On metastable regimes in stochastic Lamb system

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We consider the long time behavior of the coupled Hamilton system of one-dimensional string and nonlinear oscillator, in contact with a heat bath modeled by the white noise. For any temperature the system converges to a statistical equilibrium described by the Boltzmann equilibrium measure. The convergence is caused by radiation provided by the nonlinear coupling. If the oscillator potential has more than one well and the temperature is small, the relaxation time is large, and the system goes through a sequence of metastable states located near local minima of the potential. When both, the temperature and the radiation rate are small, the metastable states are distributions among the minima of the potential. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2189198]

I. STRING COUPLED TO A NONLINEAR OSCILLATOR

We will consider a nonlinear oscillator coupled to a heat bath and to a one-dimensional (1D) string. The string is governed by the 1D wave equation

$$\mu\ddot{u}(x,t) = Tu''(x,t), \quad x \in \mathbb{R}\setminus\{0\}, \tag{1.1}$$

where $u(x,t)$ is the real function, $\mu>0$ is the string density, and $T>0$ is its tension. The oscillator is a particle of mass $m>0$ attached to the string at the point $x=0$, so

$$u(0,t) = q(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where $q(t)$ is the deviation of the oscillator. The heat bath is modeled as white noise, so the oscillator is governed by the stochastic equation

$$m\ddot{q}(t) = F(q(t)) + T[u'(0+,t) - u'(0-,t)] + \sqrt{\sigma}\dot{W}(t); \quad q(t) \equiv u(0,t), \tag{1.3}$$

where $F(q)$ stands for the oscillator force function, $W(t)$ is the standard one-dimensional Wiener process, and $\sigma \geqslant 0$ is the temperature of the heat bath. The middle term on the right-hand side of (1.3) descibes the string-oscillator interaction. Roughly speaking, $Tu'(0+,t)$, respectively, $-Tu'(0+,t)$ is the "vertical projection" of the tension of the string to the right-handside, respectively, to the left-handside of the oscillator (see Fig. 1).

The system (1.1)–(1.3) is formally equivalent to a one-dimensional nonlinear wave equation with the nonlinear term concentrated at the single point $x=0$ and with a mass m concentrated at the same point,

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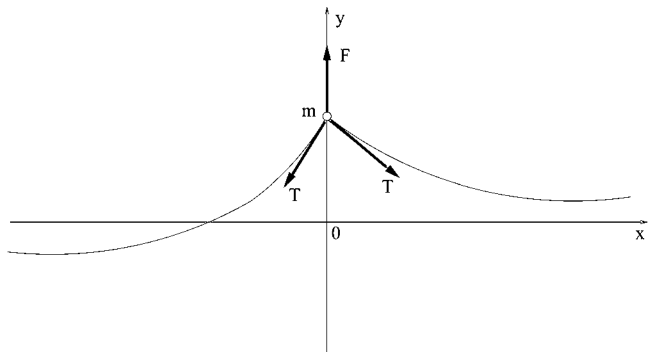

FIG. 1. String with the oscillator.

$$(\mu + m\delta(x))\ddot{u}(x,t) = Tu''(x,t) + \delta(x)[F(u(x,t)) + \sqrt{\sigma}\dot{W}(t)], \quad (x,t) \in \mathbb{R}^2. \tag{1.4}$$

Namely, the equation (1.1) follows from (1.4) with $x \neq 0$, while (1.3) follows by equating the coefficients at the delta function in both sides of (1.4).

For the linear oscillator when $F(q) = -\omega^2 q$ the system (1.1), (1.3) with $\sigma = 0$ was considered originally by Lamb.[11] For general nonlinear force function $F(q)$ and $\sigma = 0$ the system was analyzed in Refs. 8 and 9 (see also Ref. 10, pp. 26–37) where the convergence to stationary states has been proved for all finite energy solutions in the long-time limit $t \to \pm\infty$.

In the present paper we consider the long-time behavior of the Lamb system (1.1)–(1.3) with $\sigma > 0$ modeling the interaction with a heat bath of the temperature σ. We consider the Cauchy problem for the system (1.1)–(1.3) with the initial conditions

$$u|_{t=0} = u_0(x), \quad \dot{u}|_{t=0} = v_0(x), \quad \dot{q}|_{t=0} = p_0. \tag{1.5}$$

We assume that initial functions are compactly supported or decreasing fast enough at infinity. Then one can expect that, due to interplay between the energy dissipation caused by radiation and the incident energy flow from the heat bath, a stationary regime will be established in large time. If the potential $U(q) := -\int F(q)\mathrm{d}q$ has more than one well and the temperature σ is small, the relaxation time is large, and the system goes through a sequence of metastable states.

In generic situation, for a given time scale and an initial point, the system will be situated near certain local minimum of the potential. We consider also the situation when both, the temperature σ and the radiation rate (which can be characterized by the product of the string density and its tension), are small. An "additional stochasticity" appears in this case due to instability near the local maximums of potential. Therefore, even for generic potential, the metastable states are distributions among the minima of the potential. We calculate these limiting distributions.

II. NOTATIONS AND DYNAMICS

Write the Cauchy problem (1.4) and (1.5) in the form

$$\dot{Y}(t) = \mathbf{F}(Y(t),t), \quad t \in \mathbb{R}, \quad Y(0) = Y_0, \tag{2.1}$$

where $Y_0 = (u_0, v_0, p_0)$ and $Y(t) = (u(t), v(t), \dot{q}(t))$.

Let us introduce a phase space $\mathcal{E}$ of finite energy states for the system (1.1)–(1.3). Denote by $\|\cdot\|$, respectively, $\|\cdot\|_R$ the norm in the Hilbert space $L^2 := L^2(\mathbb{R})$, respectively, $L^2(-R,R)$.

Definition 2.1: (i) *$\mathcal{E}$ is the Hilbert space of the triples $(u(x), v(x), p) \in C(\mathbb{R}) \oplus L^2 \oplus \mathbb{R}$ with $u'(x) \in L^2$ and the* global energy norm

$$\|(u,v,p)\|_{\mathcal{E}} = \|u'\| + |u(0)| + \|v\| + |p|. \tag{2.2}$$

(ii) *$\mathcal{E}_F$ is the space $\mathcal{E}$ endowed with the Fréchet topology defined by the* local energy seminorms

$$\|(u,v,p)\|_{\mathcal{E},R} \equiv \|u'\|_R + |u(0)| + \|v\|_R + |p|, \quad R > 0. \tag{2.3}$$

(iii) $Y_n \xrightarrow{\mathcal{E}_F} Y$ *iff* $\|Y_n - Y\|_{\mathcal{E},R} \to 0$, $\forall R > 0$.

Remark 2.2: This convergence is equivalent to the convergence with respect to the metric

$$\rho(X,Y) = \sum_{R=1}^{\infty} 2^{-R} \frac{\|X-Y\|_{\mathcal{E},R}}{1 + \|X-Y\|_{\mathcal{E},R}}, \quad X,Y \in \mathcal{E}. \tag{2.4}$$

We assume that

$$F(q) \in C^1(\mathbb{R}), \tag{2.5}$$

$$U(q) := -\int F(q)\mathrm{d}q \to +\infty, \quad |q| \to \infty. \tag{2.6}$$

Then the system (1.1)–(1.3) for $\sigma = 0$ is formally Hamiltonian with the Hamilton functional

$$\mathcal{H}(u,v,p) = \frac{1}{2}\int [|v(x)|^2 + |u'(x)|^2]\mathrm{d}x + m\frac{|p|^2}{2} + U(u(0)) \tag{2.7}$$

for $(u,v,p) \in \mathcal{E}$. We consider solutions $u(x,t)$ such that $Y(t) = (u(\cdot,t), \dot{u}(\cdot,t), \dot{q}(t)) \in C(0,\infty;\mathcal{E})$.

Let us discuss the definition of the Cauchy problem (2.1) for the functions $Y(t) \in C(0,\infty;\mathcal{E})$. At first, $u(x,t) \in C(\mathbb{R}^2)$ due to $Y(t) \in C(0,\infty;\mathcal{E})$. Then the wave equation (1.1) is understood in the sense of distributions. This is equivalent to the d'Alembert decomposition

$$u(x,t) = f_\pm(x - at) + g_\pm(x + at), \quad \pm x > 0, \tag{2.8}$$

where $a = \sqrt{T/\mu} > 0$, and $f_- \in C(-\infty,0)$, $g_+ \in C(0,\infty)$, and $f_+, g_- \in C(-\infty,\infty)$. Therefore,

$$\dot{u}(x,t) = -f'_\pm(x - at) + g'_\pm(x + at), \quad u'(x,t) = f'_\pm(x - at) + g'_\pm(x + at) \text{ for } \pm x > 0,\ t \in \mathbb{R}, \tag{2.9}$$

where all the derivatives are understood in the sense of distributions. The condition $Y(t) \in C(0,\infty;\mathcal{E})$ implies that

$$f'_\pm, g'_\pm \in L^2_{\mathrm{loc}}(\mathbb{R}). \tag{2.10}$$

We now explain the second equation (1.3).

Definition 2.3: In the equation (1.3) *set*

$$u'(0\pm,t) \equiv f'_\pm(-at) + g'_\pm(at) \in L^2_{\mathrm{loc}}(0,\infty), \tag{2.11}$$

while the derivative $\ddot{q}(t)$ *of* $q(t) \equiv u(0,t) \in C[0,\infty)$ [*or of* $\dot{q}(t) \in L^2_{\mathrm{loc}}[0,\infty)$ *by* (2.10)] *is understood in the sense of distributions.*

Note that the functions $f_\pm$ and $g_\pm$ in (2.8) are unique up to an additive constant. Hence definition (2.11) is unambiguous.

Proposition 2.4: (*Ref.* 9) *Let the conditions* (2.5), (2.6) *be fulfilled, and* $W(t) \in C(0,\infty;\mathbb{R})$ *is a fixed function. Then*

(i) *For every* $Y_0 \in \mathcal{E}$ *the Cauchy problem* (2.1) *admits a unique solution* $Y(t) \in C(0,\infty;\mathcal{E})$.
(ii) *The map* $U(t): Y_0 \mapsto Y(t)$ *is continuous in* $\mathcal{E}$ *and* $\mathcal{E}_F$.

The proposition is proved in Ref. 9 for the case $W(t) \equiv 0$. The proof for the general case is very similar.

III. GLOBAL ATTRACTOR FOR ZERO TEMPERATURE

The stationary states $S=(s(x),0,0)\in\mathcal{E}$ for (1.1)–(1.3) with $\sigma=0$ are evidently determined. We define for every $c\in\mathbb{R}$ the constant function

$$s_c(x)=c,\quad x\in\mathbb{R}. \tag{3.1}$$

Then the set $\mathcal{S}$ of all stationary states $S\in\mathcal{E}$ is given by

$$\mathcal{S}=\{S_z=(s_z(\cdot),0,0):\ z\in Z\}, \tag{3.2}$$

where $Z=\{z\in\mathbb{R}:\ F(z)=0\}$.

The set S is a global attractor for the Lamb system (1.1)–(1.3) with $\sigma=0$.[9] Our main goal in this paper is to describe the convergence to the statistical equilibrium for the Lamb system with $\sigma>0$, and metastable rejimes for small $\sigma>0$ and $m>0$.

IV. REDUCED EQUATION FOR POSITIVE TEMPERATURE

The Lamb system (1.1)–(1.3) is equivalent to the following reduced equation:

$$m\ddot{q}(t)=F(q(t))-\frac{2T}{a}\dot{q}(t)+\frac{2}{a}\dot{w}_{\rm in}(t)+\sqrt{\sigma}\dot{W}(t),\quad t>0, \tag{4.1}$$

where $w_{\rm in}(t)\in C[0,\infty)$ is determined by the initial conditions (1.5) and the equation is understood in the sense of the corresponding integral equation (or distributions).[9,10]

For $|x|\geqslant at\geqslant 0$ at the solution of the system (1.1)–(1.3) is determined uniquely by the initial functions and is expressed by the d'Alembert formula

$$u(x,t)=\frac{u_0(x-at)+u_0(x+at)}{2}+\frac{1}{2}\int_{x-at}^{x+at}v_0(y)\mathrm{d}y,\quad |x|\geqslant at\geqslant 0. \tag{4.2}$$

For $|x|\leqslant at$ the solution cannot be expressed in the initial functions. Indeed, the waves $f_+(x-at)$, respectively, $g_-(x+at)$ [see (2.8)] in the regions $0<x<at$, respectively, $-at<x<0$ are the *reflected waves* and are not determined by the initial conditions. To determine both these two reflected waves we need two equations: first is the gluing equation $u(0+,t)=u(0-,t)$, and the second is the "jump equation" (1.3). Substituting the d'Alembert representations (2.8) to the equations, we get (see Ref. 9 or Ref. 10, Chap. 1, Lemma 4.6) the reduced equation (4.1), where $w_{\rm in}(t)$ is the sum of the *incident waves* $g_+(x+at)$ and $f_-(x-at)$ at the point $x=0$:

$$w_{\rm in}(t)=g_+(at)+f_-(-at),\quad t\geqslant 0. \tag{4.3}$$

For this function we have

$$\dot{w}_{\rm in}(t)\in L^2(0,\infty) \tag{4.4}$$

since the initial functions belong to the phase space of finite energy states. Moreover, we get the expressions [see Ref. 10, Chap. 1, (4.33)]

$$u(x,t)=\left.\begin{cases} q(t-x/a)+g_+(x+at)-g_+(at-x), & 0<x<at \\ q(t+x/a)+f_-(x-at)-f_-(-x-at), & -at<x<0\end{cases}\right|,\quad t\geqslant 0. \tag{4.5}$$

It is important to note that this formula contains only the incident waves which are constant for large time if the initial functions are constant for large $|x|$. Namely, let us consider the initial functions in (1.5) with

$$u_0(x)=C_\pm,\quad v_0(x)=0,\quad \pm x>R_0. \tag{4.6}$$

Then $g_+(z)=c_+$ for $z>R_0$, and $f_-(z)=c_-$ for $z<-R_0$. Hence (4.3) implies that

$$w_{\text{in}}(t)=0, \quad t>R_0/a. \tag{4.7}$$

Respectively, (4.1) becomes

$$m\ddot{q}(t)=F(q(t))-\frac{2T}{a}\dot{q}(t)+\sqrt{\sigma}\dot{W}(t), \quad t>R_0/a, \tag{4.8}$$

and (4.5) implies that

$$u(x,t)=q(t-|x|/a), \quad |x|<R, \quad t\geqslant\frac{R+R_0}{a} \tag{4.9}$$

for every $R>0$. Finally, take into account the value of a. Then (4.8) reads

$$m\ddot{q}(t)=F(q(t))-2\sqrt{\mu T}\dot{q}(t)+\sqrt{\sigma}\dot{W}(t), \quad t>R_0/a. \tag{4.10}$$

Our goal is to describe a long-time behavior of the solution for the cases

$$m\sim\sqrt{\mu T}\ll 1, \quad \sqrt{\sigma}\ll 1. \tag{4.11}$$

V. CONVERGENCE TO EQUILIBRIUM DISTRIBUTION

If the supports of the initial functions $u_0(x)$ and $v_0(x)$ belong to a finite interval $|x|<R_0<\infty$, then, at least after time $t_0=R_0/a$, no incident waves come to the origin. So the evolution of the oscillator can be described by Eq. (4.10) which is equivalent to the following system:

$$\begin{aligned}\dot{q}(t)&=p(t),\\ m\dot{p}(t)&=-U'(q(t))-2\sqrt{\mu T}p(t)+\sqrt{\sigma}\dot{W}(t), \quad t>t_0.\end{aligned} \tag{5.1}$$

The values $q(t_0)$ and $p(t_0)$ are defined by the initial condition and by the trajectory $W(t)$ for $0\leqslant t\leqslant t_0$.

The stochastic process $(q(t),p(t))$ defined by (5.1) is a (degenerate) diffusion process governed by the differential operator

$$Lu(q,p)=p\frac{\partial u}{\partial q}-\frac{1}{m}[U'(q)+2p\sqrt{\mu T}]\frac{\partial u}{\partial p}+\frac{\sigma}{2m^2}\frac{\partial^2 u}{\partial p^2}. \tag{5.2}$$

Solving the stationary Fokker-Planck (forward Kolmogorov) equation $L^*v(q,p)=0$, we find the stationary Boltzman distribution

$$v(q,p)=\frac{1}{Z}\exp\left\{-\frac{4\sqrt{\mu T}}{\sigma}\left(\frac{mp^2}{2}+U(q)\right)\right\}, \tag{5.3}$$

where Z is a normalizing constant,

$$Z=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{-\frac{4\sqrt{\mu T}}{\sigma}\left(\frac{mp^2}{2}+U(q)\right)\right\}\mathrm{d}p\,\mathrm{d}q.$$

Assume that the following condition (*) is satisfied: for some $Q_0>0$,

$$U'(q)\,\text{sign}\,q\geqslant\alpha>0 \quad \text{for } |q|>Q_0\,. \; (*)$$

This condition, in particular, provides finiteness of the normalizing constant Z.

Proposition 5.1: Let condition (*) *be satisfied. Then*

(i) *There exists a unique stationary solution* $(\bar{q}(t),\bar{p}(t))$ *of system* (5.1). *For any* t, *the* *distribution of* $(\bar{q}_t, \bar{p}_t)$ *is given by* (5.3).

(ii) *The solution* $(q(t), p(t))$ *of system* (5.1) *with any initial condition* $q(0)=q_0, p(0)=p_0$, *converges to* $(\bar{q}_t, \bar{p}_t)$ *as time tends to infinity, i.e.*

For every $A>0$, *the distribution of the random process* $(g(T+s), p(T+s)), 0 \leq s \leq A$, *in* $C_{0,A}$ *converges weakly to the distribution of* $(\bar{q}_s, \bar{p}_s)$ *as* $T \to \infty$.

Proof: Without loss of generality, we can assume, that $m=2\sqrt{\mu T}=\sigma=1$. Then the function

$$v(q,p) = \frac{p^2}{2} + (q - \gamma \arctan q)p + v(q) + \int_0^q (s - \gamma \arctan s)\mathrm{d}s + k$$

for a suitable choice of parameters γ, k satisfies the conditions (see Ref. 7, Sec. 3.5) $v(q,p) \geq 0, Lv(q,p) < -\alpha < 0$ for $|p|+|q|$ large enough, $\lim_{|p|+|q|\to\infty} v(q,p)=\infty$. Here L is defined in (5.2) and α is a positive constant from (*).

As we already mentioned, the Boltzman distribution is invariant for the equation (5.1). To prove uniqueness of the stationary distribution and convergence, one can use the standard construction which goes back to Ref. 7. Therefore we give just a sketch of the proof.

Let $\gamma_A, A>0$ be the boundary of the square $\{(q,p) \in \mathbb{R}^2 : |q| < A, |p| < A\}, \Gamma_A = \gamma_{A+1}$. Consider a Markov chain Z_n with the state space γ_A which is defined as follows: Starting from any point $Z_0=(q_0,p_0) \in \gamma_A$, the trajectory of the process $X_t=(q_t,p_t)$ hits at some time Γ_A and then comes back to γ_A. Let τ be the first time when X_t comes to γ_A after hitting Γ_A. Existence of the function $v(q,p)$ constructed above implies that the random variable τ is finite with probability 1, and the expected value $E\tau < \infty$, at least, if A is large enough. Set $Z_1 = X_\tau$, so that the chain Z_n in one step jumps from Z_0 to $Z_1=X_\tau$. The standard proof of uniqueness of the stationary distribution and convergence to it is given under nondegeneracy assumption of the operator L.

In our case L is degenerate. But it satisfies the Hörmander conditions, providing the existence of a positive density for X_t. Moreover, one can check that, due to the structure of the operator L, Doeblin conditions for the chain Z_n are satisfied (Ref. 2, Sec. 6.2). This implies that the chain Z_n on γ_A has a unique stationary distribution which is also the limiting distribution for Z_n.

The last property provides uniqueness of the stationary distribution for process (q_t,p_t) and convergence to this distribution (which is the Boltzman distribution) as $t\to\infty$. This implies existence and uniqueness of a stationary process $(\bar{q}_t, \bar{p}_t)$ satisfying equations (5.1) and the last statement of Proposition 5.1. The stationary process $(\bar{q}_t, \bar{p}_t)$ is the solution of (5.1) with the initial point distributed according to the Boltzman distribution.

Theorem 5.1: *Let* (*) *be satisfied*, *and the initial functions* (1.5) *have a compact support. Then*, *we have the following*:

(i) *There exists a unique random field* $\bar{u}(t,x)$, $t \in (-\infty, \infty)$, $x \in \mathbb{R}^1$, *such that* $\bar{u}(t,x)$ *is a solution of equation*

$$\mu \frac{\partial^2 \bar{u}}{\partial t^2} = T \frac{\partial^2 \bar{u}}{\partial x^2}, \quad t \in (-\infty, \infty), \quad x \in \mathbb{R}^1 \setminus \{0\}; \tag{5.4}$$

at $x=0$ *the gluing conditions* (1.2) and (1.3) *are satisfied. The distribution of* $\bar{u}(t,x)$ *is invariant with respect to time shifts*: *for any* $h \in \mathbb{R}^1$, $\bar{u}(t+h,x)$ *and* $\bar{u}(t,x)$ *have the same distribution in the space of continuous functions* $\varphi(t,x)$, $t \in (-\infty,\infty), x \in \mathbb{R}^1$.

(ii) *This unique solution is given by the formula* [cf. (4.9)]

$$\bar{u}(t,x) = \bar{q}\left(t - \frac{|x|}{a}\right), \quad (x,t) \in \mathbb{R}^2, \tag{5.5}$$

where $a=\sqrt{T/\mu}$.

(iii) *For every initial condition with a compact support*, *the solution* $u(t,x)$ *of problem* (1.1)–(1.3) *converges to the stationary solution* $\bar{u}(t,x)$.

For any $A>0$, *the random process* $\tilde{u}_T(t,x)=u(T+t,x), 0\leq t\leq A, |x|\leq A$, *converges weakly to* $\bar{u}(t,x)$ *as* $T\to\infty$ *in the space of continuous functions on* $[0,A]\times[|x|\leq A]$ *provided with uniform topology.*

Proof: As it follows from Proposition 5.1, if condition (*) is satisfied, a unique stationary solution $(\bar{q}_t,\bar{p}_t)$ of problem (5.1) exists. Then the function (5.5) satisfies equation (5.4) and the gluing conditions at $x=0$. Since the stochastic process $\bar{q}_t$ is invariant with respect to the time shifts, so is the function $\bar{u}(t,x)$. If another time shift invariant solution $\bar{\bar{u}}(t,x)$ exists, then $\bar{\bar{u}}(t,0)$ should coincide in distribution with $\bar{q}_t$, since problem (5.1) has a unique stationary solution. Convergence of the solution of problem (1.1)–(1.3) with compactly supported initial functions to $\bar{u}(t,x)$ follows by the formula (4.9) from the convergence of $q(t)$ to $\bar{q}_t$.

VI. LOW TEMPERATURE LIMIT

Let us note that the Boltzman equilibrium distribution (5.3) corresponds to the temperature proportional to

$$\frac{\sigma}{\sqrt{\mu T}}. \tag{6.1}$$

Here we discuss low temperature behavior in the Lamb system (1.1)–(1.3), when $\sigma\to 0$ for fixed μ and T.

If the potential $U(q)$ has more than one well, and $\sigma\ll 1$, the convergence to the stationary regime will be slow, and the system will go through a sequence of metastable regimes, where it spends a long time before approaching the stationary solution described above.

The sequence of metastable regimes depends on the equilibrium state of equations (5.1) with $\sigma=0$, to which the system was brought by the initial conditions (1.5). Since we assume that the initial conditions have a compact support, say, they are equal to zero for $|x|>\theta$, no incident waves come to the origin $x=0$ after time $t_0=\theta/a$. Set $u(t_0,0)=q_0^*$, $\dot{u}(t_0,0)=p_0^*$.

Assume that, for system (5.1) with $\sigma=0$, the initial point (q_0^*,p_0^*) is attracted to the stable equilibrium $O_{k(q_0^*,p_0^*)}$. The point $O_{k(q_0^*,p_0^*)}$ in the phase space $\mathbb{R}^2$ has coordinates $(q_{k(q_0^*,p_0^*)},0)$; $q_{k(q_0^*,p_0^*)}$ is a local minimum of the potential $U(q)$.

If $\sigma=0$, then the solution of system (5.1) with initial point (q_0^*,p_0^*) will stay near $O_{k(q_0^*,p_0^*)}$ forever. In the case $0<\sigma\ll 1$, the trajectory (q_t^σ,p_t^σ) of system (5.1) will stay in a neighborhood of $O_{k(q_0^*,p_0^*)}$ a time of order $\exp\{\text{const}/\sigma\}$, and then will switch to another equilibrium of system (5.1) with $\sigma=0$. It will stay there a long time and then again switches to the basin of another equilibrium and so on. It is important to underline that, in the generic case, for each stable equilibrium O_k, there exists exactly one (nonrandom) equilibrium $O_{k'}$, such that, with probability close to 1 for σ small enough, the system switches from O_k to $O_{k'}$. Since we assume that there are just a finite number of minima of $U(q)$, the sequence of transitions, after some time, becomes periodic (see Refs. 3 and 4). Thus we will have a decomposition of the set of local minima of $U(q)$ in cycles of rank 1. Moreover, the transition time $T_{k,k'}^\sigma$ between the basins of O_k and $O_{k'}$ is a random variable, but its logarithmic asymptotics as $\sigma\downarrow 0$ is not random.[3] In time scale larger than transition time in first rank cycles, transitions between the 1-cycles begin. So that in larger time scale, cycles of rank 2 appear, then cycles of rank 3, and so on, until all stable equilibriums of the nonperturbed system will be involved in the transitions.

The cycles of higher rank, as well as the logarithmic asymptotics of transition times between them, are also not random. So that one can speak on quasideterministic approximation for the long-time behavior of a dynamical system perturbed by a small noise. This hierarchy of cycles in a rather general situation was described in Refs. 3 and 4. The construction based on the large deviation theory for dynamical system perturbed by a small noise.[6]

Denote by $\mathcal{E}(O_k,E)$, $E\geq U(O_k)$, the connected component of the set $\{q\in\mathbb{R}^1 : U(q)\leq E\}$ con-

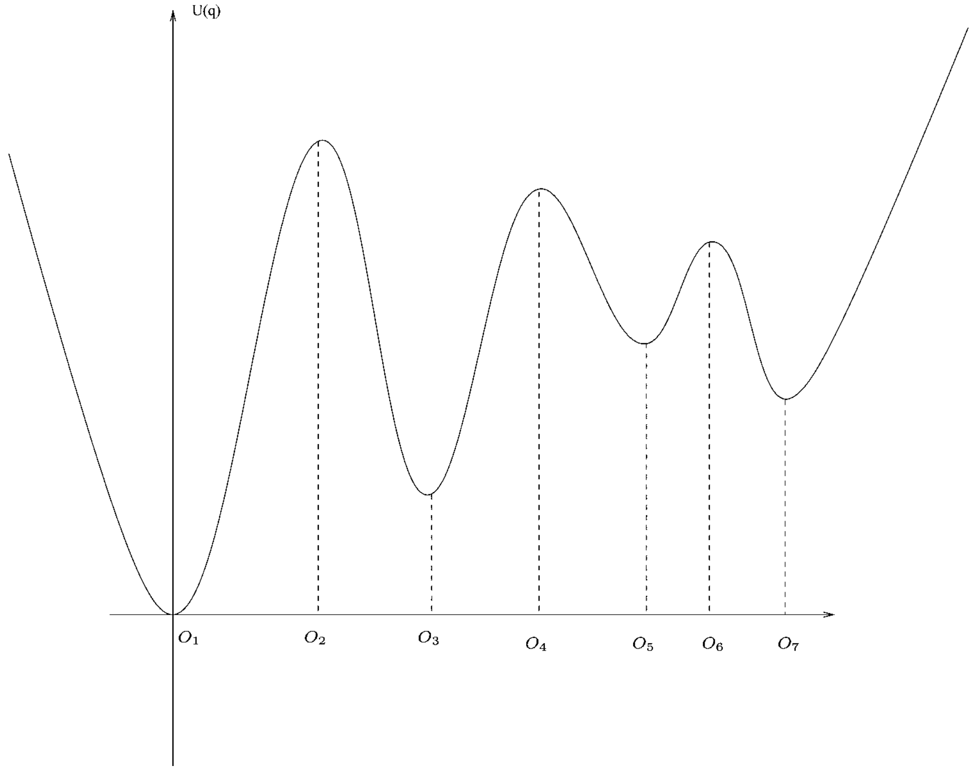

FIG. 2. Potential.

taining the stable equilibrium O_k. Let $O_{k^*(k,E)}$ be the point of $\mathcal{E}(O_k,E)$ such that $\min\{U(q):q \in \mathcal{E}(O_k,E)\}=U(O_{k^*(k,E)})$. We assume that the potential $U(q)$ is generic. Then the equilibrium $O_{k^*(k,E)}$ is defined in a unique way.

It is clear that for E close enough to $U(O_k)$, $O_{k^*(k,E)}=O_k$. If the potential is generic, and O_k is just a local, but not global, minimum of $U(q)$, then one can find E_1 such that $O_{k^*(k,E)}=O_k$ for $E \in (U(O_k),E_1)$, and $O_{k^*(k,E)} \neq O_k$ for $E>E_1$. In general, $E_1<E_2<\cdots<E_m$ exist such that $O_{k^*(k,E)} =O_{\bar{k}}$, $\bar{k}=\bar{k}(k,E_i,E_{i+1})=\text{const}$ for $E\in(E_i,E_{i+1})$, $i\in\{0,\ldots,m\}$, $E_0=U(O_k)$, $E_{m+1}=\infty$. Such an increasing sequence E_i can be defined for any stable equilibrium O_k. If $U(O_k)=\min\{U(q):q\in\mathbb{R}^1\}$, then set $E_1=\infty$.

For example, for the potential shown in Fig. 2 and $O_k=O_5$, $E_1=U(O_6)$, $E_2=U(O_4)$, $E_3 =U(O_2)$, and $E_4=\infty$; $\bar{k}(5,U(O_5),E_1)=5$, $\bar{k}(5,E_1,E_2)=7$, $\bar{k}(5,E_2,E_3)=3$, $\bar{k}(5,E_3,E_4)=1$.

Set $\lambda_1=E_1-U(O_k)$, $\lambda_2=E_2-U(O_{\bar{k}(E_1,E_2)}),\ldots,\lambda_l=E_l-U(O_{\bar{k}(k,E_{l-1},E_l)}),\ldots,\lambda_m=E_m -U(O_{\bar{k}(k,E_{m-1},E_m)})$. It is easy to see that $\lambda_1<\lambda_2<\cdots<\lambda_m$.

It follows from Refs. 6 and 4 that in the time scale $T^\sigma \asymp e^{\lambda/\sigma}$, $\lambda_l<\lambda<\lambda_{l+1}$ (here "$\asymp$" is the sign of logarithmic equivalency as $\sigma\downarrow 0$), trajectory (q_t,p_t) of system (5.1) starting at a point from the basin of O_k spends most of the time as $a\downarrow 0$ in a small neighborhood of $O_{\bar{k}(E_l,E_{l+1})}$: For each $\delta >0$, the random variable,

$$\eta_\sigma^\delta=\frac{1}{T^\sigma}\wedge\{t\in[0,T^\sigma]:d((q_t,p_t),O_{\bar{k}(k,E_lE_{l+1})})>\delta\},$$

where $\wedge\{\cdot\}$ stands for the Lebesgue measure in $\mathbb{R}^1$, and $d(\cdot,\cdot)$ is Euclidian distance in $\mathbb{R}^2$, tends to zero in probability as $\sigma\downarrow 0$.

This and formula (4.9) imply the following result.

Theorem 6.1: *Let condition* (*) *be satisfied and the initial functions* (1.5) *have a compact* *support*: $u_0(x)=v_0(x)=0$ *for* $|x|\geqslant R_0$. *Suppose the point* $(u(\theta/a,0),\dot{u}(\theta/a,0))$ *belongs to the basin of a stable equilibrium* O_k. *Let* $T^\sigma \asymp e^{-\lambda/\sigma}$, $\lambda\in(\lambda_l,\lambda_{l+1})$, $O_{\bar{k}(k,E_lE_{l+1})}=(q_{\bar{k}(k,E_l,E_{l+1})},0)$. *Then, for every* $A>0$,

$$\int_{-A}^{A}\mathrm{d}x\int_0^A |u(tT^\sigma,x)-q_{\bar{k}(k,E_l,E_{l+1})}|^2\,\mathrm{d}t$$

tends to zero in probability as $\sigma\downarrow 0$.

VII. LOW TEMPERATURE AND RADIATION LIMIT

Consider now the case

$$m\ll 1,\quad 2m^{-1}\sqrt{T\mu}\sim 1,\quad k=\frac{\sigma}{m}\ll 1. \tag{7.1}$$

Then the Eq. (4.10) can be written as follows:

$$\ddot{q}(t)=-\frac{1}{m}U'(q(t))-\dot{q}(t)+\sqrt{\frac{\kappa}{m}}\dot{W}_t. \tag{7.2}$$

Hence, the first two conditions in (7.1) mean that the Hamiltonian vector field is large with respect to the "radiative effects" described by the friction term in (7.2), so we can apply the averaging arguments. The last two conditions in (7.1) mean the low temperature limit $\sigma\to 0$, as above [see (6.1)].

In the preceding section, under certain conditions, we described metastable regimes of our system: For a given initial state and a time scale, the system spends most of the time near certain stationary state of the system without noise (zero temperature). Under conditions of this section, the metastable state, in general, is not a stationary state of the zero temperature system, but a certain distribution among such states. This distribution is determined by the initial conditions and the time scale.

To be specific, we assume that the potential has four minima. Then the Hamiltonian $H(p,q)=p^2/2m+U(q)$ has the wells as it is shown in Fig. 3. The case of general potential can be treated similarly.

The level set $C(z)=\{(q,p):H(q,p)=z\}$ consists, in general of several connected components $C_k(z):C(z)=\cup_{k=1}^N C_k(z)$. We denote by $G_k(z)$ the domain bounded by $C_k(z)$ (compare with Ref. 6, Chap. 8). Let Γ be the graph homeomorphic to the set of connected components of all level sets of the Hamiltonian $H(q,p)$ [Fig. 3(b)] provided with natural topology.

The connected component of the level set of a saddle point O_2, containing O_2, is an eight-shaped curve γ [Fig. 3(c) and 3(d)] consisting of two parts G_1 and G_2.

Equation (7.2) can be written as the system

$$\begin{aligned}\dot{q}^{m,\kappa}(t)&=\frac{1}{\sqrt{m}}p^{m,\kappa}(t),\\ \dot{p}^{m,\kappa}(t)&=-\frac{1}{\sqrt{m}}U'(q^{m,\kappa}(t))-p^{m,\kappa}(t)+\sqrt{\kappa}\dot{W}(t).\end{aligned} \tag{7.3}$$

Let, first, $m\to 0$, then we are in the situation when the averaging principle should be applied. The fast component of the process $(q^{m,\kappa}(t),p^{m,\kappa}(t))$ is, roughly speaking, the motion along the trajectories of the Hamiltonian system with

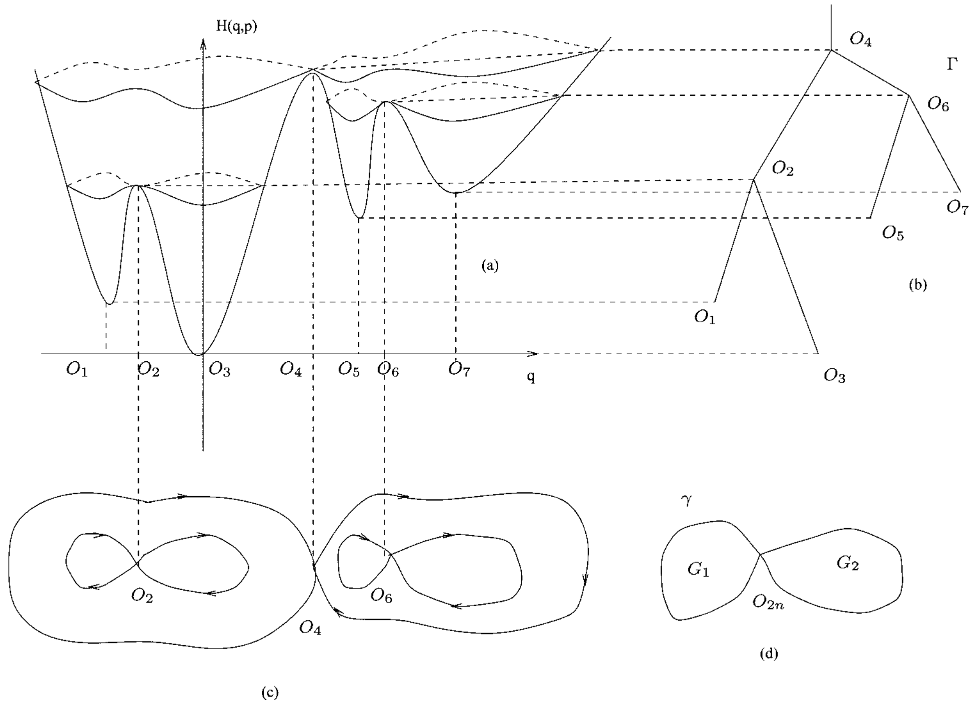

FIG. 3. Hamiltonian, graphs and level sets.

$$H = \frac{p^2}{2} + U(q),$$

and the slow component is the projection $Y(q^{m,\kappa}(t), p^{m,\kappa}(t))$ on the graph Γ: $Y(x)$ is the point of the graph corresponding to the connected component of $H(x)$-level set containing the point $x \in \mathbb{R}^2$.[5,6] It is shown in Ref. 5 that $Y(q^{m,\kappa}(t),\, p^{m,\kappa}(t))$ converges weakly (in the space of continuous functions on any finite time interval $[0,M]$ with the values in Γ) to a diffusion process $Y^\kappa(t)$ on Γ. The process $Y^\kappa(t)$ is defined by the family of second order operators L_k^κ, one on each edge of the graph, and by gluing conditions at the vertices.

The operator L_k^κ on the edge I_k has the form

$$L_k^\kappa(q,p)f(z) = \frac{\kappa}{2T_k(z)} \frac{d}{\mathrm{d}z}\left(\bar{a}_k(z) \frac{\mathrm{d}f}{\mathrm{d}z} \right) - \frac{\bar{\beta}_k(z)}{T_k(z)} \frac{\mathrm{d}f}{\mathrm{d}z},$$

where $T_k(z)$ is the period of rotation along the level set component $C_k(z)$ corresponding to the point (z,k) of the graph: k is the number of the edge containing this point, and z is the corresponding value of the Hamiltonian $H(q,p)$ on the level set of component corresponding to the point of Γ.

Further, $T_k(z) = S'(z)$, where $S_k(z)$ is the area of the domain $G_k(z)$ bounded by $C_k(z)$, and

$$\bar{a}_k(z) = \bar{\beta}_k(z) = S_k(z).$$

To define the process $Y^\kappa(t)$ on Γ in a unique way one should add gluing conditions at the vertices. These conditions were calculated in Ref. 5, but we do not need their form, so we will not describe them here.

Now we want to take κ to zero in the process $Y^\kappa(t)$ on Γ. Then (see Ref. 1) $Y^\kappa(t)$ converges weakly as $\kappa \to 0$ to a process $Y(t)$ on Γ, which has the following structure. Inside an edge $I_k \subset \Gamma$, this is a nonrandom motion with the speed $-\bar{\beta}_k(z)/T_k(z) = -S_k(z)/S_k'(z)$. When a trajectory comes to a vertex O corresponding to a saddle point (vertices O_2, O_4, and O_6 in Fig. 3), it proceeds without any delay at O in one of the edges below (in energy) of O with certain probabilities $P_1(O)$ and $P_2(O)$, $P_1(O)+P_2(O)=1$. To find these probabilities, one should consider the eight-shaped curve $\gamma(O)$ corresponding to the vertex O [see Fig. 3(d)]. It has two components $G_1(O)$ and $G_2(O)$. Then

$$P_i(O) = \frac{S(G_i)}{S(G_1) + S(G_2)}, \quad i = 1,2,$$

$S(G_i)$ being the area of G_i.

Assume that at time t_0 the oscillator has the energy H_0 greater than the level set of the highest saddle point [$H_0 > H(O_4)$ in Fig. 3]. Then because of "friction" (=radiation) it will lose the energy until it comes to the level $H(O_4)$; this will happen in a finite time. Then trajectory goes to the left (to the edge $[O_2, O_4]$) with probability

$$P_1(O_4) = \frac{S(G_1(O_4))}{S(G_1(O_4)) + S(G_2(O_4))},$$

and to the right (to $[O_4, O_6]$) with probability

$$P_2(O_4) = \frac{S(G_2(O_4))}{S(G_1(O_4)) + S(G_2(O_4))}.$$

Trajectory $Y(t)$ proceeds to go down until it meets the next saddle point (O_2 or O_6). It is scattered on those saddle points and eventually approaches one of the local minima of the potential. In a finite time for every $\delta > 0$ it approaches the δ neighborhood of one of the local minima and stays in the corresponding well a time of order $e^{C/\kappa}$, where

$$C = \min\{U(O_2) - U(O_1), U(O_2) - U(O_3), U(O_6) - U(O_5), U(O_6) - U(O_7)\}.$$

Thus if we observe $(q_t^{m,\kappa}, p_t^{m,\kappa})$ on the time interval $1 \ll t < T_E^\kappa = e^{E/\kappa}$ with $0 < E < C$, it is distributed among the local minima O_1, O_3, O_5, O_7 with probabilities, respectively, equal to

$$\begin{aligned} m_1 &= \frac{S(G_1(O_4))}{S(G_1(O_4)) + S(G_2(O_4))} \cdot \frac{S(G_1(O_2))}{S(G_1(O_2)) + S(G_2(O_2))}, \\ m_3 &= \frac{S(G_1(O_4))}{S(G_1(O_4)) + S(G_2(O_4))} \cdot \frac{S(G_2(O_2))}{S(G_1(O_2)) + S(G_2(O_2))}, \\ m_5 &= \frac{S(G_2(O_4))}{S(G_1(O_4)) + S(G_2(O_4))} \cdot \frac{S(G_1(O_6))}{S(G_1(O_6)) + S(G_2(O_6))}, \\ m_7 &= \frac{S(G_2(O_4))}{S(G_1(O_4)) + S(G_2(O_4))} \cdot \frac{S(G_2(O_6))}{S(G_1(O_6)) + S(G_2(O_6))}, \end{aligned} \tag{7.4}$$

if, first, $m \downarrow 0$ and then $\kappa \downarrow 0$. This is metastable distribution in the time scale $T_E^\kappa = e^{E/\kappa}$ for $E < C$. In larger time scales, the support of this limiting distribution will become smaller and smaller. To be specific, assume that

$$U(O_6) - U(O_7) < U(O_6) - U(O_5) < U(O_4) - U(O_5) < U(O_2) - U(O_1) < U(O_2) - U(O_3). \tag{7.5}$$

Then if $U(O_6) - U(O_7) < E < U(O_6) - U(O_5)$, trajectory already have enough time to leave O_7 so that the metastable distribution among O_1, O_3, O_5, O_7 in this time scale is $(m_1, m_3, m_5 + m_7, 0)$. If $E \in (U(O_4) - U(O_5), U(O_2) - U(O_1))$, the metastable distribution is

$$\left(\frac{S(G_1(O_2))}{S(G_1(O_2))+S(G_2(O_2))},\frac{S(G_2(O_2))}{S(G_1(O_2))+S(G_2(O_2))},0,0\right).$$

Eventually, if $E>U(O_2)-U(O_1)$, then the distribution is concentrated at point O_3, which is the absolute minimum of the potential.

Together with the equality $u(t,x)=u^{m,\kappa}(t,x)=q^{m,\kappa}(t-|x|/a)$, $a=T/m$, which holds for any $x\in\mathbb{R}^1$ and t large enough, this implies the following result.

Theorem 7.1: Let condition (*) be satisfied, and initial functions (1.5) have a compact support belonging to $\{|x|<\theta\}\subset\mathbb{R}^1$. Let the Hamiltonian $H(p,q)=p^2/2+U(q)$ be as shown in Fig. 3. Let the energy of the oscillator with $\kappa=0$ be greater than $U(O_4)$ at time $t_0=\theta/a$:

$$\frac{m}{2}\left(\frac{\partial u^{m,0}(t_0,0)}{\partial t}\right)^2+U(u^{m,0}(t_0,0))>U(O_4).$$

Assume that inequalities (7.5) are satisfied. Then for any A, $E>0$ and $T_E^\kappa\asymp\exp\{E/\kappa\}$, the random function $u^{m,\kappa}(T_E^\kappa t,x)$, $t\in[0,A]$, $x\in[-A,A]$, converges weakly in $L^2_{[0,A]\times[-A,A]}$ to a random variable η_E as, first, $m\downarrow 0$ and then $\kappa\downarrow 0$.

The random variable η_E has values O_1,O_3,O_5,O_7 with probabilities m_1,m_3,m_5,m_7, respectively, if $0<E<C$, with probabilities $(m_1,m_3,m_5+m_7,0)$ if $E\in(U(O_6)-U(O_7),U(O_6),-U(O_5))$, with probabilities $m_1+m_3,m_5+m_7,0,0)$ if $E\in(U(O_4)-U(O_5),U(O_2)-U(O_1))$, and $P\{\eta_E=O_3\}=1$ if $E>U(O_2)-U(O_1)$.

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