the solution of equation (1.1) with the initial state* $(u_0, v_0) = (\Psi^1, \Psi^0)$.

Proof. Differentiating (5.2) with $Y, \Psi \in \mathcal{S}$, we obtain

$$\langle Y, \dot{U}_0'(t)\Psi\rangle = \langle \dot{U}_0(t)Y, \Psi\rangle.$$

Group $U_0(t)$ has the generator (3.3). The generator of $U_0'(t)$ is the conjugate operator

$$\mathcal{A}_0' = \begin{pmatrix} 0 & A_0 \\ 1 & 0 \end{pmatrix}. \tag{5.4}$$

Hence, equation (5.3) holds with $\ddot{\phi} = A_0\phi$. For the group $U'(t)$ the proof is similar. □

Denote $\Phi(\cdot, t) = U_0'(t)\Psi$. Then (5.2) means that

$$\langle Y(t), \Psi\rangle = \langle Y_0, \Phi(\cdot, t)\rangle, \quad t \in \mathbb{R}. \tag{5.5}$$

In fact, (5.4) and (A.1) imply that in the Fourier representation, $\dot{\hat{\Phi}}(k, t) = \hat{\mathcal{A}}_0'(k)\hat{\Phi}(k, t)$ and $\hat{\Phi}(k, t) = \hat{\mathcal{G}}_t'(k)\hat{\Psi}(k)$, where $\hat{\mathcal{G}}_t'$ is the conjugate matrix to $\hat{\mathcal{G}}_t$, defined in (A.2). Therefore,

$$\Phi(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ikx} \hat{\mathcal{G}}_t'(k)\hat{\Psi}(k)\, dk. \tag{5.6}$$

Now we can obtain the asymptotics of function $\Phi(x, t)$. The following lemma is true.

LEMMA 5.2. *Let (5.1) hold and* $n \ge 2$ *be even. Then, for all* $N \in \mathbb{N}$,

$$|\Phi(x, t)| \le C(N, \Psi) \frac{t^{-(n-1)/2}}{|t - |x||^N + 1}, \quad (x, t) \in \mathbb{R}^{n+1}. \tag{5.7}$$

Proof. In fact, (5.6) and (A.2) imply that Φ can be written as the sum

$$\Phi(x, t) = \frac{1}{(2\pi)^n} \sum_{\pm} \int_{\mathbb{R}^n} \exp\{-i(kx \mp |k|t)\} a^{\pm}(|k|)\hat{\Psi}(k)\, dk, \tag{5.8}$$

where $a^{\pm}(|k|)$ is a matrix whose entries are linear functions in $|k|$ or $1/|k|$. According to the method of stationary phase [**11**], we have two estimates:

$$\sup_{x \in \mathbb{R}^n} |\Phi(x, t)| \le C(\Psi) t^{-(n-1)/2}$$

and

$$|\Phi(x, t)| \le C(\varepsilon, N, \Psi)(1 + t + |x|)^{-N}, \quad ||x| - t| \ge \varepsilon t, \quad 0 < \varepsilon < 1. \tag{5.9}$$

Therefore, it remains to prove (5.7) for $t+1 \le |x| \le 1+t(1+\varepsilon)$ or $t(1-\varepsilon)-1 \le |x| \le t-1$. We consider only the first case, the second is done in a similar way.

Let us prove the asymptotics (5.7) along each ray $x = vt + x_0$, $x_0 = v/|v|$ with $1 \le |v| \le 1 + \varepsilon$. Then we get, from (5.8),

$$\Phi(vt + x_0, t) = \frac{1}{(2\pi)^n} \sum_{\pm} \int_{\mathbb{R}^n} \exp\{-it(kv \mp |k|) - ikx_0\} a^{\pm}(|k|)\hat{\Psi}(k)\, dk. \tag{5.10}$$

This is a sum of oscillatory integrals with the phase functions $\phi_\pm(k) = kv \pm |k|$. For $|v| = 1$, the standard form of the stationary phase method is not applicable here as the set of stationary points $\{k \in \mathbb{R}^n : \nabla\phi_\pm(k) = 0\}$ is a ray $v = \pm k/|k|$ and the Hessian is degenerate everywhere. Therefore, we rewrite (5.10) in polar coordinates $r = |k|$, $\omega = k/|k|$ as

$$\Phi(vt + x_0, t) = \sum_{\pm} \int_0^\infty \left(\int_{|\omega|=1} \exp\{-irt(\omega v \mp 1) - ir\omega x_0\} f^\pm(r\omega)\, d\omega \right) dr, \quad (5.11)$$

where

$$f^\pm(r\omega) = \frac{1}{(2\pi)^n} a^\pm(r) r^{n-1} \hat{\Psi}(r\omega)$$

are smooth functions on $[0, \infty] \times S_1$, where $S_1 = (k \in \mathbb{R}^n : |k| = 1)$. Each phase function $\phi^1_\pm(\omega) = \omega v \pm 1$ restricted to the sphere $|\omega| = 1$ has two stationary points $\omega_{\rm cr} = \pm v/|v|$ and the Hessian is non-degenerate everywhere:

$$\operatorname{rank}(\operatorname{Hess} \phi^1_\pm(\omega)) = n - 1, \quad |\omega| = 1.$$

Next we apply the standard method of the stationary phase [**5**] to the inner integral $I^\pm$ in (5.11). For example, the integral I^+ has the asymptotics

$$\begin{aligned} I^+ &= \sum_{\omega_{\rm cr}} (rt)^{-(n-1)/2} \exp\{-itr(\omega_{\rm cr} v - 1) - ir\omega_{\rm cr} x_0\} \sum_{j=0}^{k-1} \mathcal{P}_j f^+(r\omega_{\rm cr})(rt)^{-j} \\ &\quad + R_k(r,t) = \sum_{\pm} (rt)^{-(n-1)/2} \exp\{-itr(\pm|v| - 1) \mp ir\} \sum_{j=0}^{k-1} \mathcal{P}_j f^+(\pm rv/|v|)(rt)^{-j} \\ &\quad + R_k(r,t), \end{aligned}$$

where $\mathcal{P}_j$ is some linear differential operator of order $2j$ and

$$|R_k(r,t)| < C(k) \| f^+(r\omega) \|_{C^\beta(S_1)} (rt)^{-(n-1)/2-k},$$

for some $\beta = \beta(k) < \infty$. (The integral I^- has the same asymptotics.) Note that $f^\pm(r\omega)$ decays rapidly at infinity. Therefore,

$$\begin{aligned} \int_0^\infty I^+\, dr = t^{-(n-1)/2} \sum_{\pm} \int_0^\infty & \exp\{-ir(\pm t|v| - t \pm 1)\} r^{-(n-1)/2} \\ & \times \sum_{j=0}^{k-1} \mathcal{P}_j f^+(\pm rv/|v|)(rt)^{-j}\, dr + C(k, \Psi) t^{-(n-1)/2-k}. \end{aligned} \quad (5.12)$$

Condition (5.1) allows us to integrate by parts N times and (5.12) implies the estimate

$$|\Phi(vt + x_0, t)| \le C(N, \Psi) \frac{t^{-(n-1)/2}}{(t|v| - t + 1)^N},$$

which implies the asymptotics (5.7) for $t + 1 \le |x| \le 1 + t(1 + \varepsilon)$. □

5.2. *Rooms and corridors.* Next we partition the ball $|x| \le 2t$ into N_t 'rooms' R_t^j, separated by 'corridors' C_t^j:

$$B_{2t} = \bigcup_{j=1}^{N_t} (R_t^j \cup C_t^j).$$

Which 'rooms' and 'corridors' we specify later. Denote by χ_t^j, ξ_t^j, η_t the indicators of the sets R_t^j, C_t^j and $\mathbb{R}^n \setminus B_{2t}$ respectively. Consider random variables r_t^j, c_t^j, l_t, where

$$r_t^j = \langle Y_0, \chi_t^j \Phi(\cdot, t)\rangle, \quad c_t^j = \langle Y_0, \xi_t^j \Phi(\cdot, t)\rangle, \quad l_t = \langle Y_0, \eta_t \Phi(\cdot, t)\rangle, \quad 1 \le j \le N_t. \tag{5.13}$$

Then (5.5) implies

$$\langle U_0(t) Y_0, \Psi\rangle = \sum_1^{N_t} (r_t^j + c_t^j) + l_t. \tag{5.14}$$

LEMMA 5.3. *Let (S1)–(S4) hold. Then, for a suitable 'room–corridor' partition, the following bounds hold for $t > 1$ and $1 \le j \le N_t$:*

$$E|r_t^j|^2 \le C(\Psi) d_t / t, \tag{5.15}$$

$$E|c_t^j|^2 \le C(\Psi) \rho_t / t, \tag{5.16}$$

$$E|l_t|^2 \le C_p(\Psi) t^{-p}, \quad \forall p > 0. \tag{5.17}$$

Proof. (5.17) follows from (5.9). Here we consider the case $n \ge 4$. The case $n = 2$ we discuss in Appendix B. Given $t > 1$, choose $d \equiv d_t \ge 1$ and $\rho \equiv \rho_t > 0$. Asymptotically relations between t, d_t and ρ_t are specified later. Define $h = d + \rho$ and

$$a^j = -2t + (j-1)h, \quad b^j = a^j + d, \quad 1 \le j \le N_t, \quad N_t \sim \frac{t}{h}. \tag{5.18}$$

Then $R_t^j = \{x \in B_{2t} : a^j \le x^n \le b^j\}$ and $C_t^j = \{x \in B_{2t} : b^j \le x^n \le a^{j+1}\}$. Here $x = (x^1, \ldots, x^n)$, d is the width of a room and ρ of a corridor. We discuss the first bound in (5.15) only, the second is done in a similar way. Rewrite the left-hand side as the integral of correlation matrices. Definition (5.13) and Corollary 4.2 imply by Fubini's theorem that

$$E|r_t^j|^2 = \langle \chi_t^j(x) \chi_t^j(y) q_0(x-y), \Phi(x,t) \otimes \Phi(y,t)\rangle. \tag{5.19}$$

According to (5.7) with $N = 2$, equation (5.19) implies that

$$\begin{aligned} E|r_t^j|^2 &\le C t^{-n+1} \int_{R_t^j \times R_t^j} \frac{\|q_0(x-y)\|\, dx\, dy}{((t-|x|)^2+1)((t-|y|)^2+1)} \\ &\le C t^{-n+1} \int_{R_t^j} \frac{dx}{(t-|x|)^2+1} \int_{\mathbb{R}^n} \|q_0(z)\|\, dz, \end{aligned} \tag{5.20}$$

where $\|q_0(z)\|$ stands for the norm of a matrix $(q_0^{ij}(z))$ and $\|q_0(z)\| \in L^1(\mathbb{R}^n)$ by (2.6) and (2.4). It remains to verify that

$$I_t(j) = \int_{R_t^j} \frac{dx}{(t-|x|)^2+1} \le C\, d t^{n-2}. \tag{5.21}$$

Let $x = (x', x^n)$, $t' = \sqrt{(2t)^2 - (x^n)^2}$. In polar coordinates $r = |x'|$, $\omega = x'/|x'|$, we have

$$I_t(j) = \int_{a^j}^{a^j+d} dx^n \int_{S^{n-2}} d\omega \int_0^{t'} \frac{r^{n-2}\, dr}{(t - \sqrt{r^2 + (x^n)^2})^2 + 1}.$$

Next, we change variable: $r \mapsto \alpha = t - \sqrt{r^2 + (x^n)^2}$, $r\, dr = -(t-\alpha)\, d\alpha$, then

$$\begin{aligned} I_t(j) &= C \int_{a^j}^{a^j+d} dx^n \int_{-t}^{t-|x^n|} \frac{((t-\alpha)^2 - (x^n)^2)^{(n-3)/2}(t-\alpha)\, d\alpha}{\alpha^2+1} \\ &\le C_1\, dt^{n-2} \int_{-t}^{t} \frac{d\alpha}{\alpha^2+1} \le C_2\, dt^{n-2}. \end{aligned}$$

□

6. *Convergence of characteristic functionals*

In this section we complete the proof of Proposition 3.2 for functions Ψ with spectral condition (5.1): we will remove it in §8. As stated, we use a version of the CLT developed by Ibragimov and Linnik. This gives the convergence to an equilibrium Gaussian measure. If $\mathcal{Q}_\infty(\Psi, \Psi) = 0$, Proposition 3.2 is obvious. Thus, we may assume that

$$\mathcal{Q}_\infty(\Psi, \Psi) \neq 0. \tag{6.1}$$

Choose $0 < \sigma < 1$ and

$$\rho_t \sim t^{1-\sigma}, \quad d_t \sim \frac{t}{\ln t}, \quad t \to \infty. \tag{6.2}$$

LEMMA 6.1. *The following limit holds true:*

$$N_t \left(\varphi(\rho_t) + \left(\frac{\rho_t}{t}\right)^{1/2} \right) + N_t^2 \left(\varphi^{1/2}(\rho_t) + \frac{\rho_t}{t} \right) \to 0, \quad t \to \infty. \tag{6.3}$$

Proof. Function $\varphi(r)$ is non-increasing; hence, by (2.4),

$$r^n \varphi^{1/2}(r) = n \int_0^r s^{n-1} \varphi^{1/2}(r)\, ds \le n \int_0^r s^{n-1} \varphi^{1/2}(s)\, ds \le C\overline{\varphi} < \infty. \tag{6.4}$$

Then equation (6.3) follows as (6.2) and (5.18) imply that $N_t \sim \ln t$. □

By the triangle inequality,

$$\begin{aligned} |\hat{\mu}_t(\Psi) - \hat{\mu}_\infty(\Psi)| \le &\left| E \exp\{i \langle U_0(t) Y_0, \Psi \rangle\} - E \exp\left\{ i \sum_t r_t^j \right\} \right| \\ &+ \left| \exp\left\{ -\frac{1}{2} \sum_t E|r_t^j|^2 \right\} - \exp\left\{ -\frac{1}{2} \mathcal{Q}_\infty(\Psi, \Psi) \right\} \right| \\ &+ \left| E \exp\left\{ i \sum_t r_t^j \right\} - \exp\left\{ -\frac{1}{2} \sum_t E|r_t^j|^2 \right\} \right| \\ \equiv &\; I_1 + I_2 + I_3, \end{aligned}$$

where the sum $\sum_t$ stands for $\sum_{j=1}^{N_t}$. We are going to show that all summands I_1, I_2, I_3 tend to zero as $t \to \infty$.

Step 1. Equation (5.14) implies

$$I_1 = \left| E \exp\left\{ i \sum_t r_t^j \right\} \left(\exp\{ i \sum_t c_t^j + i l_t \} - 1 \right) \right| \\ \le \sum_t E|c_t^j| + E|l_t| \\ \le \sum_t (E|c_t^j|^2)^{1/2} + (E|l_t|^2)^{1/2}. \tag{6.5}$$

From (6.5), (5.16), (5.17) and (6.3), we obtain that

$$I_1 \le C_1 N_t \left(\frac{\rho_t}{t}\right)^{1/2} + C_2 t^{-p} \to 0, \quad t \to \infty.$$

Step 2. By the triangle inequality,

$$I_2 \le \frac{1}{2}\left| \sum_t E|r_t^j|^2 - \mathcal{Q}_\infty(\Psi,\Psi) \right| \le \frac{1}{2}|\mathcal{Q}_t(\Psi,\Psi) - \mathcal{Q}_\infty(\Psi,\Psi)| \\ + \frac{1}{2}\left| E\left(\sum_t r_t^j\right)^2 - \sum_t E|r_t^j|^2 \right| + \frac{1}{2}\left| E\left(\sum_t r_t^j\right)^2 - \mathcal{Q}_t(\Psi,\Psi) \right| \\ \equiv I_{21} + I_{22} + I_{23}, \tag{6.6}$$

where $\mathcal{Q}_t$ is a quadratic form with the integral kernel $(Q_t^{ij}(x,y))$. Equation (4.6) implies that $I_{21} \to 0$. As to I_{22}, we first have that

$$I_{22} \le \sum_{j<l} E|r_t^j r_t^l|. \tag{6.7}$$

The next lemma is a corollary of [**7**, Lemma 17.2.3].

LEMMA 6.2. *Let ξ be a complex random value measurable with respect to σ-algebra $\sigma(\mathcal{A})$, η with respect to σ-algebra $\sigma(\mathcal{B})$, and* $\mathrm{dist}(\mathcal{A},\mathcal{B}) \ge r > 0$.

(1) *Let $(E|\xi|^2)^{1/2} \le a$, $(E|\eta|^2)^{1/2} \le b$. Then*

$$|E\xi\eta - E\xi E\eta| \le Cab\varphi^{1/2}(r).$$

(2) *Let $|\xi| \le a$, $|\eta| \le b$ almost sure. Then*

$$|E\xi\eta - E\xi E\eta| \le Cab\varphi(r).$$

We apply Lemma 6.2 to deduce that $I_{22} \to 0$ as $t \to \infty$. Note that $r_t^j = \langle Y_0, \chi_t^j \Phi(\cdot,t)\rangle$ is measurable with respect to the σ-algebra $\sigma(R_t^j)$. The distance between the different rooms R_t^j is greater or equal to ρ_t according to (5.18). Then (6.7) and (S2), (S4) imply, together with Lemma 6.2(1), that

$$I_{22} \le C N_t^2 \varphi^{1/2}(\rho_t), \tag{6.8}$$

which goes to 0 as $t \to \infty$ because of (6.3). Finally, it remains to check that $I_{23} \to 0$, $t \to \infty$. By the Cauchy–Schwartz inequality,

$$I_{23} \le \left| E\left(\sum_t r_t^j\right)^2 - E\left(\sum_t r_t^j + \sum_t c_t^j + l_t\right)^2 \right| \le C N_t \sum_t E|c_t^j|^2 \\ + C\left(E\left(\sum_t r_t^j\right)^2\right)^{1/2} \left(N_t \sum_t E|c_t^j|^2 + E|l_t|^2\right)^{1/2} + C E|l_t|^2. \tag{6.9}$$

Then (5.15), (6.7) and (6.8) imply

$$E\Big(\sum_t r_t^j\Big)^2 \le \sum_t E|r_t^j|^2 + 2\sum_{j<l} E|r_t^j r_t^l| \le CN_t\frac{d_t}{t} + C_1N_t^2\varphi^{1/2}(\rho_t) \le C_2 < \infty.$$

Now (5.15), (6.9) and (6.3) yield

$$I_{23} \le C_1N_t^2\frac{\rho_t}{t} + C_2N_t\left(\frac{\rho_t}{t}\right)^{1/2} + C_3t^{-p} \to 0, \quad t\to\infty.$$

So, all terms I_{21}, I_{22}, I_{23} in (6.6) tend to zero. Then (6.6) implies that

$$I_2 \le \frac{1}{2}\Big|\sum_t E|r_t^j|^2 - \mathcal{Q}_\infty(\Psi,\Psi)\Big| \to 0, \quad t\to\infty. \tag{6.10}$$

Step 3. It remains to verify that

$$I_3 = \Big|E\exp\Big\{i\sum_t r_t^j\Big\} - \exp\Big\{-\frac{1}{2}E\sum_t |r_t^j|^2\Big\}\Big| \to 0, \quad t\to\infty.$$

Using Lemma 6.2(2), we obtain

$$\begin{aligned}
&\Big|E\exp\Big\{i\sum_t r_t^j\Big\} - \prod_1^{N_t} E\exp\{ir_t^j\}\Big| \\
&\quad\le \Big|E\exp\{ir_t^1\}\exp\Big\{i\sum_2^{N_t} r_t^j\Big\} - E\exp\{ir_t^1\}E\exp\Big\{i\sum_2^{N_t} r_t^j\Big\}\Big| \\
&\qquad + \Big|E\exp\{ir_t^1\}E\exp\Big\{i\sum_2^{N_t} r_t^j\Big\} - \prod_1^{N_t} E\exp\{ir_t^j\}\Big| \\
&\quad\le C\varphi(\rho_t) + \Big|E\exp\Big\{i\sum_2^{N_t} r_t^j\Big\} - \prod_2^{N_t} E\exp\{ir_t^j\}\Big|.
\end{aligned}$$

We then apply Lemma 6.2(2) recursively and get, according to Lemma 6.1,

$$\Big|E\exp\Big\{i\sum_t r_t^j\Big\} - \prod_1^{N_t} E\exp\{ir_t^j\}\Big| \le CN_t\varphi(\rho_t) \to 0, \quad t\to\infty.$$

It remains to check that

$$\Big|\prod_1^{N_t} E\exp\Big\{ir_t^j\Big\} - \exp\Big\{-\frac{1}{2}\sum_t E|r_t^j|^2\Big\}\Big| \to 0, \quad t\to\infty.$$

According to the standard statement of the CLT (see, e.g., [**10**, Theorem 4.7]), it suffices to verify the Lindeberg condition, for all $\varepsilon > 0$,

$$\frac{1}{\sigma_t}\sum_t E_{\varepsilon\sqrt{\sigma_t}}|r_t^j|^2 \to 0, \quad t\to\infty.$$

Here $\sigma_t \equiv \sum_t E|r_t^j|^2$ and $E_\delta f \equiv E(X_\delta f)$, where X_δ is the indicator of the event $|f| > \delta^2$. Note that (6.10) and (6.1) imply that

$$\sigma_t \to \mathcal{Q}_\infty(\Psi,\Psi) \ne 0, \quad t\to\infty.$$

Hence, it remains to verify that, for all $\varepsilon > 0$,

$$\sum_t E_\varepsilon|r_t^j|^2 \to 0, \quad t\to\infty. \tag{6.11}$$

We check (6.11) in §7. This will complete the proof of Proposition 3.2.

7. *The Lindeberg condition*

The proof of (6.11) can be reduced to the case when for some $\Lambda \geq 0$ we have, almost surely, that

$$|u_0(x)| + |v_0(x)| \leq \Lambda < \infty, \quad x \in \mathbb{R}^n. \tag{7.1}$$

Then the proof of (6.11) is reduced to the convergence

$$\sum_t E|r_t^j|^4 \to 0, \quad t \to \infty \tag{7.2}$$

by using Chebyshev's inequality. The general case can be covered by standard cutoff arguments taking into account that bound (5.15) for $E|r_t^j|^2$ depends only on e_0 and φ. We deduce (7.2) from the following theorem.

THEOREM 7.1. *Let the conditions of Theorem B hold and assume that (7.1) is fulfilled. Then, for any $\Psi \in \mathcal{D}$, there exists a constant $C(\Psi)$ such that*

$$E|r_t^j|^4 \leq C(\Psi)\Lambda^4 d_t^2/t^2, \quad t > 1. \tag{7.3}$$

Proof. Step 1. Given four points $x_1, x_2, x_3, x_4 \in \mathbb{R}^n$, set

$$M_0^{(4)}(x_1, \ldots, x_4) = E(Y_0(x_1) \otimes \cdots \otimes Y_0(x_4)).$$

Then, similarly to (5.19), equations (7.1) and (5.13) imply by the Fubini theorem that

$$E|r_t^j|^4 = \langle \chi_t^j(x_1) \ldots \chi_t^j(x_4) M_0^{(4)}(x_1, \ldots, x_4), \Phi(x_1, t) \otimes \cdots \otimes \Phi(x_4, t)\rangle. \tag{7.4}$$

Let us analyze the domain of the integration $(R_t^j)^4$ in the right-hand side of (7.4). We partition $(R_t^j)^4$ into three parts W_2, W_3 and W_4:

$$(R_t^j)^4 = \bigcup_{i=2}^4 W_i, \quad W_i = \left\{\bar{x} = (x_1, x_2, x_3, x_4) \in (R_t^j)^4 : |x_1 - x_i| = \max_{p=2,3,4} |x_1 - x_p|\right\}.$$

Furthermore, given $\bar{x} = (x_1, x_2, x_3, x_4) \in W_i$, divide R_t^j into three parts S_j, $j = 1, 2, 3$: $R_t^j = S_1 \cup S_2 \cup S_3$, by two hyperplanes orthogonal to the segment $[x_1, x_i]$ and partitioning it into three equal segments, where $x_1 \in S_1$ and $x_i \in S_3$. Denote by x_p, x_q the two remaining points with $p, q \neq 1, i$. Set $\mathcal{A}_i = \{\bar{x} \in W_i : x_p \in S_1, x_q \in S_3\}$, $\mathcal{B}_i = \{\bar{x} \in W_i : x_p, x_q \notin S_1\}$ and $\mathcal{C}_i = \{\bar{x} \in W_i : x_p, x_q \notin S_3\}$, $i = 2, 3, 4$. Then $W_i = \mathcal{A}_i \cup \mathcal{B}_i \cup \mathcal{C}_i$. Define the function $\mathrm{m}_0^{(4)}(\bar{x})$, $\bar{x} \in (R_t^j)^4$, in the following way:

$$\mathrm{m}_0^{(4)}(\bar{x})|_{W_i} = \begin{cases} M_0^{(4)}(\bar{x}) - q_0(x_1 - x_p) \otimes q_0(x_i - x_q), & \bar{x} \in \mathcal{A}_i, \\ M_0^{(4)}(\bar{x}), & \bar{x} \in \mathcal{B}_i \cup \mathcal{C}_i. \end{cases}$$

This determines $\mathrm{m}_0^{(4)}(\bar{x})$ correctly for almost all quadruples $\bar{x}$. Note that

$$\begin{aligned} &\langle \chi_t^j(x_1) \cdots \chi_t^j(x_4) q_0(x_1 - x_p) \otimes q_0(x_i - x_q), \Phi(x_1, t) \otimes \cdots \otimes \Phi(x_4, t)\rangle \\ &\quad = \langle \chi_t^j(x_1)\chi_t^j(x_p) q_0(x_1 - x_p), \Phi(x_1, t) \otimes \Phi(x_p, t)\rangle \\ &\qquad \times \langle \chi_t^j(x_i)\chi_t^j(x_q) q_0(x_i - x_q), \Phi(x_i, t) \otimes \Phi(x_q, t)\rangle. \end{aligned}$$

Each factor here is bounded by $C(\Psi)\,d_t/t$. Similarly to (5.15), this can be deduced from an expression of type (5.19) for the factors. Therefore, the proof of (7.3) reduces to the proof of the bound

$$\begin{aligned} J_t(j) &:= |\langle \chi_t^j(x_1)\cdots\chi_t^j(x_4)\mathrm{m}_0^{(4)}(x_1,\dots,x_4), \Phi(x_1,t)\otimes\cdots\otimes\Phi(x_4,t)\rangle| \\ &\le C(\Psi)\Lambda^4 d_t^2/t^2, \quad t>1. \end{aligned}$$

Step 2. Similarly to (5.20), asymptotics (5.7) with $N=2$ implies

$$J_t(j) \le C(\Psi)\,t^{-2n+2}\int_{(R_t^j)^4}\frac{|\mathrm{m}_0^{(4)}(x_1,\dots,x_4)|\,dx_1\,dx_2\,dx_3\,dx_4}{((t-|x_1|)^2+1)\cdots((t-|x_4|)^2+1)}.$$

Let us estimate $\mathrm{m}_0^{(4)}$ using Lemma 6.2(2).

LEMMA 7.1. *For each $i=2,3,4$ and almost all $\overline{x}\in W_i$, the following bound holds:*

$$|\mathrm{m}_0^{(4)}(x_1,\dots,x_4)| \le C\Lambda^4\varphi(|x_1-x_i|/3).$$

Proof. For $\bar{x}\in\mathcal{A}_i$ we apply Lemma 6.2(2) to $\mathbb{C}^2\otimes\mathbb{C}^2$-valued random variables $\xi = Y_0(x_1)\otimes Y_0(x_p)$ and $\eta = Y_0(x_i)\otimes Y_0(x_q)$. Then (7.1) implies the bound for almost all $\bar{x}\in\mathcal{A}_i$:

$$|\mathrm{m}_0^{(4)}(\bar{x})| \le C\Lambda^4\varphi(|x_1-x_i|/3).$$

For $\bar{x}\in\mathcal{B}_i$, we apply Lemma 6.2(2) to $\xi=Y_0(x_1)$ and $\eta = Y_0(x_p)\otimes Y_0(x_q)\otimes Y_0(x_i)$. Then (S1) implies a similar bound for almost all $\bar{x}\in\mathcal{B}_i$,

$$|\mathrm{m}_0^{(4)}(\bar{x})| = |M_0^{(4)}(\bar{x}) - EY_0(x_1)\otimes E(Y_0(x_p)\otimes Y_0(x_q)\otimes Y_0(x_i))| \le C\Lambda^4\varphi(|x_1-x_i|/3),$$

and the same for almost all $\bar{x}\in\mathcal{C}_i$. □

Step 3. It remains to prove the following bounds for each $i=2,3,4$:

$$V_i(t) := \int_{(R_t^j)^4}\frac{X_i(\overline{x})\varphi(|x_1-x_i|/3)\,dx_1\,dx_2\,dx_3\,dx_4}{((t-|x_1|)^2+1)\cdots((t-|x_4|)^2+1)} \le Cd_t^2 t^{2n-4}, \tag{7.5}$$

where X_i is an indicator of the set W_i. In fact, this integral does not depend on i; hence, set $i=2$ in the integrand:

$$V_2(t) \le C\int_{(R_t^j)^2}\frac{\varphi(|x_1-x_2|/3)}{(t-|x_1|)^2+1}\left[\int_{R_t^j}\frac{1}{(t-|x_3|)^2+1}\left(\int_{R_t^j}X_2(\overline{x})\,dx_4\right)dx_3\right]dx_1\,dx_2.$$

Now a key observation is that the inner integral in dx_4 is $\mathcal{O}(|x_1-x_2|^n)$ as $X_2(\overline{x})=0$ for $|x_4-x_1|>|x_1-x_2|$. This implies

$$V_2(t) \le C_1\int_{R_t^j}\left(\int_{R_t^j}\varphi(|x_1-x_2|/3)|x_1-x_2|^n\,dx_2\right)\frac{dx_1}{(t-|x_1|)^2+1}\int_{R_t^j}\frac{dx_3}{(t-|x_3|)^2+1}. \tag{7.6}$$

The inner integral in dx_2 is bounded as

$$\begin{aligned} \int_{R_t^j}\varphi(|x_1-x_2|/3)|x_1-x_2|^n dx_2 &\le C(n)\int_0^{4t} r^{2n-1}\varphi(r/3)\,dr \\ &\le C_1(n)\sup_{r\in[0,4t]} r^n\varphi^{1/2}(r/3)\int_0^{4t} r^{n-1}\varphi^{1/2}(r/3)\,dr. \end{aligned}$$

The 'sup' and the last integral are bounded by (6.4) and (2.4), respectively. Therefore, (7.5) follows from (7.6) by (5.21) for $n \geq 4$ and by (B.1) for $n = 2$. This completes the proof of Theorem 7.1. $\square$

Proof of convergence (7.2). Estimate (7.3) implies, since $d_t \leq h \sim t/N_t$,

$$\sum_t E|r_t^j|^4 \leq \frac{C\Lambda^4 d_t^2}{t^2} N_t \leq \frac{C_1\Lambda^4}{N_t} \to 0, \quad t \to \infty. \qquad \square$$

8. *Removing the spectral condition*

Now we remove spectral condition (5.1) by a partition of unity in Fourier space. We must prove Proposition 3.2 for any $\Psi \in \mathcal{S}$. Let us split Ψ in to the sum of two functions:

$$\Psi = \Psi_\varepsilon + \Theta_\varepsilon, \tag{8.1}$$

where

$$\hat{\Psi}_\varepsilon(k) = (1 - \alpha_\varepsilon(k))\hat{\Psi}(k), \quad \hat{\Theta}_\varepsilon(k) = \alpha_\varepsilon(k)\hat{\Psi}(k)$$

and

$$\alpha_\varepsilon(k) \in C_0^\infty(\mathbb{R}^n), \quad \alpha_\varepsilon(k) = \begin{cases} 1, & |k| \leq \varepsilon, \\ 0, & |k| \geq 2\varepsilon. \end{cases}$$

Then, by the triangle inequality,

$$\begin{aligned}
&|E\exp\{i\langle U(t)Y_0, \Psi\rangle\} - \exp\{-\tfrac{1}{2}\mathcal{Q}_\infty(\Psi, \Psi)\}| \\
&\quad = |E\exp\{i\langle U(t)Y_0, \Psi_\varepsilon\rangle\}\exp\{i\langle U(t)Y_0, \Theta_\varepsilon\rangle\} - \exp\{-\tfrac{1}{2}\mathcal{Q}_\infty(\Psi, \Psi)\}| \\
&\quad \leq |E\exp\{i\langle U(t)Y_0, \Psi_\varepsilon\rangle\}\exp\{i\langle U(t)Y_0, \Theta_\varepsilon\rangle\} - E\exp\{i\langle U(t)Y_0, \Psi_\varepsilon\rangle\}| \\
&\qquad + |E\exp\{i\langle U(t)Y_0, \Psi_\varepsilon\rangle\} - \exp\{-\tfrac{1}{2}\mathcal{Q}_\infty(\Psi_\varepsilon, \Psi_\varepsilon)\}| \\
&\qquad + |\exp\{-\tfrac{1}{2}\mathcal{Q}_\infty(\Psi_\varepsilon, \Psi_\varepsilon)\} - \exp\{-\tfrac{1}{2}\mathcal{Q}_\infty(\Psi, \Psi)\}| = I_1 + I_2 + I_3.
\end{aligned} \tag{8.2}$$

We must estimate each term in the right-hand side.

Step 1. Applying the Cauchy–Schwartz inequality, we get, with the summation in the repeating indices,

$$\begin{aligned}
I_1 &= |E\exp\{i\langle U(t)Y_0, \Psi_\varepsilon\rangle\}(\exp\{i\langle U(t)Y_0, \Theta_\varepsilon\rangle\} - 1)| \\
&\leq E|\exp\{i\langle U(t)Y_0, \Theta_\varepsilon\rangle\} - 1| \leq E|\langle U(t)Y_0, \Theta_\varepsilon\rangle| \leq (E\langle U(t)Y_0, \Theta_\varepsilon\rangle^2)^{1/2} \\
&\leq \langle Q_t^{ij}(x, y), \Theta_\varepsilon^i(x) \otimes \Theta_\varepsilon^j(y)\rangle^{1/2}.
\end{aligned} \tag{8.3}$$

The right-hand side of (8.3) in Fourier space can be written as

$$\langle \hat{q}_t^{ij}(k), \hat{\Theta}_\varepsilon^i(k) \otimes \hat{\Theta}_\varepsilon^j(k)\rangle^{1/2} = \langle \hat{q}_t^{ij}(k), \alpha_\varepsilon^2(k)\hat{\Psi}^i(k) \otimes \hat{\Psi}^j(k)\rangle^{1/2} \leq \mu(\varepsilon), \quad t > 0.$$

Here $\mu(\varepsilon) \to 0$, $\varepsilon \to 0$, uniformly in $t \geq 0$ since the functions $\hat{q}_t^{ij}(k)$ admit the summable (in k) dominant independent of t by Proposition 4.1 and formulas of type (4.4) and (4.5) for $\hat{q}_t^{ij}(k)$.

Step 2. The second term I_2 in the right-hand side of (8.2) converges to zero as $t \to \infty$ according results of §6 since $\hat{\Psi}_\varepsilon(k) = 0$ for $|k| \leq \varepsilon$.

Step 3. It remains to verify that the third summand I_3 is $o(1)$ or that the difference $\mathcal{Q}_\infty(\Psi,\Psi) - \mathcal{Q}_\infty(\Psi_\varepsilon,\Psi_\varepsilon)$ is $o(1)$. According to (8.1),

$$\begin{aligned}\mathcal{Q}_\infty(\Psi,\Psi) - \mathcal{Q}_\infty(\Psi_\varepsilon,\Psi_\varepsilon) &= \mathcal{Q}_\infty(\Psi,\Psi-\Psi_\varepsilon) + \mathcal{Q}_\infty(\Psi-\Psi_\varepsilon,\Psi_\varepsilon)\\ &= \mathcal{Q}_\infty(\Psi,\Theta_\varepsilon) + \mathcal{Q}_\infty(\Theta_\varepsilon,\Psi_\varepsilon).\end{aligned}$$

Using FT, we obtain by (2.7)

$$\mathcal{Q}_\infty(\Psi,\Theta_\varepsilon) = \sum_{i,j=0,1}\int_{|k|\le 2\varepsilon} \hat{q}^{ij}_\infty(k)\hat{\Psi}^i(k)\hat{\Theta}^j_\varepsilon(k)\,dk = o(1),$$

since $\hat{\Psi}^i$, $\hat{\Theta}^j_\varepsilon$ are bounded and $\hat{q}^{ij}_\infty \in L^1(\mathbb{R}^n)\otimes M^2$ according to Corollary 4.1. The second summary in the right-hand side of (8.1) is $o(1)$ by the same argument.

Therefore, $I_1+I_2+I_3$ converge to zero as $t\to\infty$ since $\varepsilon>0$ is arbitrary.

9. *Scattering theory for infinite energy solutions*

We prove Theorem A for non-zero magnetic field

$$B(x) = \operatorname{curl} A(x) \not\equiv 0. \tag{9.1}$$

The proof for zero magnetic field is given in Appendix C. In this section we develop a version of scattering theory to deduce Theorem A from Theorem B. The main step is to establish asymptotics of type (1.13) for adjoint groups by using Vainberg's results [**13**].

Consider complex spaces $\mathring{H}^1(\mathbb{R}^n)$ which is a completion of $\mathcal{D}$ in the norm $\|\nabla u\|_{L^2}$ and $\mathring{H} = L^2(\mathbb{R}^n)\oplus\mathring{H}^1(\mathbb{R}^n)$ with the norm

$$\|\Psi\|_1^2 = \|\Psi\|^2_{\mathring{H}} = \int_{\mathbb{R}^n}(|\Psi^0(x)|^2 + |\nabla\Psi^1(x)|^2)\,dx < \infty.$$

Let

$$\|\Psi\|_2^2 = \int_{\mathbb{R}^n}(|\Psi^0(x)|^2 + |(\nabla - iA(x))\Psi^1(x)|^2)\,dx < \infty,$$

where $A(x) = (A_1(x),\ldots,A_n(x))$. These norms are equivalent, which follows from the following lemma.

LEMMA 9.1. *Let (9.1) hold and $n>2$. Then, for all $u\in C_0^\infty(\mathbb{R}^n)$,*
(1) $\|(\nabla - iA(x))u\|_{L^2} \le C_1\|\nabla u\|_{L^2}$,
(2) $\|\nabla u\|_{L^2} \le C_2\|(\nabla - iA(x))u\|_{L^2}$.

Proof of Lemma 9.1. The first bound follows as the function $A(x)$ vanish for $|x|\ge R_0$ and by the embedding theorem (see, e.g., [**8**, ch. 11]):

$$\|u\|_{L^2(B_{R_0})} \le C(R_0)\|u\|_{L^{n'}(B_{R_0})} \le C(R_0)\|\nabla u\|_{L^2}, \quad n' = 2n/(n-2) > 2.$$

To prove the second bound, note that

$$\nabla u = (\nabla - iA(x))u + iA(x)u.$$

Therefore,

$$\|\nabla u\|_{L^2} \le \|(\nabla - iA(x))u\|_{L^2} + C_3\|u\|_{L^2(B_{R_0})}.$$

It remains to prove that

$$\|u\|_{L^2(B_{R_0})} \le C_4 \|(\nabla - iA(x))u\|_{L^2(B_{R_0})}.$$

We establish a stronger property:

$$\|u\|_{H^1(B_{R_0})} \le C_5 \|(\nabla - iA(x))u\|_{L^2(B_{R_0})}, \forall u \in H^1(B_{R_0}).$$

Indeed, assume that such a constant C_5 does not exist. Then, for all $m \in \mathbb{N}$, there exists $u_m \in H^1(B_{R_0}) : \|u_m\|_{H^1(B_{R_0})} > m\|(\nabla - iA(x))u_m\|_{L^2(B_{R_0})}$. Set $v_m = u_m/\|u_m\|_{H^1(B_{R_0})}$, then

$$\|v_m\|_{H^1(B_{R_0})} = 1, \tag{9.2}$$

and

$$\|(\nabla - iA(x))v_m\|_{L^2(B_{R_0})} < \frac{1}{m}. \tag{9.3}$$

The sequence v_m is bounded in $H^1(B_{R_0})$ by (9.2), so we can choose a fundamental subsequence in $L^2(B_{R_0})$. We will suppose that it is v_m itself, i.e.

$$\|v_m - v_k\|_{L^2(B_{R_0})} \to 0, \quad m, k \to \infty.$$

Then (9.3) implies

$$\begin{aligned}\|\nabla(v_m - v_k)\|_{L^2(B_{R_0})} &\le \|(\nabla - iA(x))(v_m - v_k)\|_{L^2(B_{R_0})} \\ &\quad + \|A(x)(v_m - v_k)\|_{L^2(B_{R_0})} \to 0, \quad \text{as } m, k \to \infty.\end{aligned}$$

Then the sequence v_m is fundamental in $H^1(B_{R_0})$ too. So v_m converge in $H^1(B_{R_0})$ to some $v \in H^1(B_{R_0})$. Thus, (9.2) and (9.3) imply

$$\|v\|_{H^1(B_{R_0})} = 1 \tag{9.4}$$

and

$$\|(\nabla - iA(x))v\|_{L^2(B_{R_0})} = 0. \tag{9.5}$$

Now (9.5) implies $\nabla v - iA(x)v \equiv 0$ in B_{R_0}. Then for all j

$$v = C \exp\left\{ i \int_{-\infty}^{x_j} A_j(x)\, dx_j \right\}.$$

Finally, (9.4) implies that $C \not\equiv 0$, hence $A = \nabla\chi$ which contradicts (9.1). □

Consider operators $U_0'(t)$, $U'(t)$ in the complex space $\mathring{H}$. The energy conservation for the wave equations (1.1) and (3.1) implies that $U_0'(t)$ and $U'(t)$ are unitary operators with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Therefore, Lemma 9.1 implies the following corollary.

COROLLARY 9.1.

$$\|U_0'(t)\Psi\|_{\mathring{H}} = \|\Psi\|_{\mathring{H}}, \quad \|U'(t)\Psi\|_{\mathring{H}} \le C\|\Psi\|_{\mathring{H}}.$$

Given $t \ge 0$, denote

$$\varepsilon(t) = (t+1)^{-1} \ln^{-2}(t+2). \tag{9.6}$$

Vainberg's results [**12, 13**] imply the following lemma.

LEMMA 9.2. *Let Assumptions (E1)–(E2) hold, and n is even. Then for any $R, R_1 > 0$ there exists a constant $C = C(R, R_1)$ such that for $\Psi \in H = L^2(\mathbb{R}^n) \otimes H^1(\mathbb{R}^n)$ with a support in the ball B_R*

$$\|U'(t)\Psi\|_{(R_1)} \le C\varepsilon(t)\|\Psi\|_{(R)}, \quad t \ge 0, \tag{9.7}$$

where

$$\|\Psi\|^2_{(R)} = \int_{|x|\le R} (|\Psi^0(x)|^2 + |\Psi^1(x)|^2 + |\nabla\Psi^1(x)|^2)\, dx, \quad R > 0.$$

The main result of this section is Theorem 9.1.

THEOREM 9.1. *Let Assumptions (E1)–(E2) and (S1)–(S4) hold, and $n > 2$ is even. Then there exist isomorphism $W : \mathring{H} \to \mathring{H}$ such that for $\Psi \in \mathring{H}$*

$$U'(t)\Psi = U_0'(t)W\Psi + r(t)\Psi, \quad t \ge 0, \tag{9.8}$$

and

$$\begin{aligned} \|r(t)\Psi\|_{\mathring{H}} &\to 0, \quad t \to \infty \\ E\langle Y_0, r(t)\Psi\rangle^2 &\to 0, \quad t \to \infty. \end{aligned} \tag{9.9}$$

Proof.

Step 1. We apply the standard Cook method: see, e.g., [**11**, Theorem XI.4]. Fix $\Psi \in D$ with a support in B_R and define $W\Psi$, formally, as

$$W\Psi = \lim_{t\to\infty} U_0'(-t)U'(t)\Psi = \Psi + \int_0^\infty \frac{d}{dt} U_0'(-t)U'(t)\Psi\, dt. \tag{9.10}$$

We have to prove the convergence of the integral in norm in space $\mathring{H}_1$. First, observe that

$$\frac{d}{dt} U_0'(t)\Psi = \mathcal{A}_0' U_0'(t)\Psi, \quad \frac{d}{dt} U'(t)\Psi = \mathcal{A}' U'(t)\Psi,$$

where $\mathcal{A}_0'$ and $\mathcal{A}'$ are the generators to groups $U_0'(t)$ and $U'(t)$, respectively. Similarly to (5.4), we have

$$\mathcal{A}' = \begin{pmatrix} 0 & A \\ 1 & 0 \end{pmatrix}, \tag{9.11}$$

where

$$A = \sum_{j=1}^n (\partial_j - iA_k)^2.$$

Therefore,

$$\frac{d}{dt} U_0'(-t)U_1'(t)\Psi = U_0'(-t)(\mathcal{A}' - \mathcal{A}_0')U'(t)\Psi. \tag{9.12}$$

Now (9.11) and (5.4) imply

$$\mathcal{A}' - \mathcal{A}_0' = \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}.$$

Furthermore, (E2) implies that $L = \sum_{j=1}^{n}(\partial_j - iA_j)^2 - \Delta$ is a first-order partial differential operator with the coefficients vanishing for $|x| \geq R_0$. Thus, (9.7), Corollary 9.1 and the Sobolev inequality imply

$$\begin{aligned}\|U_0'(-t)(\mathcal{A}' - \mathcal{A}_0')U'(t)\Psi\|_1 &= \|(\mathcal{A}' - \mathcal{A}_0')U'(t)\Psi\|_1\\ &= \|((\mathcal{A}' - \mathcal{A}_0')U'(t)\Psi)^0\|_{L^2(B_{R_0})}\\ &\leq C\|(U'(t)\Psi)^1\|_{H^1(B_{R_0})}, \quad t \geq 0.\end{aligned}$$

Hence, (9.12) implies that

$$\int_s^\infty \left\| \frac{d}{dt} U_0'(-t)U'(t)\Psi \right\|_{\mathring{H}} dt \leq C(R)\varepsilon_1(s)\|\Psi\|_{\mathring{H}}, \quad s \geq 0, \tag{9.13}$$

where $\varepsilon_1(t) = \ln^{-1}(t+2)$. Therefore, (9.13) provides the convergence in (9.10) for $\Psi \in D$. The convergence for $\Psi \in \mathring{H}$ follows from uniform bound of the family of operators $U_0'(-t)U'(t)$ (see Corollary 9.1) and $\varepsilon/3$ argument. Moreover,

$$\begin{aligned}\|r(t)\Psi\|_{\mathring{H}} &= \|U'(t)\Psi - U_0'(t)W\Psi\|_{\mathring{H}} = \|U_0'(t)(U_0'(-t)U'(t)\Psi - W\Psi)\|_{\mathring{H}}\\ &= \|U_0'(-t)U'(t)\Psi - W\Psi\|_{\mathring{H}} \to 0, \quad t \to \infty.\end{aligned}$$

Step 2. Now we prove the convergence (9.9). First, similarly to (5.19),

$$E\langle Y_0, r(t)\Psi\rangle^2 = \langle q_0^{ij}(x-y), (r(t)\Psi(x))^i \otimes (r(t)\Psi(y))^j\rangle.$$

The Young identity and (2.4), (2.6) imply for $i = j = 0$ and all even $n \geq 2$

$$\begin{aligned}\langle q_0^{00}(x-y), (r(t)\Psi(x))^0 \otimes (r(t)\Psi(y))^0\rangle &\leq \|q_0^{00}\|_{L^1}\|(r(t)\Psi)^0\|_{L^2}\|(r(t)\Psi)^0\|_{L^2}\\ &\leq C\|r(t)\Psi\|_{\mathring{H}}\|r(t)\Psi\|_{\mathring{H}} \to 0, \quad t \to \infty.\end{aligned}$$

Note that $(r(t)\Psi(x))^1 \in \mathring{H}^1$, so we obtain, for $i = j = 1$, $n' = 2n/(n-2)$, $n'' = n/(n+2)$,

$$\begin{aligned}\langle q_0^{11}(x-y), (r(t)\Psi(x))^1 \otimes (r(t)\Psi(y))^1\rangle &\leq \|q_0^{11}\|_{L^{n''}}\|(r(t)\Psi)^1\|_{L^{n'}}\|(r(t)\Psi)^1\|_{L^{n'}}\\ &\leq C\|r(t)\Psi\|_{\mathring{H}}\|r(t)\Psi\|_{\mathring{H}} \to 0, \quad t \to \infty.\end{aligned}$$

In the cases $(i, j) = (0, 1)$ and $(i, j) = (1, 0)$ the proof is similar.

Step 3. It remains to prove that W is an isomorphism. Note, that for $n \geq 4$, the decay estimate of type (9.7) holds for the group $U_0'(t)$, as well as for $U'(t)$, and it provides the existence of operator $\Omega = s - \lim_{t\to\infty} U'(-t)U_0'(t)$. This operator is inverse to the operator W, since the composition of strong convergent bounded operators converges strongly. □

10. *Convergence to equilibrium for variable coefficients*

Theorem A follows from the following two propositions.

PROPOSITION 10.1. *The family of the measures $\{\mu_t, t \in \mathbb{R}\}$ is weakly compact in $\mathcal{H}^{-\varepsilon}$, for all $\varepsilon > 0$.*

PROPOSITION 10.2. *There exists a dense subspace* Π *in* $\mathring{H}$ *such that, for any* $\Psi \in \Pi$,

$$\hat{\mu}_t(\Psi) \equiv \int \exp\{i\langle Y, \Psi\rangle\}\mu_t(dY) \to \exp\left\{-\frac{1}{2}\mathcal{Q}_\infty(W\Psi, W\Psi)\right\}, \quad t \to \infty.$$

Proposition 10.1 provides the existence of the limiting measures of the family μ_t and Proposition 10.2 provides the uniqueness of the limiting measure, hence the convergence (2.2). We deduce these propositions from Propositions 3.1 and 3.2, respectively, by means of Theorem 9.1.

Proof of Proposition 10.1. Similarly to Proposition 3.1, Proposition 10.1 follows from the bounds

$$\sup_{t\geq 0} E\|U(t)Y_0\|_R < \infty, \quad R > 0. \tag{10.1}$$

For the proof, write the solution to (1.1) in the form

$$u(x,t) = v(x,t) + w(x,t). \tag{10.2}$$

Here $v(x,t)$ is the solution to (3.1), and $w(x,t)$ is the solution to the following Cauchy problem:

$$\begin{aligned}\ddot{w}(x,t) &= \sum_{k=1}^{n}(\partial_k - iA_k(x))^2 w(x,t)\\ &\quad - \sum_{k=1}^{n} 2iA_k(x)\partial_k v(x,t) - \sum_{k=1}^{n}(i\partial_k A_k(x) + A_k^2(x))v(x,t),\\ &\qquad w|_{t=0} = 0, \quad \dot{w}|_{t=0} = 0, \quad x \in \mathbb{R}^n.\end{aligned} \tag{10.3}$$

Then (10.2) implies

$$E\|U(t)Y_0\|_R \leq E\|U_0(t)Y_0\|_R + E\|(w(\cdot,t), \dot{w}(\cdot,t))\|_R. \tag{10.4}$$

By Proposition 3.1, we have

$$\sup_{t\geq 0} E\|U_0(t)Y_0\|_R < \infty. \tag{10.5}$$

It remains to estimate the second term in the right-hand side of (10.4). The Duhamel representation for the solution to (10.3) gives

$$(w, \dot{w}) = \int_0^t U(t-s)(0, \psi(\cdot,s))\,ds, \tag{10.6}$$

where

$$\psi(x,s) = -2i\sum_{k=1}^{n} A_k(x)\partial_k v(x,s) - \sum_{k=1}^{n}(i\partial_k A_k(x) + A_k^2(x))v(x,s).$$

Assumption (E2) implies that supp $\psi(\cdot,s) \subset B_{R_0}$. Moreover,

$$\|(0, \psi(\cdot,s))\|_{R_0} \leq C\|v(\cdot,s)\|_{H^1(B_{R_0})} \leq C\|U_0(s)Y_0\|_{R_0}. \tag{10.7}$$

Decay estimates of type (9.7) hold for the group $U(t)$, as well as for $U'(t)$, as both groups correspond to the same wave equation by Lemma 5.1. Hence, we have from (10.7),

$$\|U(t-s)(0,\psi(\cdot,s))\|_R \le C(R)\varepsilon(t-s)\|(0,\psi(\cdot,s))\|_{R_0} \le C_1(R)\varepsilon(t-s)\|U_0(s)Y_0\|_{R_0},$$

where $\varepsilon(\cdot)$ is defined in (9.6). Therefore, (10.6) and (10.5) imply

$$E\|(w(\cdot,t),\dot w(\cdot,t))\|_R \le C(R)\int_0^t \varepsilon(t-s)E\|U_0(s)Y_0\|_{R_0}\,ds \le C_2(R) < \infty, \quad t \ge 0.$$

Then (10.5) and (10.4) imply (10.1). □

Proof of Proposition 10.2. Equations (9.8) and (9.9), by the Cauchy–Schwartz inequality, imply

$$\begin{aligned}|E\exp i\langle U(t)Y_0,\Psi\rangle - E\exp i\langle Y_0, U_0'(t)W\Psi\rangle| &\le E|\langle Y_0, r(t)\Psi\rangle| \\ &\le (E|\langle Y_0, r(t)\Psi\rangle|^2)^{1/2} \to 0, \quad t\to\infty.\end{aligned}$$

It suffices to prove that for any Ψ from a dense set $\Pi \subset \mathring{H}$, we have

$$E\exp i\langle Y_0, U_0'(t)W\Psi\rangle \to \exp\{-\tfrac{1}{2}\mathcal{Q}_\infty(W\Psi, W\Psi)\}, \quad t\to\infty.$$

This follows directly from Proposition 3.2 in the case when $W\Psi \in \mathcal{D}$. Finally, the set $\Pi = W^{-1}\mathcal{D}$ is dense in $\mathring{H}$ since W is an isomorphism. □

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A. *Appendix. Fourier transform calculations*

Consider the CFs of the solutions to the system (3.2). Let $F : w \mapsto \hat w$ denote the FT of a tempered distribution $w \in S'(\mathbb{R}^n)$ (see, e.g., [**4**]). We also use this notation for vector- and matrix-valued functions.

A.1. *Dynamics in the FT space.* In the FT representation, system (3.2) becomes $\dot{\hat Y}(k,t) = \hat{\mathcal{A}}_0(k)\hat Y(k,t)$, hence

$$\hat Y(k,t) = \hat{\mathcal{G}}_t(k)\hat Y_0(k), \quad \hat{\mathcal{G}}_t(k) = \exp\{\hat{\mathcal{A}}_0(k)t\}. \tag{A.1}$$

Here we denote

$$\hat{\mathcal{A}}_0(k) = \begin{pmatrix} 0 & 1 \\ -|k|^2 & 0\end{pmatrix}, \quad \hat{\mathcal{G}}_t(k) = \begin{pmatrix} \cos|k|t & \dfrac{\sin|k|t}{|k|} \\ -|k|\sin|k|t & \cos|k|t\end{pmatrix}. \tag{A.2}$$

A.2. *Covariance matrices in the FT space.*

LEMMA A.1. *In the sense of matrix-valued distributions,*

$$q_t(x-y) := E(Y(x,t)\otimes Y(y,t)) = F^{-1}_{k\to x-y}\hat{\mathcal{G}}_t(k)\hat{q}_0(k)\hat{\mathcal{G}}'_t(k), \quad t\in\mathbb{R}. \tag{A.3}$$

Proof. Translation invariance (1.3) implies

$$E(Y_0(x)\otimes_{\mathbb{C}} Y_0(y)) = C_0^+(x-y), \quad E(Y_0(x)\otimes_{\mathbb{C}} \overline{Y_0(y)}) = C_0^-(x-y)$$

where $\otimes_{\mathbb{C}}$ stands for the tensor product of complex vectors. Therefore,

$$E(\hat{Y}_0(k)\otimes_{\mathbb{C}} \hat{Y}_0(k')) = F_{x\to k}F_{y\to k'}C_0^+(x-y) = (2\pi)^n\delta(k+k')\hat{C}_0^+(k),$$
$$E(\hat{Y}_0(k)\otimes_{\mathbb{C}} \overline{\hat{Y}_0(k')}) = F_{x\to k}F_{y\to k'}C_0^-(x-y) = (2\pi)^n\delta(k+k')\hat{C}_0^-(k).$$

Now (A.1) and (A.2) give, in matrix notation,

$$E(\hat{Y}(k,t)\otimes_{\mathbb{C}} \hat{Y}(k',t)) = (2\pi)^n\delta(k+k')\hat{\mathcal{G}}_t(k)\hat{C}_0^+(k)\hat{\mathcal{G}}'_t(k),$$
$$E(\hat{Y}(k,t)\otimes_{\mathbb{C}} \overline{\hat{Y}(k',t)}) = (2\pi)^n\delta(k+k')\hat{\mathcal{G}}_t(k)\hat{C}_0^-(k)\hat{\mathcal{G}}'_t(k).$$

Therefore, by the inverse FT formula, we get

$$E(Y(x,t)\otimes_{\mathbb{C}} Y(y,t)) = F^{-1}_{k\to x-y}\hat{\mathcal{G}}_t(k)\hat{C}_0^+(k)\hat{\mathcal{G}}'_t(k),$$
$$E(Y(x,t)\otimes_{\mathbb{C}} \overline{Y(y,t)}) = F^{-1}_{k\to x-y}\hat{\mathcal{G}}_t(k)\hat{C}_0^-(k)\hat{\mathcal{G}}'_t(k).$$

Then (A.3) follows by linearity. □

B. *Appendix. The 'rooms–corridors' partition in the case $n=2$*

We prove Lemma 5.3 in the case $n=2$. The 'room–corridor' partition by hyperplanes, orthogonal to the axis x^n (as in the case $n\ge 4$) is not now suitable, because this partition does not allow us to obtain bound (5.21). So we choose another partition: it is more optimal to divide the circle $|x|\le 2t$ into symmetric sectors.

Given $t>1$, choose σ as in the general case $n\ge 4$, and then

$$N_t \sim \ln t,\, h_t = \frac{4\pi t}{N_t-1} \sim \frac{t}{\ln t}.$$

We assume the same asymptotically relations between t, d_t and ρ_t as in (6.2):

$$h_t = d_t + \rho_t, \quad d_t \sim \frac{t}{\ln t}, \quad \rho_t \sim t^{1-\sigma}.$$

Set

$$\theta^j = (j-1)\frac{h_t}{2t}, \quad \gamma^j = \theta^j + \frac{d_t}{2t}, \quad j = 1,\dots,N_t-1,$$

and then define 'rooms' and 'corridors' in polar coordinates (r,ϕ) as follows:

$$\begin{cases} R_t^j = \{(r,\phi): t/2\le r\le 2t, \theta^j\le\phi\le\gamma^j\}, \\ C_t^j = \{(r,\phi): t/2\le r\le 2t, \gamma^j\le\phi\le\theta^{j+1}\}, \end{cases} \quad 1\le j\le N_t-1,$$

and, for $j = N_t$, we set

$$R_t^{N_t} = \emptyset, \quad C_t^{N_t} = \left\{(r, \phi) : |r| \le \frac{t}{2}\right\}.$$

It remains to prove estimates of the type (5.21):

$$I_t(j) = \int_{R_t^j} \frac{dx}{(t - |x|)^2 + 1} \le C d_t, \quad J_t(j) = \int_{C_t^j} \frac{dx}{(t - |x|)^2 + 1} \le C \rho_t.$$

In polar coordinates (r, ϕ), we have

$$I_t(j) = \int_{\theta^j}^{\theta^j + d_t/(2t)} d\phi \int_{t/2}^{2t} \frac{r\,dr}{(t - r)^2 + 1} = \frac{d_t}{2t} \int_{t/2}^{2t} \frac{r\,dr}{(t - r)^2 + 1}.$$

Set $\rho = t - r$, then

$$I_t(j) = \frac{d_t}{2t} \int_{-t}^{t/2} \frac{(t - \rho)\,d\rho}{\rho^2 + 1} \le C \frac{d_t}{t} t = C d_t. \tag{B.1}$$

The integrals $J_t(j)$ can be estimated in a similar way to that for $1 \le j \le N_t - 1$. For $j = N_t$ we have

$$J_t(N_t) = \int_0^{2\pi} d\phi \int_0^{t/2} \frac{r\,dr}{(t - r)^2 + 1} \le 2\pi \int_0^{t/2} \frac{r\,dr}{1 + (t/2)^2} \le C.$$

C. *Appendix. Proof of Theorem A for zero magnetic field*

Let us prove Theorem A in the case

$$B(x) = \operatorname{curl} A(x) \equiv 0.$$

In this case there exists a function $\chi(x) \in C_0^\infty(B_{R_0})$ such that $A(x) = \nabla \chi(x)$. Let us write the solution to (1.1) in the form

$$u(x, t) = \exp\{i\chi(x)\} v(x, t).$$

When $v(x, t)$ is the solution to (3.1) with the initial state

$$v|_{t=0} = \exp\{-i\chi(x)\} u_0(x), \quad \dot{v}|_{t=0} = \exp\{-i\chi(x)\} v_0(x).$$

Therefore,

$$U(t)Y_0 = \exp\{i\chi\} U_0(t)(\exp\{-i\chi\} Y_0) = \exp\{i\chi\} U_0(t) Y_0 + \rho(t) Y_0, \tag{C.1}$$

where

$$\rho(t) Y_0 = \exp\{i\chi\} U_0(t)((\exp\{-i\chi\} - 1) Y_0).$$

The support of the function $(\exp\{-i\chi\} - 1) Y_0$ belongs to the ball B_{R_0} and decay estimates of type (9.7) hold for the group $U_0(t)$ too. Therefore,

$$\|\rho(t) Y_0\|_R = \|U_0(t)((\exp\{-i\chi\} - 1) Y_0)\|_R \le C(R, R_0) \varepsilon(t) \|Y_0\|_{R_0} \tag{C.2}$$

Theorem A follows from next two propositions.

PROPOSITION C.1. *The family of the measures* $\{\mu_t, t \in \mathbb{R}\}$, *is weakly compact in* $\mathcal{H}^{-\varepsilon}$, *for all* $\varepsilon > 0$, *and the bounds hold:*

$$\sup_{t \geq 0} E\|U(t)Y_0\|_R^2 < \infty, \quad R > 0. \tag{C.3}$$

PROPOSITION C.2. *For any* $\Psi \in D$,

$$\hat{\mu}_t(\Psi) \equiv \int \exp\{i\langle Y, \Psi\rangle\}\mu_t(dY) \to \exp\left\{-\frac{1}{2}\mathcal{Q}_\infty(W\Psi, W\Psi)\right\}, \quad t \to \infty, \tag{C.4}$$

where $W\Psi = \exp\{i\chi\}\Psi$.

Proof of Proposition C.1. Representation (C.1) and bound (C.2) imply

$$\begin{aligned} E\|U(t)Y_0\|_R^2 &\leq 2E\|U_0(t)Y_0\|_R^2 + 2E\|\rho(t)Y_0)\|_R^2 \\ &\leq 2E\|U_0(t)Y_0\|_R^2 + C_1(R, R_0)E\|Y_0\|_{R_0}^2. \end{aligned}$$

Then (C.3) follows from (1.4) and (3.4). Then the compactness follows similarly to Proposition 3.1. □

Proof of Proposition C.2. Let $\Psi \in D$ with support in B_R. Equation (C.1) implies

$$\begin{aligned} \hat{\mu}_t(\Psi) &\equiv E\exp\{i\langle U(t)Y_0, \Psi\rangle\} \\ &= E\exp\{i\langle \exp\{i\chi\}U_0(t)Y_0 + \rho(t)Y_0, \Psi\rangle\}E\exp\{i\langle U_0(t)Y_0, \exp\{i\chi\}\Psi\rangle\} + \nu(t), \end{aligned} \tag{C.5}$$

where

$$\nu(t) = E[\exp\{i\langle \exp\{i\chi\}U_0(t)Y_0, \Psi\rangle\}(\exp\{i\langle\rho(t)Y_0, \Psi\rangle\} - 1)].$$

Note that $\nu(t)$ vanishes as $t \to \infty$. In fact, the bound (C.2) implies, as before,

$$|\nu(t)| \leq E|\langle\rho(t)Y_0, \Psi\rangle| \leq C(R)\|\Psi\|_{(R)}E\|\rho(t)Y_0\|_R \to 0, \quad t \to \infty. \tag{C.6}$$

Finally, Proposition 3.2 implies that

$$E\exp\{i\langle U_0(t)Y_0, \exp\{i\chi\}\Psi\rangle\} \to \exp\{-\tfrac{1}{2}\mathcal{Q}_\infty(\exp\{i\chi\}\Psi, \exp\{i\chi\}\Psi)\}, \tag{C.7}$$

as $t \to \infty$, and (C.5)–(C.7) imply (C.4). □

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Annotation 1
Au:
`CVs' or `CFs'?