Scattering of Solitons for Coupled Wave-Particle Equations

Running head: Soliton Scattering for Wave-Particle Eqs

Valery Imaykin [1]

Faculty of Mathematics
TU Munich, Garching, 85747 Germany

Alexander Komech[1 2]

Faculty of Mathematics of Vienna University and
Institute for Information Transmission Problems RAS.

Boris Vainberg [3]

Department of Mathematics and Statistics,
UNC at Charlotte, Charlotte, NC 28223, USA

Abstract

We establish a long time soliton asymptotics for a nonlinear system of wave equation coupled to a charged particle. The coupled system has a six dimensional manifold of soliton solutions. We show that in the large time approximation, any solution, with an initial state close to the solitary manifold, is a sum of a soliton and a dispersive wave which is a solution to the free wave equation. It is assumed that the charge density satisfies Wiener condition which is a version of Fermi Golden Rule, and that the momenta of the charge distribution vanish up to the fourth order. The proof is based on a development of the general strategy introduced by Buslaev and Perelman: symplectic projection in Hilbert space onto the solitary manifold, modulation equations for the parameters of the projection, and decay of the transversal component.

AMS subject classification: 35Q51, 35Q70, 37K40

[1]Supported partially by Austrian Science Foundation (FWF), by research grants of DFG and RFBR.
[2]Supported by the Alexander von Humboldt Research Award.
[3]Supported partially by NSF grant DMS-0706928.

1 Introduction

Our paper concerns an old mathematical problem of nonlinear field-particle interaction. A charged particle radiates a field which acts on the particle etc. This self-action is probably responsible for some crucial features of the process: asymptotically uniform motion of the particle, increment of the particle's mass etc. (see [36], Part I). The problem has many different appearances: for a classical particle coupled to a scalar or Maxwell field, for coupled Maxwell-Schrödinger or Maxwell-Dirac equations, for the corresponding second-quantized equations etc.

One of the main goals of mathematical investigation of this problem is studying soliton-type long time asymptotics and asymptotic stability of soliton solutions. The first results in this direction have been discovered for KdV equation and other *complete integrable equations*. For KdV equation, any solution with sufficiently smooth and rapidly decaying initial data converges to a finite sum of solitons moving to the right, and a dispersive wave moving to the left. A complete survey and proofs can be found in [11].

For non-integrable equations, the long time convergence of the solution to a soliton part and dispersive wave was obtained first by Soffer and Weinstein in the context of $U(1)$-invariant Schrödinger equation with potential, [31, 32, 33]. The extension to translation invariant equations was obtained by Buslaev and Perelman [4, 5] for 1D Schrödinger equation, and by Miller, Pego and Weinstein for 1D modified KdV and RLW equations, [26, 27, 28]. Later the results of Pego and Weinstein were developed by Martel and Merle [25]. The problem initiated by Soffer, Weinstein and Buslaev, Perelman is almost completely solved by Cuccagna in [9], where the proof works in all dimensions. Bambusi and Cuccagna [2] have solved completely the problem initiated in [33]

In [4, 5] the long time convergence is obtained for translation invariant and $U(1)$-invariant nonlinear Schrödinger equation: for any finite-energy solution $\psi(x,t)$ with initial data close to a soliton $\psi_{v_0}(x - v_0 t - a_0)e^{i\omega_0 t}$, the following asymptotics hold:

$$\psi(x,t) = \psi_{v_\pm}(x - v_\pm t - a_\pm)e^{i\omega_\pm t} + W_0(t)\psi_\pm + r_\pm(x,t), \;\; t \to \pm\infty. \tag{1.1}$$

Here the first term of the right hand side is a soliton with parameters $v_\pm$, $a_\pm$, $\omega_\pm$ close to v_0, a_0, ω_0, the function $W_0(t)\psi_\pm$ is a dispersive wave which is a solution to the free Schrödinger equation, and the remainder $r_\pm(x,t)$ converges to zero in the global L^2-norm. Recently Cuccagna has extended the results to nD Schrödinger equations with $n \geq 3$, [7, 8].

In the present paper we consider a scalar real-valued wave field $\psi(x)$ in $\mathbb{R}^3$, coupled to a relativistic particle with position q and momentum p, governed by

$$\begin{aligned} &\dot\psi(x,t) = \pi(x,t), \qquad \dot\pi(x,t) = \Delta\psi(x,t) - \rho(x - q(t)), \quad x \in \mathbb{R}^3, \\ &\dot q(t) = p(t)/\sqrt{1+p^2}, \quad \dot p(t) = \int \psi(x,t)\, \nabla\rho(x - q(t))dx. \end{aligned} \tag{1.2}$$

This is a Hamilton system with the Hamilton functional

$$\mathcal{H}(\psi,\pi,q,p) = \frac{1}{2}\int \Big(|\pi(x)|^2 + |\nabla\psi(x)|^2\Big)dx + \int \psi(x)\rho(x-q)dx + \sqrt{1+p^2}. \tag{1.3}$$

The first two equations for the fields are equivalent to the wave equation with the source $\rho(x-q)$. We have set the mechanical mass of the particle and the speed of wave propagation equal to one. The case of the point particle corresponds to $\rho(x) = \delta(x)$ and then the interaction term in the Hamiltonian is simply $\psi(q)$. However, in this case the Hamiltonian is unbounded from below which leads to the ill-posedness of the problem, that is also known as ultraviolet divergence. Therefore we smooth the coupling by the function $\rho(x)$ following the "extended electron" strategy proposed by M. Abraham for the Maxwell field. In analogy to the Maxwell-Lorentz equations we call ρ the "charge distribution". Finally, the form of the last two equations in (1.2) is determined by the choice of the relativistic kinetic energy $\sqrt{1+p^2}$ in (1.3).

Let us write the system (1.2) as

$$\dot Y(t) = F(Y(t)), \qquad t \in \mathbb{R}, \tag{1.4}$$

where $Y(t) := (\psi(x,t), \pi(x,t), q(t), p(t))$. The system (1.2) is translation-invariant and admits soliton solutions

$$Y_{a,v}(t) = (\psi_v(x - vt - a), \pi_v(x - vt - a), vt + a, p_v), \qquad p_v = v/\sqrt{1-v^2} \tag{1.5}$$

for all $a, v \in \mathbb{R}^3$ with $|v| < 1$ (see (2.8) below). The states $S_{a,v} := Y_{a,v}(0)$ form the solitary manifold

$$\mathcal{S} := \{S_{a,v} : a, v \in \mathbb{R}^3, |v| < 1\}. \tag{1.6}$$

Our main result (announced in [18]) is the soliton asymptotics of type (1.1),

$$(\psi(x,t), \pi(x,t)) \sim (\psi_{v_\pm}(x - v_\pm t - a_\pm), \pi_{v_\pm}(x - v_\pm t - a_\pm)) + W_0(t)\mathbf{\Psi}_\pm, \qquad t \to \pm\infty, \tag{1.7}$$

for solutions to (1.2) with initial data close to the solitary manifold $\mathcal{S}$. Here $W_0(t)$ is the dynamical group of the free wave equation, $\mathbf{\Psi}_\pm$ are the corresponding *asymptotic scattering states*, and the remainder converges to zero *in the global energy norm*, i.e. in the norm of the Sobolev space $\dot{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, see Section 2. For the particle's trajectory we prove that

$$\dot{q}(t) \to v_\pm, \quad q(t) \sim v_\pm t + a_\pm, \qquad t \to \pm\infty. \tag{1.8}$$

The results are established under the following conditions on the charge distribution: ρ is a real-valued function of the Sobolev class $H^2(\mathbb{R}^3)$, compactly supported, and spherically symmetric, i.e.

$$\rho, \nabla\rho, \nabla\nabla\rho \in L^2(\mathbb{R}^3), \qquad \rho(x) = 0 \text{ for } |x| \geq R_\rho, \qquad \rho(x) = \rho_1(|x|). \tag{1.9}$$

We require that all "nonzero modes" of the wave field are coupled to the particle. This is formalized by the Wiener condition

$$\hat{\rho}(k) = (2\pi)^{-3/2} \int e^{ikx}\rho(x)dx \neq 0 \quad \text{for all} \quad k \in \mathbb{R}^3 \setminus \{0\}. \tag{1.10}$$

It is the nonlinear Fermi Golden Rule for our model: the coupling term $\rho(x - q)$ is not orthogonal to the eigenfunctions e^{ikx} of the continuous spectrum of the linear part of the equation (cf. [2, 6, 9, 10, 30, 34]). Note that the Wiener condition is close to the linear version of the FGR [29, pp. 67-68]. As we will see, the Wiener condition (1.10) is very essential for our asymptotic analysis. Generic examples of the coupling function ρ satisfying (1.9) and (1.10) are given in [20]. In particular, the Wiener condition allows us to identify the discrete spectral subspace corresponding to the spectral point $\lambda = 0$ of the linearized system. Thus, we do not impose any implicit spectral conditions for the linearized system. Further, we will assume that $\rho(k)$ has a fifth order zero at the point $k = 0$, i.e.

$$\hat{\rho}^{\,(\alpha)}(0) = 0 \text{ for all multiindeces } \alpha \text{ with } |\alpha| \leq 4. \tag{1.11}$$

Equivalently, the following momenta vanish:

$$\int x^\alpha \rho(x)dx = 0, \quad |\alpha| \leq 4, \tag{1.12}$$

in particular, the total charge of the particle is zero (neutrality condition). It is easy to obtain an example of ρ satisfying both (1.10) and (1.11). Indeed, take a ρ_2 satisfying the Wiener condition (1.10). Then $\rho = \Delta^3\rho_2$ satisfies both (1.10) and (1.11).

The system (1.2) describes the charged particle interacting with its "own" scalar field. The asymptotics (1.7)-(1.8) mean asymtotic stability of uniform motion, i.e. "the law of inertia". The stability is caused by "radiative damping", i.e. radiation of energy to infinity appearing analytically as a local energy decay for solutions to the linearized equation provided by the Wiener condition (1.9). The radiative damping was suggested first by M.Abraham in 1905 in the context of Classical Electrodynamics, [1]. However, the asymptotics (1.7)-(1.8) are not proved yet for the Maxwell-Lorentz equations though close results are establihsed in [22] and [23].

One could also expect asymptotics (1.7) for small perturbations of the solitons for the relativistic nonlinear wave equations and for the coupled nonlinear Maxwell-Dirac equations whose solitons were constructed in [3] and [12] respectively. Our result is a model of this situation though the relativistic case is still open problem.

Let us briefly comment on earlier results. In the case of weak coupling, i.e. $\|\rho\|_{L^2} \ll 1$, scattering behavior of type (1.7) for the system (1.2) is established in [15] for all finite energy solutions. In [13, 14, 16] the result was extended to the cases of Klein-Gordon field, Maxwell field, and spinning charge subject to Maxwell field respectively. The results under the Wiener condition are not established yet.

The system (1.2) under the Wiener condition was considered in [20, 21, 19]. In [20] the convergence to stationary states is proved for all finite energy solutions: in particular, the relaxation of accelleration $\ddot{q} \to 0$ as

$t\to 0$ holds. In [21] the soliton-type asymptotics for the fields is established for all finite energy solutions. In [19] the effective dynamics under the slowly varying potential is constructed for solutions sufficiently close to the solitary manifold. However, the asymptotics (1.7) and (1.8) are not established in [20, 21, 19].

Let us comment on our techniques. For the proof we develop the general strategy of [17], where similar results were obtained for Klein-Gordon equation without the conditions (1.11): symplectic projection, modulation equations, and the time decay for the linearized dynamics in the transversal directions, see Introduction in [17] for details. However, the case of the wave equation requires novel techniques. Namely, in the case of Klein-Gordon one had very fast (exponential) spatial decay of solitons. The initial data had to be close to the solitary manifold in the weighted Sobolev norm with the weight $(1+|x|)^\beta$, $\beta>3/2$, and the rate of convergence was

$$\dot q(t)=v_\pm+\mathcal{O}(t^{-2}),\;\; q(t)=v_\pm t+a_\pm+\mathcal{O}(t^{-3/2}),$$
$$(\psi(x,t),\pi(x,t))=(\psi_{v_\pm}(x-v_\pm t-a_\pm),\pi_{v_\pm}(x-v_\pm t-a_\pm))+W^0(t)\mathbf{\Psi}_\pm+\mathcal{O}(t^{-1/2}).$$

In the case of the wave equation the corresponding solitons and tangent vectors have slow spatial decay. This is related to well known features of the Coulomb potential. That is why, first, we have to impose the condition (1.11) to have the symplectic projection well defined on the solutions and to obtain some important estimates below. Second, the initial data have to be close to the solitary manifold in a stronger weighted norm with the weight $(1+|x|)^{4+\delta}$, $0<\delta<1/2$, and the rate of convergence is now as follows:

$$\dot q(t)=v_\pm+\mathcal{O}(t^{-1-\delta}),\;\; q(t)=v_\pm t+a_\pm+\mathcal{O}(t^{-2\delta}),$$
$$(\psi(x,t),\pi(x,t))=(\psi_{v_\pm}(x-v_\pm t-a_\pm),\pi_{v_\pm}(x-v_\pm t-a_\pm))+W^0(t)\mathbf{\Psi}_\pm+\mathcal{O}(t^{-\delta}).$$

The main novelty in the case of the wave equation is thorough establishing the appropriate decay of the linearized dynamics which requires the two following novel robust ideas.

I. Unlike Klein-Gordon case, for the wave equation the left and the right edge points of continuouos spectrum meet at the zero point which is moreover, the point of discrete spectrum. In other words, the spectral gap is absent. This situation never happens in all previous works on the asymptotic stability of the solitary waves for the Schrödinger and Klein-Gordon equations.

Thus, the symplectic orthogonality condition is imposed now at the interior point of the continuous spectrum in contrast to all previous works in the field. Respectively, the integrand at this point is not smooth even if the symplectic orthogonality condition holds. Hence, the integration by parts in the oscillatory integral as in the case of Klein-Gordon equation, is impossible, and we develop in Propositions 7.10 and 7.11 new more subtle technique of the convolutions.

II. This situation requires to know the exact structure (7.43) of the resolvent for the linearized dynamics, and not only its residues. In particular, we calculate the spectrum of linearized equation. In previous works in the field, a variety of the spectral assumptions were introduced but examples are mostly absent.

Our paper is organized as follows. In Section 2, we formulate the main result. In Section 3, we introduce the symplectic projection onto the solitary manifold. The linearized equation is defined and studied in Sections 4-5. In Section 6, we split the dynamics in two components: along the solitary manifold, and in transversal directions, and we justify the estimate concerning the tangential component. Section 7 concerns the time decay of the linearized dynamics. The time decay of the transversal component is established in Sections 8-11 under an assumption on the time decay of the linearized dynamics. In Section 12, we prove the main result.

Acknowledgements The authors thank V. Buslaev for detailed and numerous lectures on his results, H. Spohn and E. Kopylova for fruitful discussions.

2 Main Results

2.1 Existence of Dynamics

To formulate our results precisely, we need some definitions. We introduce a suitable phase space for the Cauchy problem corresponding to (1.2). Let L^2 be the real Hilbert space $L^2(\mathbb{R}^3)$ with the scalar product $\langle\cdot,\cdot\rangle$ and the corresponding norm $|\cdot|$, and let $\dot H^1$ be the completion of the real space $C_0^\infty(\mathbb{R}^3)$ with the norm $|\nabla\psi(x)|$. Equivalently, using Sobolev embedding theorem, $\dot H^1=\{\psi(x)\in L^6(\mathbb{R}^3):\;\; |\nabla\psi(x)|\in L^2\}$. Let us introduce the weighted Sobolev spaces L^2_α and H^1_α with the norms $|\psi|_\alpha=|(1+|x|)^\alpha\psi|$ and $\|\psi\|_{1,\alpha}=|\psi|_\alpha+|\nabla\psi|_\alpha$ respectively.

Definition 2.1 *i) $\mathcal{E}$ is the Hilbert space $\dot H^1 \oplus L^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with the finite norm*

$$\| Y \|_{\mathcal{E}} = |\nabla\psi| + |\pi| + |q| + |p| \quad \textit{for} \quad Y = (\psi, \pi, q, p)\,.$$

ii) $\mathcal{E}_\alpha$ is the space $H^1_\alpha \oplus L^2_{\alpha+1} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with the norm

$$\|Y\|_\alpha = \| Y\|_{\mathcal{E}_\alpha} = \|\psi\|_{1,\alpha} + |\pi|_{\alpha+1} + |q| + |p|. \tag{2.1}$$

iii) $\mathcal{F}$ is the space $\dot H^1 \oplus L^2$ of fields $F = (\psi, \pi)$ with the finite norm

$$\|F\|_{\mathcal{F}} = |\nabla\psi| + |\pi|. \tag{2.2}$$

iv) $\mathcal{F}_\alpha$ is the space $H^1_\alpha \oplus L^2_{\alpha+1}$ with the norm

$$\| F\|_\alpha = \| F\|_{\mathcal{F}_\alpha} = \|\psi\|_{1,\alpha} + |\pi|_{\alpha+1} \tag{2.3}$$

Note that $\dot H^1$ is not contained in L^2 and for instance $|\psi_v| = \infty$ if the neutrality condition is not imposed, see (2.11) below. However, $\mathcal{E}$ is the space of finite energy states (i.e. $\mathcal{H}(Y) < \infty$ for $Y \in \mathcal{E}$) due to the following estimates which are valid for an arbitrary smooth $\psi(x)$ vanishing at infinity

$$-\frac{1}{8\pi}\int\!\!\int dx\, dy\, \frac{\rho(x)\rho(y)}{|x-y|} = \frac{1}{2}\langle\rho, \Delta^{-1}\rho\rangle \le \frac{1}{2}|\nabla\psi|^2 + \langle\psi(x), \rho(x-q)\rangle \le \ |\nabla\psi|^2 - \frac{1}{2}\langle\rho, \Delta^{-1}\rho\rangle. \tag{2.4}$$

The Hamilton functional $\mathcal{H}$ is continuous in the space $\mathcal{E}$, and the lower bound in (2.4) implies that the energy (1.3) is bounded from below. Note that the latter is not true if ρ is delta-function.

We consider the Cauchy problem for the Hamilton system (1.2) which we write as

$$\dot Y(t) = F(Y(t)), \quad t \in \mathbb{R}; \quad Y(0) = Y_0. \tag{2.5}$$

Here $Y(t) = (\psi(t), \pi(t), q(t), p(t))$, $Y_0 = (\psi_0, \pi_0, q_0, p_0)$, and all derivatives are understood in the sense of distributions.

Proposition 2.2 [20] *Let (1.9) hold. Then*

(i) For every $Y_0 \in \mathcal{E}$, the Cauchy problem (2.5) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$.

(ii) For every $t \in \mathbb{R}$, the map $U(t) : Y_0 \mapsto Y(t)$ is continuous in $\mathcal{E}$.

(iii) The energy is conserved, i.e.

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y_0), \quad t \in \mathbb{R}, \tag{2.6}$$

and the velocity is bounded:

$$|\dot q(t)| \le \overline{v} < 1, \qquad t \in \mathbb{R}, \tag{2.7}$$

with some $\overline{v}$ which depends on Y_0.

2.2 Solitary Manifold and Main Result

Let us compute the solitons (1.5). The substitution to (1.2) gives the following stationary equations,

$$\left.\begin{array}{rclrcl} -v\cdot\nabla\psi_v(y) &=& \pi_v(y)\ , & -v\cdot\nabla\pi_v(y) &=& \Delta\psi_v(y) - \rho(y) \\ v &=& \dfrac{p_v}{\sqrt{1+p_v^2}}\ , & 0 &=& -\displaystyle\int \nabla\psi_v(y)\rho(y)\,dy \end{array}\right| \tag{2.8}$$

Then the first two equations imply

$$\Lambda_v\psi_v(y) := (-\Delta + (v\cdot\nabla)^2)\psi_v(y) = -\rho(y), \qquad y \in \mathbb{R}^3. \tag{2.9}$$

For $|v| < 1$ the equation (2.9) defines a unique function $\psi_v \in \dot H^1(\mathbb{R}^3)$. If v is given and $|v| < 1$, then p_v can be found from the third equation of (2.8). Further, functions ρ and ψ_v are even due to (1.9). Thus, $\nabla\psi_v$ is odd and the last equation of (2.8) holds. Hence, the soliton solution (1.5) exists and is defined uniquely for any couple (a, v) with $|v| < 1$.

The soliton can be computed by the Fourier transform $\hat\psi(k):=(2\pi)^{-3/2}\displaystyle\int e^{ikx}\psi(x)dx$:

$$\hat\psi_v(k)=-\frac{\hat\rho(k)}{k^2-(kv)^2},\qquad \hat\pi_v(k)=-\frac{ikv\hat\rho(k)}{k^2-(kv)^2} \tag{2.10}$$

In the coordinate space

$$\psi_v(x)=-\frac{1}{4\pi}\int\frac{\rho(y)d^3y}{|\gamma(y-x)_{\|}+(y-x)_{\perp}|},\ \ \pi_v(x)=-v\cdot\nabla\psi_v(x),\ \ p_v=\gamma v. \tag{2.11}$$

Here we set $\gamma=1/\sqrt{1-v^2}$ and $x=x_{\|}+x_{\perp}$, where $x_{\|}\|v$ and $x_{\perp}\perp v$ for $x\in\mathbb{R}^3$. From the condition (1.12) it follows that

$$\psi_v(y)\sim|y|^{-6},\ \ \pi_v(y)\sim|y|^{-7}\ \text{ as }\ |y|\to\infty$$

and thus,

$$\psi_v\in H^1_\alpha,\ \ \alpha<9/2;\ \ \pi_v\in L^2_\alpha,\ \ \alpha<11/2. \tag{2.12}$$

Definition 2.3 *A soliton state is* $S(\sigma):=(\psi_v(x-b),\pi_v(x-b),b,p_v)$, *where* $\sigma:=(b,v)$ *with* $b\in\mathbb{R}^3$ *and* $|v|<1$.

By (2.12) for the soliton states we have

$$S(\sigma)\in\mathcal{E}_\alpha,\ \ \alpha<\frac{9}{2}. \tag{2.13}$$

Obviously, the soliton solution (1.5) admits the representation $Y_{a,v}(t)=S(\sigma(t))$, where

$$\sigma(t)=(b(t),v(t))=(vt+a,v). \tag{2.14}$$

Definition 2.4 *The solitary manifold is the set* $\mathcal{S}:=\{S(\sigma):b\in\mathbb{R}^3,|v|<1\}$.

The main result of our paper is the following theorem.

Theorem 2.5 *Let (1.9), the Wiener condition (1.10), and the condition (1.11) hold. Let* $0<\delta<1/2$, *set* $\beta:=4+\delta$. *Consider the solution* $Y(t)$ *to the Cauchy problem (2.5) with the initial state* Y_0 *which is sufficiently close to the solitary manifold:*

$$Y_0=S(\sigma_0)+Z_0,\ \ d_\beta:=\|Z_0\|_\beta\ll 1. \tag{2.15}$$

Then the asymptotics hold for $t\to\pm\infty$,

$$\dot q(t)=v_\pm+\mathcal{O}(|t|^{-1-\delta}),\qquad q(t)=v_\pm t+a_\pm+\mathcal{O}(|t|^{-2\delta}), \tag{2.16}$$

$$(\psi(x,t),\pi(x,t))=(\psi_{v_\pm}(x-v_\pm t-a_\pm),\pi_{v_\pm}(x-v_\pm t-a_\pm))+W_0(t)\mathbf{\Psi}_\pm+r_\pm(x,t) \tag{2.17}$$

with

$$\|r_\pm(t)\|_{\mathcal{F}}=\mathcal{O}(|t|^{-\delta}). \tag{2.18}$$

It suffices to prove the asymptotics (2.17), (2.16) for $t\to+\infty$ since the system (1.2) is time reversible.

3 Symplectic Projection

3.1 Symplectic Structure and Hamilton Form

The system (1.2) reads as the Hamilton system

$$\dot Y=J\mathcal{D}\mathcal{H}(Y),\ \ J:=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&I_3\\0&0&-I_3&0\end{pmatrix},\ \ Y=(\psi,\pi,q,p)\in\mathcal{E}, \tag{3.1}$$

where $\mathcal{DH}$ is the Fréchet derivative of the Hamilton functional (1.3), I_3 is the 3×3 identity matrix. Let us identify the tangent space to $\mathcal{E}$, at every point, with $\mathcal{E}$. Consider the bilinear form Ω defined on $\mathcal{E}$ by $\Omega = \displaystyle\int d\psi(x)\wedge d\pi(x)\,dx + dq\wedge dp$, i.e.

$$\Omega(Y_1,Y_2)=\langle Y_1, JY_2\rangle,\ \ Y_1,Y_2\in\mathcal{E}, \tag{3.2}$$

where

$$\langle Y_1,Y_2\rangle := \langle\psi_1,\psi_2\rangle+\langle\pi_1,\pi_2\rangle+q_1q_2+p_1p_2$$

and $\langle\psi_1,\psi_2\rangle$ stands for the scalar product $\displaystyle\int\psi_1(x)\psi_2(x)dx$ or its different extensions.

Definition 3.1 *i)* $Y_1 \nmid Y_2$ *means that* Y_1 *is symplectic orthogonal to* Y_2*, i.e.* $\Omega(Y_1,Y_2)=0$.
ii) A projection operator $\boldsymbol{P}:\mathcal{E}\to\mathcal{E}$ *is called symplectic orthogonal if* $Y_1\nmid Y_2$ *for* $Y_1\in\operatorname{Ker}\boldsymbol{P}$ *and* $Y_2\in\operatorname{Im}\boldsymbol{P}$.

3.2 Symplectic Projection onto Solitary Manifold

From now on we suppose that the condition (1.11) is satisfied. Let us consider the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ to the manifold $\mathcal{S}$ at a point $S(\sigma)$, where $\sigma=(b,v)$. The vectors $\tau_j:=\partial_{\sigma_j}S(\sigma)$, where $\partial_{\sigma_j}:=\partial_{b_j}$ and $\partial_{\sigma_{3+j}}:=\partial_{v_j}$ with $j=1,2,3$, form a basis in $\mathcal{T}_\sigma\mathcal{S}$. In detail,

$$\left.\begin{array}{rcl}\tau_j=\tau_j(v) &:=& \partial_{b_j}S(\sigma)=(\,-\partial_j\psi_v(y)\,,-\partial_j\pi_v(y)\,,\ e_j\,,\quad 0\quad)\\ \tau_{3+j}=\tau_{3+j}(v) &:=& \partial_{v_j}S(\sigma)=(\ \ \partial_{v_j}\psi_v(y)\,,\ \partial_{v_j}\pi_v(y)\,,\ \ 0\ \,,\ \partial_{v_j}p_v\,)\end{array}\right|\quad j=1,2,3, \tag{3.3}$$

where $y:=x-b$ is the "coordinate in the moving frame", $e_1=(1,0,0)$ etc.

By (2.12) for the tangent vectors we have

$$\tau_j(v)\in\mathcal{E}_\alpha,\ \alpha<\frac{9}{2};\ \ j=1,\dots,6. \tag{3.4}$$

Lemma 3.2 *The matrix with the elements* $\Omega(\tau_l(v),\tau_j(v))$ *is non-degenerate for* $|v|<1$.

The proof is made by a straightforward computation, see [17], Lemma 3.2 for the case $m=0$ (note that the left hand side of the identity [17, (A.8)] is well defined by the condition (1.11)).

Now we show that in a small neighborhood of the soliton manifold $\mathcal{S}$ a "symplectic orthogonal projection" onto $\mathcal{S}$ is well defined. Let us introduce the translations $T_a:(\psi(x),\pi(x),q,p)\mapsto(\psi(x-a),\pi(x-a),q+a,p)$, $a\in\mathbb{R}^3$. Note that the manifold $\mathcal{S}$ is invariant with respect to the translations.

Definition 3.3 *Let us denote by* $v(Y):=p/\sqrt{1+p^2}$ *where* $p\in\mathbb{R}^3$ *is the last component of the vector* Y.

Lemma 3.4 *Let (1.9) hold,* $\alpha>-9/2$ *and* $\overline{v}<1$*. Then*
i) there exists a neighborhood $\mathcal{O}_\alpha(\mathcal{S})$ *of* $\mathcal{S}$ *in* $\mathcal{E}_\alpha$ *and a map* $\boldsymbol{\Pi}:\mathcal{O}_\alpha(\mathcal{S})\to\mathcal{S}$ *such that* $\boldsymbol{\Pi}$ *is uniformly continuous on* $\mathcal{O}_\alpha(\mathcal{S})\cap\{Y\in\mathcal{E}_\alpha: v(Y)\le\overline{v}\}$ *in the metric of* $\mathcal{E}_\alpha$*,*

$$\boldsymbol{\Pi}Y=Y\ \ \textit{for}\ \ Y\in\mathcal{S},\qquad \textit{and}\qquad Y-S\nmid\mathcal{T}_S\mathcal{S},\ \ \textit{where}\ \ S=\boldsymbol{\Pi}Y. \tag{3.5}$$

ii) $\mathcal{O}_\alpha(\mathcal{S})$ *is invariant with respect to the translations* T_a*, and*

$$\boldsymbol{\Pi}T_aY=T_a\boldsymbol{\Pi}Y,\qquad \textit{for}\ \ Y\in\mathcal{O}_\alpha(\mathcal{S})\ \ \textit{and}\ \ a\in\mathbb{R}^3. \tag{3.6}$$

iii) For any $\overline{v}<1$ *there exists a* $\tilde{v}<1$ *such that* $|v(\boldsymbol{\Pi}Y)|<\tilde{v}$ *when* $|v(Y)|<\overline{v}$.

iv) For any $\tilde{v}<1$ *there exists an* $r_\alpha(\tilde{v})>0$ *s.t.* $S(\sigma)+Z\in\mathcal{O}_\alpha(\mathcal{S})$ *if* $|v(S(\sigma))|<\tilde{v}$ *and* $\|Z\|_\alpha<r_\alpha(\tilde{v})$.

The proof is similar to that of [17, Lemma 3.4].

We will call $\boldsymbol{\Pi}$ the *symplectic orthogonal projection* onto $\mathcal{S}$.

Corollary 3.5 [17, Corollary 3.5]. *The condition (2.15) implies that* $Y_0=\tilde{S}+\tilde{Z}_0$ *where* $\tilde{S}=\boldsymbol{\Pi}Y_0$*, and*

$$\|\tilde{Z}_0\|_\beta\ll 1. \tag{3.7}$$

4 Linearization on the Solitary Manifold

Let us consider a solution to the system (1.2), and split it as the sum

$$Y(t) = S(\sigma(t)) + Z(t), \tag{4.1}$$

where $\sigma(t) = (b(t), v(t)) \in \mathbb{R}^3 \times \{|v| < 1\}$ is an arbitrary smooth function of $t \in \mathbb{R}$. In detail, denote $Y = (\psi, \pi, q, p)$ and $Z = (\Psi, \Pi, Q, P)$. Then (4.1) means that

$$\left.\begin{array}{rclrcl} \psi(x,t) &=& \psi_{v(t)}(x-b(t)) + \Psi(x-b(t),t), & q(t) &=& b(t)+Q(t) \\ \pi(x,t) &=& \pi_{v(t)}(x-b(t)) + \Pi(x-b(t),t), & p(t) &=& p_{v(t)} + P(t) \end{array}\right| \tag{4.2}$$

Let us substitute (4.2) to (1.2), and linearize the equations in Z. Later we will choose $S(\sigma(t)) = \mathbf{\Pi} Y(t)$, i.e. $Z(t)$ is symplectic orthogonal to $\mathcal{T}_{S(\sigma(t))}\mathcal{S}$. However, this orthogonality condition is not needed for the formal process of linearization. The orthogonality condition will be important in Section 6, where we derive "modulation equations" for the parameters $\sigma(t)$.

Let us proceed to linearization. Setting $y = x - b(t)$ which is the *coordinate in the moving frame*, we obtain from (4.2) and (1.2) that

$$\left.\begin{array}{rcl} \dot\psi &=& \dot v\cdot\nabla_v\psi_{v(t)}(y) - \dot b\cdot\nabla\psi_{v(t)}(y) + \dot\Psi(y,t) - \dot b\cdot\nabla\Psi(y,t) = \pi_{v(t)}(y) + \Pi(y,t) \\ \dot\pi &=& \dot v\cdot\nabla_v\pi_{v(t)}(y) - \dot b\cdot\nabla\pi_{v(t)}(y) + \dot\Pi(y,t) - \dot b\cdot\nabla\Pi(y,t) \\ &=& \Delta\psi_{v(t)}(y) + \Delta\Psi(y,t) - \rho(y-Q) \\ \dot q &=& \dot b + \dot Q = \dfrac{p_v + P}{\sqrt{1+(p_v+P)^2}} \\ \dot p &=& \dot v\cdot\nabla_v p_{v(t)} + \dot P = -\langle\nabla(\psi_{v(t)}(y) + \Psi(y,t)), \rho(y-Q)\rangle \end{array}\right| \tag{4.3}$$

The equations are linear in Ψ and Π, hence it remains to extract linear terms in Q and P. Within this section, we estimate the remainders in the norms of the space $\mathcal{E}_\alpha$ with an arbitrary $\alpha > 0$.

First note that $\rho(y-Q) = \rho(y) - Q\cdot\nabla\rho(y) - N_2(Q)$, where $-N_2(Q) = \rho(y-Q) - \rho(y) + Q\cdot\nabla\rho(y)$; for $N_2(Q)$ the bound holds,

$$|N_2(Q)|_\alpha \le C(\overline{Q})Q^2 \tag{4.4}$$

uniformly in $|Q| \le \overline{Q}$ for any fixed $\overline{Q}$. Second, the Taylor expansion gives

$$\frac{p_v + P}{\sqrt{1+(p_v+P)^2}} = v + \nu(P - v(v\cdot P)) + N_3(v,P),$$

where $\nu = \nu_v := (1+p_v^2)^{-1/2} = \sqrt{1-v^2}$, and

$$|N_3(v,P)| \le C(\tilde v)P^2 \tag{4.5}$$

uniformly in v with $|v| \le \tilde v < 1$. Using the equations (2.8), we obtain from (4.3) the following equations for the components of the vector $Z(t)$:

$$\left.\begin{array}{ll} \dot\Psi(y,t) = \Pi(y,t) + \dot b\cdot\nabla\Psi(y,t) + (\dot b - v)\cdot\nabla\psi_v(y) - \dot v\cdot\nabla_v\psi_v(y) & \\ \dot\Pi(y,t) = \Delta\Psi(y,t) + \dot b\cdot\nabla\Pi(y,t) + Q\cdot\nabla\rho(y) + (\dot b - v)\cdot\nabla\pi_v(y) - \dot v\cdot\nabla_v\pi_v(y) + N_2 & \\ \dot Q(t) \quad = \nu_v(I_3 - v\otimes v)P + (v - \dot b) + N_3 & \\ \dot P(t) \quad = \langle\Psi(y,t), \nabla\rho(y)\rangle + \langle\nabla\psi_v(y), Q\cdot\nabla\rho(y)\rangle - \dot v\cdot\nabla_v p_v + N_4(v,Z) & \end{array}\right| \tag{4.6}$$

where $N_4(v,Z) = \langle\nabla\psi_v, N_2(Q)\rangle + \langle\nabla\Psi, Q\cdot\nabla\rho\rangle + \langle\nabla\Psi, N_2(Q)\rangle$. Clearly, $N_4(v,Z)$ satisfies the following estimate

$$|N_4(v,Z)| \le C_\beta(\rho, \tilde v, \overline{Q})\Big[Q^2 + \|\Psi\|_{-\alpha}|Q|\Big], \tag{4.7}$$

uniformly in v, Q with $|v| \le \tilde{v}$ and $|Q| \le \overline{Q}$. We can write the equations (4.6) as

$$\dot{Z}(t) = A(t)Z(t) + T(t) + N(t), \quad t \in \mathbb{R}. \tag{4.8}$$

Here the operator $A(t)$ depends on $\sigma(t) = (b(t), v(t))$. We will use the parameters $v = v(t)$ and $w := \dot{b}(t)$. Then $A(t)$ can be written in the form

$$A(t)\begin{pmatrix}\Psi\\ \Pi\\ Q\\ P\end{pmatrix} = A_{v,w}\begin{pmatrix}\Psi\\ \Pi\\ Q\\ P\end{pmatrix} := \begin{pmatrix} w\cdot\nabla & 1 & 0 & 0\\ \Delta & w\cdot\nabla & \nabla\rho\cdot & 0\\ 0 & 0 & 0 & B_v\\ \langle\cdot,\nabla\rho\rangle & 0 & \langle\nabla\psi_v,\cdot\nabla\rho\rangle & 0\end{pmatrix}\begin{pmatrix}\Psi\\ \Pi\\ Q\\ P\end{pmatrix}, \tag{4.9}$$

where $B_v = \nu_v(Id_3 - v\otimes v)$. Furthermore, $T(t)$ and $N(t)$ in (4.8) stand for

$$T(t) = T_{v,w} = \begin{pmatrix} (w-v)\cdot\nabla\psi_v - \dot{v}\cdot\nabla_v\psi_v\\ (w-v)\cdot\nabla\pi_v - \dot{v}\cdot\nabla_v\pi_v\\ v-w\\ -\dot{v}\cdot\nabla_v p_v\end{pmatrix}, \quad N(t) = N(\sigma, Z) = \begin{pmatrix}0\\ N_2(Z)\\ N_3(v,Z)\\ N_4(v,Z)\end{pmatrix}, \tag{4.10}$$

where $v = v(t)$, $w = w(t)$, $\sigma = \sigma(t) = (b(t), v(t))$, and $Z = Z(t)$. Since $|Q| \le \|Z\|_{-\alpha}$ for any α, the estimates (4.4) with $\overline{Q} = r_{-\alpha}(\tilde{v})$, (4.5) and (4.7) imply the following

Lemma 4.1 *For any $\alpha > 0$*

$$\|N(\sigma, Z)\|_\alpha \le C(\tilde{v})\|Z\|_{-\alpha}^2 \tag{4.11}$$

uniformly in σ, Z with $\|Z\|_{-\alpha} \le r_{-\alpha}(\tilde{v})$ and $|v| < \tilde{v}$.

Remarks 4.2 i) The term $A(t)Z(t)$ in the right hand side of the equation (4.8) is linear in $Z(t)$, and $N(t)$ is a *high order term* in $Z(t)$. On the other hand, $T(t)$ is a *zero order term* which does not vanish at $Z(t) = 0$ since $S(\sigma(t))$ generally is not a soliton solution if (2.14) does not hold (though $S(\sigma(t))$ belongs to the solitary manifold).
ii) Formulas (3.3) and (4.10) imply:

$$T(t) = -\sum_{l=1}^{3}[(w-v)_l\tau_l + \dot{v}_l\tau_{l+3}] \tag{4.12}$$

and hence $T(t) \in \mathcal{T}_{S(\sigma(t))}\mathcal{S}$, $t \in \mathbb{R}$. This fact suggests an unstable character of the nonlinear dynamics *along the solitary manifold.*

5 The Linearized Equation

Here we collect some Hamiltonian and spectral properties of the operator (4.9). The statements of the section are particular cases of those in [17], Section 5 for $m = 0$. First, we consider the linear equation

$$\dot{X}(t) = A_{v,w}X(t), \qquad t \in \mathbb{R} \tag{5.1}$$

with an arbitrary fixed v such that $|v| < 1$, and $w \in \mathbb{R}^3$. Let us define the space

$$\mathcal{E}^+ = H^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \oplus \mathbb{R}^3 \oplus \mathbb{R}^3.$$

Lemma 5.1 *i) For any v, $|v| < 1$ and $w \in \mathbb{R}^3$ the equation (5.1) formally can be written as the Hamilton system (cf. (3.1)),*

$$\dot{X}(t) = JD\mathcal{H}_{v,w}(X(t)), \qquad t \in \mathbb{R}, \tag{5.2}$$

where $D\mathcal{H}_{v,w}$ is the Fréchet derivative of the Hamilton functional

$$\begin{aligned}\mathcal{H}_{v,w}(X) \;&=\; \frac{1}{2}\int\Big[|\Pi|^2 + |\nabla\Psi|^2\Big]dy + \int \Pi w\cdot\nabla\Psi dy + \int\rho(y)Q\cdot\nabla\Psi dy\\ &+\; \frac{1}{2}P\cdot B_vP - \frac{1}{2}\langle Q\cdot\nabla\psi_v(y), Q\cdot\nabla\rho(y)\rangle, \qquad X = (\Psi,\Pi,Q,P) \in \mathcal{E}.\end{aligned} \tag{5.3}$$

ii) Energy conservation law holds for the solutions $X(t) \in C(\mathbb{R}, \mathcal{E})$,

$$\mathcal{H}_{v,w}(X(t)) = \text{const}, \qquad t \in \mathbb{R}. \tag{5.4}$$

iii) The skew-symmetry relation holds,

$$\Omega(A_{v,w}X_1, X_2) = -\Omega(X_1, A_{v,w}X_2), \qquad X_1 \in \mathcal{E}, \;\; X_2 \in \mathcal{E}^+. \tag{5.5}$$

Lemma 5.2 *The operator* $A_{v,w}$ *acts on the tangent vectors* $\tau_j(v)$ *to the solitary manifold as follows,*

$$A_{v,w}[\tau_j(v)] = (w - v) \cdot \nabla\tau_j(v), \;\; A_{v,w}[\tau_{j+3}(v)] = (w - v) \cdot \nabla\tau_{j+3}(v) + \tau_j(v), \;\; j = 1, 2, 3. \tag{5.6}$$

We will apply Lemmas 5.1 and 5.2 mainly to the operator $A_{v,v}$ corresponding to $w = v$. In that case (5.6) reads

$$A_{v,v}[\tau_j(v)] = 0, \;\; A_{v,v}[\tau_{j+3}(v)] = \tau_j(v), \;\; j = 1, 2, 3. \tag{5.7}$$

Moreover, the linearized equation acquires the additional essential feature.

Lemma 5.3 *Let us assume that* $w = v$ *and* $|v| < 1$. *Then the Hamilton functional (5.3) reads, see* [17, (5.14)]

$$\mathcal{H}_{v,v}(X) = \frac{1}{2}\int \Big(|\Pi + v \cdot \nabla\Psi|^2 + |\Lambda_v^{1/2}\Psi - \Lambda_v^{-1/2}Q \cdot \nabla\rho|^2\Big)\, dx + \frac{1}{2}P \cdot B_v P \geq 0. \tag{5.8}$$

Here Λ_v *is the operator defined by (2.9).*

Remark 5.4 Lemma 5.3 together with energy conservation (5.4) imply the analyticity of the resolvent $(A_{v,v} - \lambda)^{-1}$ for $\operatorname{Re}\lambda > 0$, see below.

Remark 5.5 For a soliton solution of the system (1.2) we have $\dot{b} = v$, $\dot{v} = 0$, and hence $T(t) \equiv 0$. Thus, the equation (5.1) is the linearization of the system (1.2) on a soliton solution. In fact, we do not linearize (1.2) on a soliton solution, but on a trajectory $S(\sigma(t))$ with $\sigma(t)$ being nonlinear in t. We will show later that $T(t)$ is quadratic in $Z(t)$ if we choose $S(\sigma(t))$ to be the symplectic orthogonal projection of $Y(t)$. Then (5.1) is again the linearization of (1.2).

6 Symplectic Decomposition of the Dynamics

Here we decompose the dynamics in two components: along the manifold $\mathcal{S}$ and in transversal directions. The equation (4.8) is obtained without any assumption on $\sigma(t)$ in (4.1). We are going to choose $S(\sigma(t)) := \mathbf{\Pi} Y(t)$, but then we need to know that

$$Y(t) \in \mathcal{O}_\alpha(\mathcal{S}), \qquad t \in \mathbb{R}, \tag{6.1}$$

with some $\mathcal{O}_\alpha(\mathcal{S})$ defined in Lemma 3.4. It is true for $t = 0$ and $\alpha \leq \beta := 4 + \delta$ by (3.7), if $d_\beta > 0$ in (2.15) is sufficiently small. Then $S(\sigma(0)) = \mathbf{\Pi} Y(0)$ and $Z(0) = Y(0) - S(\sigma(0))$ are well defined.

We set $\alpha = -\beta$ and will prove below that (6.1) holds if d_β is sufficiently small. First, the a priori estimate (2.7) together with Lemma 3.4 iii) imply that $\mathbf{\Pi} Y(t) = S(\sigma(t))$ with $\sigma(t) = (b(t), v(t))$, and

$$|v(t)| \leq \tilde{v} < 1, \qquad t \in \mathbb{R} \tag{6.2}$$

if $Y(t) \in \mathcal{O}_\alpha(\mathcal{S})$. Denote by $r_\alpha(\tilde{v})$ the positive number from Lemma 3.4 iv) which corresponds to the chosen $\alpha = -\beta$. Then $S(\sigma) + Z \in \mathcal{O}_\alpha(\mathcal{S})$ if $\sigma = (b, v)$ with $|v| < \tilde{v}$ and $\|Z\|_\alpha < r_\alpha(\tilde{v})$. Note that (3.7) implies $\|Z(0)\|_\alpha < r_\alpha(\tilde{v})$ if d_β is sufficiently small. Therefore, $S(\sigma(t)) = \mathbf{\Pi} Y(t)$ and $Z(t) = Y(t) - S(\sigma(t))$ are well defined for $t \geq 0$ so small that $\|Z(t)\|_\alpha < r_\alpha(\tilde{v})$. This is formalized by the following standard definition.

Definition 6.1 t_* *is the "exit time",*

$$t_* = \sup\{t > 0 : \|Z(s)\|_{-\beta} < r_{-\beta}(\tilde{v}), \;\; 0 \leq s \leq t\}, \;\; Z(s) = Y(s) - S(\sigma(s)). \tag{6.3}$$

One of our main goals is to prove that $t_* = \infty$ if d_β is sufficiently small. This would follow if we show that

$$\|Z(t)\|_{-\beta} < r_{-\beta}(\tilde{v})/2, \qquad 0 \leq t < t_*. \tag{6.4}$$

Now $N(t)$ in (4.8) satisfies, by (4.11) with $\alpha = \beta$, the following estimate:

$$\|N(t)\|_\beta \leq C(\tilde{v})\|Z(t)\|_{-\beta}^2, \;\; 0 \leq t < t_*. \tag{6.5}$$

6.1 Longitudinal Dynamics: Modulation Equations

From now on we fix the decomposition $Y(t) = S(\sigma(t)) + Z(t)$ for $0 < t < t_*$ by setting $S(\sigma(t)) = \mathbf{\Pi} Y(t)$ which is equivalent to the symplectic orthogonality condition of type (3.5),

$$Z(t) \nmid \mathcal{T}_{S(\sigma(t))}\mathcal{S}, \ \ 0 \le t < t_*. \tag{6.6}$$

This allows us to simplify drastically the asymptotic analysis of the dynamical equations (4.8) for the transversal component $Z(t)$. As the first step, we derive the longitudinal dynamics, i.e. the *modulation equations* for the parameters $\sigma(t)$. Let us derive a system of ordinary differential equations for the vector $\sigma(t)$. For this purpose, let us write (6.6) in the form

$$\Omega(Z(t), \tau_j(t)) = 0, \ \ j = 1, \dots, 6, \qquad 0 \le t < t_*, \tag{6.7}$$

where the vectors $\tau_j(t) = \tau_j(\sigma(t))$ span the tangent space $\mathcal{T}_{S(\sigma(t))}\mathcal{S}$. Note that $\sigma(t) = (b(t), v(t))$, where

$$|v(t)| \le \tilde{v} < 1, \qquad 0 \le t < t_*, \tag{6.8}$$

by Lemma 3.4 iii). It would be convenient for us to use some other parameters (c, v) instead of $\sigma = (b, v)$, where $c(t) = b(t) - \displaystyle\int_0^t v(\tau) d\tau$ and

$$\dot{c}(t) = \dot{b}(t) - v(t) = w(t) - v(t), \qquad 0 \le t < t_*. \tag{6.9}$$

We do not need an explicit form of the equations for (c, v) but the following statement.

Lemma 6.2 *Let $Y(t)$ be a solution to the Cauchy problem (2.5), and (1.11), (4.1), (6.7) hold. Then $(c(t), v(t))$ satisfies the equation*

$$\begin{pmatrix} \dot{c}(t) \\ \dot{v}(t) \end{pmatrix} = \mathcal{N}(\sigma(t), Z(t)), \qquad 0 \le t < t_*, \tag{6.10}$$

where

$$\mathcal{N}(\sigma, Z) = \mathcal{O}(\|Z\|^2_{-\beta}) \tag{6.11}$$

uniformly in $\sigma \in \{(b, v) : |v| \le \tilde{v}\}$.

Proof We differentiate (6.7) in t and take the equation (4.8) into account. Then (see details of computation in [17], Lemma 6.2) we obtain, in the vector form [17, (6.18)]:

$$0 = \Omega(v) \begin{pmatrix} \dot{c} \\ \dot{v} \end{pmatrix} + \mathcal{M}_0(\sigma, Z) \begin{pmatrix} \dot{c} \\ \dot{v} \end{pmatrix} + \mathcal{N}_0(\sigma, Z), \ \ \mathcal{N}_{0j}(\sigma, Z) = \Omega(N, \tau_j). \tag{6.12}$$

Here the matrix $\Omega(v)$ has the matrix elements $\Omega(\tau_l, \tau_j)$ and hence is invertible by Lemma 3.2. The 6×6 matrix $\mathcal{M}_0(\sigma, Z)$ has the matrix elements $\sim \ \|Z\|_{-\beta}$ and hence we can resolve the equation (6.12) with respect to $(\dot{c}, \dot{v})$. Then (6.11) follows from Lemma 4.1 with $\alpha = \beta$, since $\mathcal{N}_0 = \mathcal{O}(\|Z\|^2_{-\beta})$. □

6.2 Decay for the Transversal Dynamics

In Section 12 we will show that our main Theorem 2.5 can be derived from the following time decay of the transversal component $Z(t)$:

Proposition 6.3 *Let all conditions of Theorem 2.5 hold. Then $t_* = \infty$, and*

$$\|Z(t)\|_{-\beta} \le \frac{C(\rho, \overline{v}, d_\beta)}{(1 + t)^{1+\delta}}, \qquad t \ge 0. \tag{6.13}$$

We will derive (6.13) in Sections 8 – 11 from our equation (4.8) for the transversal component $Z(t)$. This equation can be specified using Lemma 6.2. Namely, by (4.12) and (6.9)

$$T(t) = -\sum_{l=1}^{3}[\dot{c}_l\tau_l + \dot{v}_l\tau_{l+3}].$$

Note that the norm $\|T(t)\|_\beta$ is well-defined by (3.4). Then Lemma 6.2 implies

$$\|T(t)\|_\beta \le C(\tilde{v})\|Z(t)\|^2_{-\beta}, \qquad 0 \le t < t_*. \tag{6.14}$$

Thus, in (4.8) we should combine the terms $T(t)$ and $N(t)$ and obtain

$$\dot{Z}(t) = A(t)Z(t) + \tilde{N}(t), \qquad 0 \le t < t_*, \tag{6.15}$$

where $A(t) = A_{v(t),w(t)}$, and $\tilde{N}(t) := T(t) + N(t)$. From (6.14) and (6.5) we obtain that

$$\|\tilde{N}(t)\|_\beta \le C(\tilde{v})\|Z(t)\|^2_{-\beta}, \qquad 0 \le t < t_*. \tag{6.16}$$

In the remaining part of our paper we will analyze mainly the **basic equation** (6.15) to establish the decay (6.13). We are going to derive the decay using the bound (6.16) and the orthogonality condition (6.6).

Let us comment on two main difficulties in proving (6.13). The difficulties are common for the problems studied in [5, 7]. First, the linear part of the equation is non-autonomous, hence we cannot apply directly known methods of scattering theory. Similarly to the approach of [5, 7], we reduce the problem to the analysis of the *frozen* linear equation,

$$\dot{X}(t) = A_1 X(t), \quad t \in \mathbb{R}, \tag{6.17}$$

where A_1 is the operator A_{v_1,v_1} defined in (4.9) with $v_1 = v(t_1)$ and a fixed $t_1 \in [0, t_*)$. Then we estimate the error by the method of majorants. Let us note that recently some methods of freezing were developed by Cuccagna and Mizumachi, [9, 10].

Second, even for the frozen equation (6.17), the decay of type (6.13) for all solutions does not hold without the orthogonality condition of type (6.6). Namely, by (5.7) the equation (6.17) admits the *secular solutions*

$$X(t) = \sum_1^3 C_j\tau_j(v_1) + \sum_1^3 D_j[\tau_j(v_1)t + \tau_{j+3}(v_1)] \tag{6.18}$$

which arise also by differentiation of the soliton (1.5) in the parameters a and v_1 in the moving coordinate $y = x - v_1 t$. Hence, we have to take into account the orthogonality condition (6.6) in order to avoid the secular solutions. For this purpose we will apply the corresponding symplectic orthogonal projection which kills the "runaway solutions" (6.18).

Remark 6.4 The solution (6.18) lies in the tangent space $\mathcal{T}_{S(\sigma_1)}\mathcal{S}$ with $\sigma_1 = (b_1, v_1)$ (for an arbitrary $b_1 \in \mathbb{R}$) that suggests an unstable character of the nonlinear dynamics *along the solitary manifold* (cf. Remark 4.2 ii)).

Definition 6.5 *i) Denote by $\mathbf{\Pi}_v$, $|v| < 1$, the symplectic orthogonal projection of $\mathcal{E}$ onto the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$, and $\boldsymbol{P}_v = \mathbf{I} - \mathbf{\Pi}_v$.*
ii) Denote by $\mathcal{Z}_v = \boldsymbol{P}_v\mathcal{E}$ the space symplectic orthogonal to $\mathcal{T}_{S(\sigma)}\mathcal{S}$ with $\sigma = (b, v)$ (for an arbitrary $b \in \mathbb{R}$).

Note that by the linearity,

$$\mathbf{\Pi}_v Z = \sum \mathbf{\Pi}_{jl}(v)\tau_j(v)\Omega(\tau_l(v), Z), \qquad Z \in \mathcal{E}, \tag{6.19}$$

with some smooth coefficients $\mathbf{\Pi}_{jl}(v)$. Hence, the projector $\mathbf{\Pi}_v$, in the variable $y = x - b$, does not depend on b, and this explains the choice of the subindex in $\mathbf{\Pi}_v$ and $\boldsymbol{P}_v$. Now we have the symplectic orthogonal decomposition

$$\mathcal{E} = \mathcal{T}_{S(\sigma)}\mathcal{S} + \mathcal{Z}_v, \qquad \sigma = (b, v), \tag{6.20}$$

and the symplectic orthogonality (6.6) can be written in the following equivalent forms,

$$\mathbf{\Pi}_{v(t)}Z(t) = 0, \quad \boldsymbol{P}_{v(t)}Z(t) = Z(t), \qquad 0 \le t < t_*. \tag{6.21}$$

Remark 6.6 The tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ is invariant under the operator $A_{v,v}$ by Lemma 5.3 i), hence the space $\mathcal{Z}_v$ is also invariant by (5.5): $A_{v,v}Z \in \mathcal{Z}_v$ for *sufficiently smooth* $Z \in \mathcal{Z}_v$.

In the next section we prove the following proposition which is one of the main ingredients for proving (6.13). Let us consider the Cauchy problem for the equation (6.17) with $A = A_{v,v}$ for a fixed v, $|v| < 1$. Recall that the $\beta = 4 + \delta$, $0 < \delta < 1/2$.

Proposition 6.7 *Let (1.9), the Wiener condition (1.10), and the condition (1.11) hold,* $|v_1| \le \tilde{v} < 1$, *and* $X_0 \in \mathcal{E}$. *Then*
i) The equation (6.17), with $A_1 = A_{v_1,v_1}$, *admits the unique solution* $e^{A_1 t}X_0 := X(t) \in C_b(\mathbb{R}, \mathcal{E})$ *with the initial condition* $X(0) = X_0$.
ii) If $X_0 \in \mathcal{Z}_{v_1} \cap \mathcal{E}_\beta$ *the solution* $X(t)$ *has the following decay,*

$$\|e^{A_1 t}X_0\|_{-2-\delta} \le \frac{C(\tilde{v})}{(1+|t|)^{1+\delta}}\|X_0\|_\beta, \quad t \in \mathbb{R}. \tag{6.22}$$

Remark 6.8 The decay is provided by two fundamental facts which we will establish below:
i) the null root space of the generator A_1 coincides with the $\mathcal{T}_{S(\sigma_1)}\mathcal{S}$ with $\sigma_1 = (b_1, v_1)$ (for an arbitrary $b_1 \in \mathbb{R}$), and
ii) the spectrum of A_1 in the space $\mathcal{Z}_{v_1}$ is absolutely continuous.

7 Proof of Proposition 6.7

Part i) follows by standard arguments using the positivity (5.8) of the Hamilton functional.

Part ii) follows by the general strategy developed in [17, 18]. The equation (6.17) is a system of four equations involving field components, Ψ and Π as well as vector components, Q and P. We apply Fourier-Laplace transform, express the field components in terms of the vector components from the first two equations and substitute to the third and the fourth equations. Then we obtain a closed system for the vector components alone and prove their decay. Finally, for the field components we come to a wave equation with a right hand side which has the established decay. This implies the corresponding decay for the field components.

So let us apply the Laplace transform

$$\Lambda X = \tilde{X}(\lambda) = \int_0^\infty e^{-\lambda t}X(t)dt, \qquad \operatorname{Re}\lambda > 0 \tag{7.1}$$

to (6.17). The integral converges in $\mathcal{E}$, since $\|X(t)\|_{\mathcal{E}}$ is bounded by Proposition 6.7, i). The analyticity of $\tilde{X}(\lambda)$ and Paley-Wiener arguments should provide the existence of a $\mathcal{E}$-valued distribution $X(t) = (\Psi(t), \Pi(t), Q(t), P(t))$, $t \in \mathbb{R}$, with a support in $[0,\infty)$. Formally,

$$\Lambda^{-1}\tilde{X} = X(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}\tilde{X}(i\omega + 0)d\omega, \qquad t \in \mathbb{R}. \tag{7.2}$$

To prove the decay (6.22) we have to study the smoothness of $\tilde{X}(i\omega+0)$ at $\omega \in \mathbb{R}$. After the Laplace transform the equation (6.17) becomes

$$\lambda\tilde{X}(\lambda) = A_1\tilde{X}(\lambda) + X_0, \quad \operatorname{Re}\lambda > 0.$$

In detail (for simplicity we write v instead of v_1 in this section),

$$\left.\begin{aligned} \tilde{\Pi}(y) + v\cdot\nabla\tilde{\Psi}(y) - \lambda\tilde{\Psi}(y) &= -\tilde{\Psi}_0(y) \\ \Delta\tilde{\Psi}(y) + v\cdot\nabla\tilde{\Pi}(y) + \tilde{Q}\cdot\nabla\rho(y) - \lambda\tilde{\Pi}(y) &= -\Pi_0(y) \\ B_v\tilde{P} - \lambda\tilde{Q} &= -Q_0 \\ -\langle\nabla\tilde{\Psi}(y), \rho(y)\rangle + \langle\nabla\psi_v(y), \tilde{Q}\cdot\nabla\rho(y)\rangle - \lambda\tilde{P} &= -P_0 \end{aligned}\right| \quad y \in \mathbb{R}^3. \tag{7.3}$$

Let us consider the first two equations. In Fourier space they become

$$\left.\begin{array}{rcl}\hat\Pi(k)-ivk\hat\Psi(k)-\lambda\hat\Psi(k)&=&-\hat\Psi_0(k)\\ -k^2\hat\Psi(k)-(ivk+\lambda)\hat\Pi(k)&=&-\hat\Pi_0(k)+i\tilde Qk\hat\rho(k)\end{array}\right|\quad k\in\mathbb{R}^3. \tag{7.4}$$

This implies

$$\hat\Psi=\frac{1}{\hat D}((ikv+\lambda)\hat\Psi_0+\hat\Pi_0-ik\tilde Q\hat\rho), \tag{7.5}$$

$$\hat\Pi=\frac{1}{\hat D}(-k^2\hat\Psi_0+(ikv+\lambda)\hat\Pi_0-i(ikv+\lambda)k\tilde Q\hat\rho), \tag{7.6}$$

where

$$\hat D=\hat D(\lambda)=k^2+(ikv+\lambda)^2. \tag{7.7}$$

From now on we use the system of coordinates in x-space in which $v=(|v|,0,0)$, hence $vk=|v|k_1$. Substitute (7.5) to the 4-th equation of (7.3) and obtain

$$\int\frac{ik}{\hat D}((ikv+\lambda)\hat\Psi_0+\Pi_0-ik\tilde Q\hat\rho)\overline{\hat\rho}dk+\int k\hat\psi_v k\tilde Q\overline{\hat\rho}dk-\lambda\tilde P=-P_0.$$

Since $\hat\psi_v=-\hat\rho/(k^2-(kv)^2)$, we come to

$$(K-H(\lambda))\tilde Q+\lambda\tilde P=P_0+\Phi(\lambda).$$

Here

$$\Phi(\lambda)=\Phi(\Psi_0,\Pi_0)(\lambda):=i\int\frac{k}{\hat D}((ikv+\lambda)\hat\Psi_0+\hat\Pi_0)\overline{\hat\rho}dk=i\langle\frac{(ikv+\lambda)\hat\Psi_0+\hat\Pi_0}{\hat D},k\hat\rho\rangle, \tag{7.8}$$

and K, $H(\lambda)$ are 3×3-matrices with the matrix elements

$$K_{ij}=\int\frac{k_ik_j|\hat\rho(k)|^2dk}{k^2-(|v|k_1)^2},\quad H_{ij}(\lambda)=\int\frac{k_ik_j|\hat\rho(k)|^2dk}{k^2+(i|v|k_1+\lambda)^2}. \tag{7.9}$$

The matrix K is diagonal and positive definite since $\hat\rho(k)$ is spherically symmetric and not identically zero by (1.10). The matrix H is well defined for $\operatorname{Re}\lambda>0$ since the denominator does not vanish. The matrix H is diagonal similarly to K. Indeed, if $i\neq j$, then at least one of these indexes is not equal to one, and the integrand in (7.9) is odd with respect to the corresponding variable by (1.9). Finally the 3-rd and the 4-th equations of (7.3) become

$$M(\lambda)\begin{pmatrix}\tilde Q\\ \tilde P\end{pmatrix}=\begin{pmatrix}Q_0\\ P_0+\Phi(\lambda)\end{pmatrix},\ \text{where } M(\lambda)=\begin{pmatrix}\lambda I_3&-B_v\\ -F(\lambda)&\lambda I_3\end{pmatrix},\quad F(\lambda):=H(\lambda)-K. \tag{7.10}$$

Remark 7.1 Note that

$$\Phi(\lambda)=\Phi(\Psi_0,\Pi_0)(\lambda)=\Lambda\langle W^1(t)(\Psi_0,\Pi_0),\nabla\rho\rangle, \tag{7.11}$$

where $W^1(t)$ is the first component of the dynamical group $W(t)$ defined below by (7.54) and Λ is the Laplace transform (7.1).

Let us proceed to x-representation. We invert the matrix of the system (7.4) and obtain

$$\begin{pmatrix}-(ivk+\lambda)&1\\ -k^2&-(ivk+\lambda)\end{pmatrix}^{-1}=[(ivk+\lambda)^2+k^2]^{-1}\begin{pmatrix}-(ivk+\lambda)&-1\\ k^2&-(ivk+\lambda)\end{pmatrix}.$$

Taking the inverse Fourier transform, we obtain the corresponding fundamental solution

$$G_\lambda(y)=\begin{pmatrix}v\cdot\nabla-\lambda&-1\\ -\Delta&v\cdot\nabla-\lambda\end{pmatrix}g_\lambda(y), \tag{7.12}$$

where $g_\lambda(y)$ is the unique tempered fundamental solution of the determinant

$$D=D(\lambda)=-\Delta+(-v\cdot\nabla+\lambda)^2. \tag{7.13}$$

Thus,

$$g_\lambda(y)=F^{-1}_{k\to y}\frac{1}{k^2+(ivk+\lambda)^2}=F^{-1}_{k\to y}\frac{1}{k^2+(i|v|k_1+\lambda)^2},\quad y\in\mathbb{R}^3. \tag{7.14}$$

Note that the denominator does not vanish for $\operatorname{Re}\lambda>0$. This implies

Lemma 7.2 *The operator G_λ with the integral kernel $G_\lambda(y-y')$ is continuous operator $H^1(\mathbb{R}^3)\oplus L^2(R^3)\to H^2(\mathbb{R}^3)\oplus H^1(R^3)$ for $\operatorname{Re}\lambda>0$.*

From (7.5) and (7.6) we obtain the convolution representation

$$\left.\begin{array}{rcl}\Psi &=& -(v\cdot\nabla-\lambda)g_\lambda*\Psi_0+g_\lambda*\Pi_0+(g_\lambda*\nabla\rho)\cdot Q\\[2ex] \Pi &=& \Delta g_\lambda*\Psi_0-(v\cdot\nabla-\lambda)g_\lambda*\Pi_0-(v\cdot\nabla-\lambda)(g_\lambda*\nabla\rho)\cdot Q\end{array}\right| \tag{7.15}$$

Let us compute $g_\lambda(y)$ explicitly. First consider the case $v=0$. The fundamental solution of the operator $-\Delta+\lambda^2$ is

$$g_\lambda(y)=\frac{e^{-\lambda|y|}}{4\pi|y|}. \tag{7.16}$$

Thus, in the case $v=0$ we have

$$G_\lambda(y-y')=\left(\begin{array}{cc}-\lambda & -1\\ -\Delta & -\lambda\end{array}\right)\frac{e^{-\lambda|y-y'|}}{4\pi|y-y'|}.$$

For general $v=(|v|,0,0)$ with $|v|<1$ the denominator in (7.14), which is the Fourier symbol of D, reads

$$\begin{aligned}\hat D(k) &= k^2+(i|v|k_1+\lambda)^2=(1-v^2)k_1^2+k_2^2+k_3^2+2i|v|k_1\lambda+\lambda^2\\ &= (1-v^2)(k_1+\frac{i|v|\lambda}{1-v^2})^2+k_2^2+k_3^2+\varkappa^2,\end{aligned} \tag{7.17}$$

where

$$\varkappa^2=\frac{v^2\lambda^2}{1-v^2}+\lambda^2=\frac{\lambda^2}{1-v^2}. \tag{7.18}$$

Therefore, setting $\gamma:=1/\sqrt{1-v^2}$, we have

$$\varkappa=\gamma\lambda. \tag{7.19}$$

Return to x-space:

$$D=-\frac{1}{\gamma^2}(\nabla_1+\gamma\varkappa_1)^2-\nabla_2^2-\nabla_3^2+\varkappa^2,\qquad \varkappa_1:=\gamma|v|\lambda \tag{7.20}$$

Define $\tilde y_1:=\gamma y_1$ and $\tilde\nabla_1:=\partial/\partial\tilde y_1$. Then

$$D=-(\tilde\nabla_1+\varkappa_1)^2-\nabla_2^2-\nabla_3^2+\varkappa^2. \tag{7.21}$$

Thus, its fundamental solution is

$$g_\lambda(y)=\frac{e^{-\varkappa|\tilde y|-\varkappa_1\tilde y_1}}{4\pi|\tilde y|},\quad \tilde y:=(\gamma y_1,y_2,y_3). \tag{7.22}$$

By (7.19), (7.20) we obtain

$$0<\operatorname{Re}\varkappa_1<\operatorname{Re}\varkappa,\qquad \operatorname{Re}\lambda>0. \tag{7.23}$$

Let us state the result which we have got above.

Lemma 7.3 *i) The operator $D=D(\lambda)$ is invertible in $L^2(\mathbb{R}^3)$ for $\operatorname{Re}\lambda>0$ and its fundamental solution (7.22) decays exponentially in y.*

ii) The formulas (7.22) and (7.19), (7.20) imply that for every fixed y, the Green function $g_\lambda(y)$ admits an analytic continuation in λ to the entire complex plane $\mathbb{C}$.

Lemma 7.4 *The matrix function $M(\lambda)$ (respectively, $M^{-1}(\lambda)$) admits an analytic (respectively meromorphic) continuation to the entire complex plane $\mathbb{C}$.*

Proof From the first equation of (7.15) and the last equation of (7.3) it follows that

$$H_{jj}(\lambda) = \langle g_\lambda * \partial_j \rho, \partial_j \rho \rangle \tag{7.24}$$

and thus, by (7.22),

$$|H_{jj}(\lambda)| = |\langle g_\lambda * \partial_j \rho, \partial_j \rho \rangle| \le \int \frac{C}{|x-y|} \partial_j \rho(x) \partial_j \rho(y) dx dy < \infty.$$

By (7.9), (7.10) this implies

$$\sup_{\mathrm{Re}\,\lambda \ge 0} |F(\lambda)| < \infty. \tag{7.25}$$

The analytic continuation of $M(\lambda)$ exists by the expressions (7.24) and Lemma 7.3 ii), since the function $\rho(x)$ is compactly supported by (1.9). The inverse matrix is then meromorphic since it exists for large $\mathrm{Re}\,\lambda$. The latter follows from (7.10) since $H(\lambda) \to 0$, $\mathrm{Re}\,\lambda \to \infty$, by (7.9). □

Proposition 7.5 *The matrix $M^{-1}(i\omega)$ is analytic in $\omega \in \mathbb{R} \setminus \{0\}$.*

Proof It suffices to prove that the limit matrix $M(i\omega) := M(i\omega + 0)$ is invertible for $\omega \ne 0$, $\omega \in \mathbb{R}$ if ρ satisfies the Wiener condition (1.10), and $|v| < 1$. Let $|\omega| > 0$. One has

$$\det M(i\omega) = \det \begin{pmatrix} i\omega I_3 & -B_v \\ -F(i\omega) & i\omega I_3 \end{pmatrix} = -(\omega^2 + \nu^3 f_1)(\omega^2 + \nu f)^2, \quad \omega \in \mathbb{R}, \tag{7.26}$$

where $F(i\omega) := F(i\omega + 0)$, $f_1 := F_{11}(i\omega)$, and $f := F_{22}(i\omega) = F_{33}(i\omega)$. The invertibility of $M(i\omega)$ follows from (7.26) by the following lemma, whose proof is based on the Sokhotsky-Plemelj formula, see [37, Chapter VII, formula (58)].

Lemma 7.6 *If (1.10) holds, then for $\omega \in \mathbb{R}$ the imaginary part of the matrix $\frac{\omega}{|\omega|} F(i\omega)$ is negative definite, i.e. $\frac{\omega}{|\omega|} \mathrm{Im}\, F_{jj}(i\omega) < 0$, $j = 1, 2, 3$.*

Proof Since $F(i\omega) = H(i\omega + 0) - K$, where the matrix K is real, we will consider only the matrix $H(i\omega + 0)$. For $\varepsilon > 0$ we have

$$H_{jj}(i\omega + \varepsilon) = \int \frac{k_j^2 |\hat\rho(k)|^2 dk}{k^2 - (|v|k_1 + \omega - i\varepsilon)^2}, \quad j = 1, 2, 3. \tag{7.27}$$

Consider the denominator

$$\hat D(i\omega + \varepsilon, k) = k^2 - (|v|k_1 + \omega - i\varepsilon)^2.$$

$\hat D(i\omega, k) = 0$ on the ellipsoid T_ω if $|\omega| > 0$, where

$$T_\omega = \{k : (\nu k_1 - \frac{|v|\omega}{\nu})^2 + k_2^2 + k_3^2 = \frac{\omega^2}{\nu^2}\},$$

here $\nu = \sqrt{1 - v^2}$. From the Sokhotsky-Plemelj formula for C^1-functions it follows that

$$\mathrm{Im}\, H_{jj}(i\omega + 0) = -\frac{\omega}{|\omega|} \pi \int_{T_\omega} \frac{k_j^2 |\hat\rho(k)|^2}{|\nabla \hat D(i\omega, k)|} dS, \tag{7.28}$$

where dS is the element of the surface area. This immediately implies the statement of the Lemma since the integrand in (7.28) is positive by the Wiener condition (1.10). This completes the proofs of the lemma and the Proposition 7.5. □

Remark 7.7 The proof of Lemma 7.6 is the unique point in the paper where the Wiener condition is indispensable.

7.1 Time Decay of the Vector Components

Here we prove the decay (6.22) for the vector components $Q(t)$ and $P(t)$ of the solution $e^{A_1 t}X_0$. Formula (7.10) expresses the Laplace transforms $\tilde{Q}(\lambda), \tilde{P}(\lambda)$. Hence, the components are given by the integral

$$\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \frac{1}{2\pi}\int e^{i\omega t} M^{-1}(i\omega)\begin{pmatrix} Q_0 \\ P_0 + \Phi(i\omega) \end{pmatrix} d\omega. \tag{7.29}$$

Let us recall that in Proposition 6.7 ii) we assume that

$$X_0 \in \mathcal{Z}_v \cap \mathcal{E}_\beta, \ \ \beta = 4+\delta, \ \ 0<\delta<1/2. \tag{7.30}$$

Theorem 7.8 *The functions $Q(t)$, $P(t)$ are continuous and*

$$|Q(t)| + |P(t)| \le \frac{C(\rho,\tilde{v})}{(1+|t|)^{1+\delta}}\|X_0\|_\beta, \qquad t\ge 0. \tag{7.31}$$

Proof Note that the Proposition 7.5 alone is not sufficient for the proof of the convergence and decay of the integral (7.29). We need an additional information about the regularity of the matrix $M^{-1}(i\omega)$ at its singular point $\omega=0$, and some bounds at $|\omega|\to\infty$.

Let us split the integral (7.29) in two terms using the partition of unity $\zeta_1(\omega)+\zeta_2(\omega)=1$, $\omega\in\mathbb{R}$:

$$\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \frac{1}{2\pi}\int e^{i\omega t}(\zeta_1(\omega)+\zeta_2(\omega))\begin{pmatrix} \tilde{Q}(i\omega) \\ \tilde{P}(i\omega) \end{pmatrix} d\omega = \begin{pmatrix} Q_1(t) \\ P_1(t) \end{pmatrix} + \begin{pmatrix} Q_2(t) \\ P_2(t) \end{pmatrix}, \tag{7.32}$$

where the functions $\zeta_k(\omega)\in C^\infty(\mathbb{R})$ are supported by

$$\operatorname{supp}\zeta_1 \subset \{\omega\in\mathbb{R} : |\omega|<r+1\}, \ \ \operatorname{supp}\zeta_2 \subset \{\omega\in\mathbb{R} : |\omega|>r\}, \tag{7.33}$$

where r is introduced below in Lemma 7.10. We prove the decay (7.31) for (Q_1,P_1) and (Q_2,P_2) in Propositions 7.11 and 7.9 respectively.

Proposition 7.9 *The functions $Q_2(t)$, $P_2(t)$ are continuous for $t\ge 0$ and*

$$|Q_2(t)| + |P_2(t)| \le \frac{C(\rho,\tilde{v})}{(1+|t|)^{3+\delta}}\|X_0\|_\beta. \tag{7.34}$$

Proof We study the asymptotic behavior of $M^{-1}(\lambda)$ at infinity. Let us recall that $M^{-1}(\lambda)$ was originally defined for $\operatorname{Re}\lambda>0$, but it admits a meromorphic continuation to $\mathbb{C}$ (see Lemma 7.4).

Lemma 7.10 *There exist a matrix R_0 and a matrix-function $R_1(\omega)$, such that*

$$M^{-1}(i\omega) = \frac{R_0}{\omega} + R_1(\omega), \quad |\omega|>r>0, \qquad \omega\in\mathbb{R}, \tag{7.35}$$

where, for every $k=0,1,2,...$,

$$|\partial_\omega^k R_1(\omega)| \le \frac{C_k}{|\omega|^2}, \qquad |\omega|>r>0, \qquad \omega\in\mathbb{R}, \tag{7.36}$$

r is sufficiently large.

Proof The statement follows from the expilicit formulas (A.4) to (A.8) for the inverse matrix $M^{-1}(i\omega)$ and from the bound (7.25). □

By (7.35) and (7.11) we obtain that

$$\begin{pmatrix} Q_2(t) \\ P_2(t) \end{pmatrix} = \frac{1}{2\pi}\int e^{i\omega t}\zeta_2(\omega)\left[\frac{R_0}{\omega}\begin{pmatrix} Q_0 \\ P_0 \end{pmatrix} + \frac{R_0}{\omega}\begin{pmatrix} 0 \\ \Phi(i\omega) \end{pmatrix} + R_1\begin{pmatrix} Q_0 \\ P_0 \end{pmatrix} + R_1\begin{pmatrix} 0 \\ \Phi(i\omega) \end{pmatrix}\right]$$

$$= s(t)\begin{pmatrix} Q_0 \\ P_0 \end{pmatrix} + s(t) * \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \tag{7.37}$$

where (see(7.2))

$$s(t) := \Lambda^{-1}\left[\zeta_2(\omega)(\frac{R_0}{\omega} + R_1(\omega))\right]$$

and

$$f(t) := \Lambda^{-1}\Phi(i\omega) = \langle W^1(t)(\Psi_0, \Pi_0), \nabla\rho\rangle, \tag{7.38}$$

since Φ is given by (7.8) and (7.11). Note that $s(t)$ is continuous for $t \geq 0$, and

$$|s(t)| = \mathcal{O}(t^{-N}), \;\; t \to \infty, \;\; \forall\, N > 0. \tag{7.39}$$

Further,

$$|f(t)| \leq \frac{C(\rho, \tilde{v})}{(1+|t|)^{3+\delta}}\|X_0\|_\beta \tag{7.40}$$

by Lemma 7.15 below with $\alpha = \beta$. By (7.39), the formula (7.37) and the bound (7.40) imply (7.34). □

Now let us prove the decay for $Q_1(t)$ and $P_1(t)$. The proof will rely substantially on the symplectic orthogonality conditions. Namely, (7.30) implies that

$$\Omega(X_0, \tau_j) = 0, \;\; j = 1 \ldots 6. \tag{7.41}$$

Proposition 7.11 *The functions $Q_1(t)$ and $P_1(t)$ are continuous for $t \geq 0$ and*

$$|Q_1(t)| + |P_1(t)| \leq \frac{C(\rho, \tilde{v})\|X_0\|_\beta}{(1+t)^{1+\delta}}, \;\; t \geq 0. \tag{7.42}$$

Proof We provide the detailed computation of the vector components $\tilde{Q}(i\omega)$ and $\tilde{P}(i\omega)$.

Lemma 7.12

$$L(\omega) := M^{-1}(\omega) = \begin{pmatrix} \frac{1}{\omega}\mathcal{L}_{11} & \frac{1}{\omega^2}\mathcal{L}_{12} \\ \mathcal{L}_{21} & \frac{1}{\omega}\mathcal{L}_{22} \end{pmatrix}, \tag{7.43}$$

where $\mathcal{L}_{ij}(\omega)$, $i,j = 1,2$ are smooth diagonal 3×3-matrices, $\mathcal{L}_{ij}(\omega) \in C^\infty(-r-1; r+1)$, and

$$\mathcal{L}_{11} = i\mathcal{L}_{12}B_v^{-1}. \tag{7.44}$$

For proof see Appendix A. Now (7.10) implies that the vector components are given by

$$\tilde{Q}(i\omega) = \frac{1}{\omega}\mathcal{L}_{11}(\omega)Q_0 + \frac{1}{\omega^2}\mathcal{L}_{12}(\omega)(P_0 + \Phi(i\omega)), \tag{7.45}$$

$$\tilde{P}(i\omega) = \mathcal{L}_{21}(\omega)Q_0 + \frac{1}{\omega}\mathcal{L}_{22}(\omega)(P_0 + \Phi(i\omega)), \tag{7.46}$$

Lemma 7.13 *The symplectic orthogonality conditions (7.41) read*

$$P_0 + \Phi(0) = 0 \;\; \text{and} \;\; B_v^{-1}Q_0 + \Phi'(0) = 0. \tag{7.47}$$

For proof see Appendix B.

Step i) Let us prove (7.42) for $P_1(t)$ relying on the representation (7.46). Namely, (7.32) and (7.46) imply

$$P_1(t) = \Lambda^{-1}\zeta_1(\omega)\mathcal{L}_{21}(\omega)Q_0 + \Lambda^{-1}\zeta_1(\omega)\mathcal{L}_{22}(\omega)\frac{P_0 + \Phi(i\omega)}{\omega} = P_1'(t) + P_1''(t).$$

Obviously, the first term $P_1'(t)$ decays like $Ct^{-\infty}\|X_0\|_\beta$ by Lemma 7.12. The second term admits the convolution representation $P_1''(t) = \Lambda^{-1}\zeta_1\mathcal{L}_{22} * g(t)$, where

$$g(t) := \Lambda^{-1}\frac{P_0 + \Phi(i\omega)}{\omega}.$$

Now we use the symplectic orthogonality conditions (7.47) and obtain

$$g(t)=\Lambda^{-1}\frac{\Phi(i\omega)-\Phi(0)}{\omega}=i\int\limits_{\infty}^{t}f(s)ds. \tag{7.48}$$

Therefore, $P_1''(t)$ decays like $Ct^{-(2+\delta)}\|X_0\|_\beta$ for $t\geq 0$, since by (7.40)

$$|g(t)|\leq C(\rho,\tilde{v})(1+t)^{-(2+\delta)}\|X_0\|_\beta,\ \ t\geq 0. \tag{7.49}$$

Step ii) Now let us prove (7.42) for $Q_1(t)$. By (7.45), (7.44), and the symplectic orthogonality conditions (7.47),

$$\tilde{Q}(i\omega)=\frac{\mathcal{L}_{12}}{\omega}\left(iB_v^{-1}Q_0+\frac{P_0+\Phi(i\omega)}{\omega}\right)=\frac{\mathcal{L}_{12}}{\omega}\left(iB_v^{-1}Q_0+\frac{\Phi(i\omega)-\Phi(0)}{\omega}\right)=$$

$$\frac{\mathcal{L}_{12}}{\omega}(iB_v^{-1}Q_0+\tilde{g}(\omega))=\mathcal{L}_{12}\frac{iB_v^{-1}Q_0+\tilde{g}(0)+\tilde{g}(i\omega)-\tilde{g}(0)}{\omega}=\mathcal{L}_{12}\frac{\tilde{g}(i\omega)-\tilde{g}(0)}{\omega},$$

since $iB_v^{-1}Q_0+\tilde{g}(0)=0$ by the symplectic orthogonality conditions (7.47), because $\tilde{g}(0)=i\Phi'(0)$. Thus, $Q_1(t)=\Lambda^{-1}\zeta_1(\omega)\mathcal{L}_{12}*h(t)$ by (7.32), where

$$h(t):=\Lambda^{-1}\frac{\tilde{g}(i\omega)-\tilde{g}(0)}{\omega}=i\int\limits_{\infty}^{t}g(s)ds,$$

similarly to (7.48). This integral decays like $Ct^{-1-\delta}\|X_0\|_\beta$ for $t\geq 0$ by (7.49), hence (7.42) for $Q_1(t)$ is proved. The proof of Proposition 7.11 and Theorem 7.8 is complete. □

7.2 Time Decay of Fields

Here we construct the field components $\Psi(x,t),\Pi(x,t)$ of the solution $X(t)$ and prove their decay corresponding to (6.22). Let us denote $F(t)=(\Psi(\cdot,t),\Pi(\cdot,t))$. We will construct the fields solving the first two equations of (6.17), where A is given by (4.9). These two equations have the form

$$\dot{F}(t)=\begin{pmatrix} v\cdot\nabla & 1\\ \Delta & v\cdot\nabla\end{pmatrix}F+\begin{pmatrix}0\\ \nabla\rho\cdot Q(t)\end{pmatrix}. \tag{7.50}$$

By Theorem 7.8 we know that $Q(t)$ is continuous and

$$|Q(t)|\leq\frac{C(\rho,\tilde{v})\|X_0\|_\beta}{(1+t)^{1+\delta}},\qquad t\geq 0. \tag{7.51}$$

Hence, the Proposition 6.7 is reduced now to the following

Proposition 7.14 *i) Let a function $Q(t)\in C([0,\infty);\mathbb{R}^3)$, and $F_0\in\mathcal{F}$. Then the equation (7.50) admits a unique solution $F(t)\in C[0,\infty;\mathcal{F})$ with the initial condition $F(0)=F_0$.*
ii) If $X_0=(F_0;Q_0,P_0)\in\mathcal{E}_\beta$ and the decay (7.51) holds, the corresponding fields also decay uniformly in v:

$$\|F(t)\|_{-2-\delta}\leq\frac{C(\rho,\tilde{v})\|X_0\|_\beta}{(1+t)^{1+\delta}},\qquad t\geq 0, \tag{7.52}$$

for $|v|\leq\tilde{v}$ with any $\tilde{v}\in(0;1)$.

Proof The statement i) follows from the Duhamel representation

$$F(t)=W(t)F_0+\left[\int_0^t W(t-s)\begin{pmatrix}0\\ \nabla\rho\cdot Q(s)\end{pmatrix}ds\right],\qquad t\geq 0, \tag{7.53}$$

where $W(t)$ is the dynamical group of the modified wave equation

$$\dot{F}(t)=\begin{pmatrix} v\cdot\nabla & 1\\ \Delta & v\cdot\nabla\end{pmatrix}F(t). \tag{7.54}$$

The group $W(t)$ can be expressed through the group $W_0(t)$ of the wave equation

$$\dot\Phi(t) = \left(\begin{array}{cc} 0 & 1 \\ \Delta & 0 \end{array}\right)\Phi(t). \quad (7.55)$$

Namely, the problem (7.55) corresponds to (7.54), when $v = 0$, and it is easy to see that

$$[W(t)F(0)](x) = [W_0(t)F(0)](x + vt), \qquad x \in \mathbb{R}^3, \quad t \in \mathbb{R}. \quad (7.56)$$

The identity (7.56) implies the energy conservation law for the group $W(t)$: for $(\Psi(\cdot,t),\Pi(\cdot,t)) = W(t)F(0)$ we have

$$\int \left(|\Pi(x,t) - v\cdot\nabla\Psi(x,t)|^2 + |\nabla\Psi(x,t)|^2\right)dx = \text{const}, \qquad t \in \mathbb{R}.$$

In particular, this gives by (2.2)

$$\|W(t)F_0\|_{\mathcal{F}} \le C(\tilde v)\|F_0\|_{\mathcal{F}}, \;\; t \in \mathbb{R}. \quad (7.57)$$

This estimate and (7.53) imply the statement i).

Let us proceed to the statement ii).

Lemma 7.15 *For $\tilde v < 1$ and $F_0 \in \mathcal{F}_\alpha$, $\alpha > 1$, the following decay holds,*

$$\|W(t)F_0\|_{-\alpha} \le \frac{C(\alpha,\tilde v)}{(1+t)^{\alpha-1}}\|F_0\|_\alpha, \qquad t \ge 0, \quad (7.58)$$

for the dynamical group $W(t)$ corresponding to the modified wave equation (7.54) with $|v| < \tilde v$.

Proof For the case $v = 0$ the proof is provided in [24]. For a nonzero v with $|v| < \tilde v$ the proof is similar, we provide it for convenience.

We should estimate $\|W(t)F_0\|_{-\alpha}$ for large $t > 0$. Set $\varepsilon = (1-\tilde v)/2$. For an arbitrary sufficiently large $t \ge 1$ let us split the initial function F_0 in two terms, $F_0 = F'_{0,t} + F''_{0,t}$ such that

$$\|F'_{0,t}\|_\alpha + \|F''_{0,t}\|_\alpha \le C\|F_0\|_\alpha, \qquad t \gg 1, \quad (7.59)$$

where C does not depend on t, and

$$F'_{0,t}(x) = 0, \;\; |x| > \varepsilon t, \;\; F''_{0,t}(x) = 0, \;\; |x| < \varepsilon t - 1. \quad (7.60)$$

For an arbitrary $f \in \dot H^1$ and $\alpha > 1$ one has $|f|_{-\alpha} \le C\|f\|_{\dot H^1}$, see [24], formula (2.9). Now the estimate for $W(t)F''_{0,t}$ follows by (7.57), (7.60) and (7.59) :

$$\|W(t)F''_{0,t}\|_{-\alpha} \le C\|W(t)F''_{0,t}\|_{\mathcal{F}} \le C\|F''_{0,t}\|_{\mathcal{F}} \le C_1(\tilde v)\|F''_{0,t}\|_\alpha(1+|t|)^{-\alpha} \le C_2(\tilde v)\|F_0\|_\alpha(1+|t|)^{-\alpha}, \;\; t \ge 1. \quad (7.61)$$

It remains to estimate $W(t)F'_{0,t}$. First note that

$$W_0(t)F'_{0,t}(x) = 0 \;\text{ for }\; |x| < (1-\varepsilon)t \quad (7.62)$$

by the strong Huygen's principle for the group $W_0(t)$. The principle reads

$$W_0(x-y,t) = 0, \qquad |x-y| \ne t, \quad (7.63)$$

where $W_0(z,t)$ is the integral (distribution) matrix kernel of the operator $W_0(t)$. Further, from (7.62) it follows that

$$[W(t)F'_{0,t}](x) = 0 \;\text{ for }\; |x| < \varepsilon t$$

by (7.56) and since $|v| < \tilde v = 1 - 2\varepsilon$.

For an arbitrary $f \in \dot H^1$ such that $f(x) = 0$ in the region $\{|x| < \varepsilon t\}$, one has $|f|_{-\alpha} \le C(\tilde v)t^{-\alpha+1}\|f\|_{\dot H^1}$, see [24], the proof of Proposition 2.1. Applying to $f = W(t)F'_{0,t}$ we obtain by (7.57) that,

$$\|W(t)F'_{0,t}\|_{-\alpha} \le C(\tilde v)t^{-\alpha+1}\|W(t)F'_{0,t}\|_{\mathcal{F}} \le Ct^{-\alpha+1}\|F'_{0,t}\|_{\mathcal{F}} \le Ct^{-\alpha+1}\|F'_{0,t}\|_{\mathcal{F}_\alpha} \le Ct^{-\alpha+1}\|F_0\|_{\mathcal{F}_\alpha}.$$

The proof is complete. $\square$

Now the statement ii) of Proposition 7.14 follows from the Lemma 7.15 and the Duhamel representation (7.53). Indeed,

$$\|W(t)F_0\|_{-2-\delta}\le Ct^{-1-\delta}\|F_0\|_{2+\delta}\le Ct^{-1-\delta}\|X_0\|_{2+\delta}\le Ct^{-1-\delta}\|X_0\|_\beta$$

by Lemma 7.15 with $\alpha=2+\delta$. Further,

$$\|\int_0^t W(t-s)\begin{pmatrix}0\\ \nabla\rho\cdot Q(s)\end{pmatrix}ds\|_{-2-\delta}\le C\int_0^t\frac{\|(0,\nabla\rho\cdot Q(s))\|_{2+\delta}ds}{(1+(t-s))^{1+\delta}}$$

$$\le C'\int_0^t\frac{|Q(s)|ds}{(1+(t-s))^{1+\delta}}\le C''\|X_0\|_\beta\int_0^t\frac{ds}{(1+(t-s))^{1+\delta}(1+s)^{1+\delta}}$$

by Lemma 7.15 with $\alpha=2+\delta$, regularity properties of ρ, and (7.51). The last integral decays like $(1+t)^{-1-\delta}$ by a well known result on decay of a convolution. $\square$

8 Frozen Form of Transversal Dynamics

Now let us fix an arbitrary $t_1\in[0,t_*)$, and rewrite the equation (6.15) in a "frozen form"

$$\dot Z(t)=A_1Z(t)+(A(t)-A_1)Z(t)+\tilde N(t),\qquad 0\le t<t_*,\tag{8.1}$$

where $A_1=A_{v(t_1),v(t_1)}$ and

$$A(t)-A_1=\begin{pmatrix}[w(t)-v(t_1)]\cdot\nabla & 0 & 0 & 0\\ 0 & [w(t)-v(t_1)]\cdot\nabla & 0 & 0\\ 0 & 0 & 0 & B_{v(t)}-B_{v(t_1)}\\ 0 & 0 & \langle\nabla(\psi_{v(t)}-\psi_{v(t_1)}),\cdot\nabla\rho\rangle & 0\end{pmatrix}.$$

The next trick is important since it allows us to kill the "bad terms" $[w(t)-v(t_1)]\cdot\nabla$ in the operator $A(t)-A_1$.

Definition 8.1 *Let us change the variables $(y,t)\mapsto(y_1,t)=(y+d_1(t),t)$ where*

$$d_1(t):=\int_{t_1}^t(w(s)-v(t_1))ds,\qquad 0\le t\le t_1.\tag{8.2}$$

Next define

$$\begin{aligned}Z_1(t)&=(\Psi_1(y_1,t),\Pi_1(y_1,t),Q(t),P(t)):=(\Psi(y,t),\Pi(y,t),Q(t),P(t))\\ &=(\Psi(y_1-d_1(t),t),\Pi(y_1-d_1(t),t),Q(t),P(t)).\end{aligned}\tag{8.3}$$

Then we obtain the final form of the "frozen equation" for the transversal dynamics

$$\dot Z_1(t)=A_1Z_1(t)+B_1(t)Z_1(t)+N_1(t),\ \ 0\le t\le t_1,\tag{8.4}$$

where $N_1(t)=\tilde N(t)$ from the basic equation (6.15) expressed in terms of $y=y_1-d_1(t)$, and

$$B_1(t)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&B_{v(t)}-B_{v(t_1)}\\0&0&\langle\nabla(\psi_{v(t)}-\psi_{v(t_1)}),\cdot\nabla\rho\rangle&0\end{pmatrix}.$$

At the end of this section, we will derive appropriate bounds for the "remainder terms" $B_1(t)Z_1(t)$ and $N_1(t)$ in (8.4). First, note that we have by Lemma 6.2,

$$|B_{v(t)}-B_{v(t_1)}|\le|\int_{t_1}^t\dot v(s)\cdot\nabla_vB_{v(s)}ds|\le C\int_t^{t_1}\|Z(s)\|^2_{-\beta}ds.\tag{8.5}$$

Similarly,

$$|\langle\nabla(\psi_{v(t)}-\psi_{v(t_1)}),\cdot\nabla\rho\rangle|\le C\int_t^{t_1}\|Z(s)\|^2_{-\beta}ds. \tag{8.6}$$

Let us recall the following well-known inequality: for any $\alpha\in\mathbb{R}$

$$(1+|y+x|)^\alpha\le(1+|y|)^\alpha(1+|x|)^{|\alpha|},\qquad x,y\in\mathbb{R}^3. \tag{8.7}$$

Lemma 8.2 [17, Lemma 7.2]. *For $(\Psi,\Pi,Q,P)\in\mathcal{E}_\alpha$ with any $\alpha\in\mathbb{R}$ the following estimate holds:*

$$\|(\Psi(y_1-d_1),\Pi(y_1-d_1),Q,P)\|_\alpha\le\|(\Psi,\Pi,Q,P)\|_\alpha(1+|d_1|)^{|\alpha|}\ ,\qquad d_1\in\mathbb{R}^3. \tag{8.8}$$

Corollary 8.3 *The following bound holds*

$$\|N_1(t)\|_\beta\le\|Z_1(t)\|^2_{-\beta}(1+|d_1(t)|)^{3\beta}\ ,\quad 0\le t\le t_1. \tag{8.9}$$

Indeed, applying the previous lemma twice, once for $\alpha=\beta$ and once for $\alpha=-\beta$, we obtain from (6.16) that

$$\|N_1(t)\|_\beta\le(1+|d_1(t)|)^\beta\|\tilde N(t,Z(t))\|_\beta\le(1+|d_1(t)|)^\beta\|Z\|^2_{-\beta}\le(1+|d_1(t)|)^{3\beta}\|Z_1(t)\|^2_{-\beta}\ .$$

Corollary 8.4 *The following bound holds*

$$\|B_1(t)Z_1(t)\|_\beta\le C\|Z_1(t)\|_{-\beta}\int_t^{t_1}(1+|d_1(\tau)|)^{2\beta}\|Z_1(\tau)\|^2_{-\beta}d\tau\ ,\quad 0\le t\le t_1. \tag{8.10}$$

For proof we apply Lemma 8.2 with $\alpha=-\beta$ to (8.5) and (8.6) and use the fact that $B_1(t)Z_1(t)$ depends only on the finite-dimensional components of $Z_1(t)$.

9 Integral Inequality

Recall that $0<\delta<1/2$. The equation (8.4) can be written in the integral form:

$$Z_1(t)=e^{A_1t}Z_1(0)+\int_0^t e^{A_1(t-s)}[B_1Z_1(s)+N_1(s)]ds,\ \ 0\le t\le t_1. \tag{9.1}$$

We apply the symplectic orthogonal projection $\boldsymbol{P}_1:=\boldsymbol{P}_{v(t_1)}$ to both sides, and get

$$\boldsymbol{P}_1Z_1(t)=e^{A_1t}\boldsymbol{P}_1Z_1(0)+\int_0^t e^{A_1(t-s)}\boldsymbol{P}_1[B_1Z_1(s)+N_1(s)]ds.$$

We have used here that $\boldsymbol{P}_1$ commutes with the group e^{A_1t} since the space $\mathcal{Z}_1:=\boldsymbol{P}_1\mathcal{E}$ is invariant with respect to e^{A_1t}, see Remark 6.6. Applying (6.22) we obtain that

$$\|\boldsymbol{P}_1Z_1(t)\|_{-2-\delta}\le\frac{C}{(1+t)^{1+\delta}}\|Z_1(0)\|_\beta+C\int_0^t\frac{1}{(1+|t-s|)^{1+\delta}}\|B_1Z_1(s)+N_1(s)\|_\beta ds, \tag{9.2}$$

since the operator $\boldsymbol{P}_1$ is continuous in $\mathcal{E}_\beta$. Hence, from (9.2) and (8.9), (8.10) we obtain that

$$\begin{aligned}&\|\boldsymbol{P}_1Z_1(t)\|_{-2-\delta}\le\frac{C}{(1+t)^{1+\delta}}\|Z_1(0)\|_\beta\\&+C(\overline{d}_1)\int_0^t\frac{1}{(1+|t-s|)^{1+\delta}}\left[\|Z_1(s)\|_{-\beta}\int_s^{t_1}\|Z_1(\tau)\|^2_{-\beta}d\tau+\|Z_1(s)\|^2_{-\beta}\right]ds,\ \ 0\le t\le t_1,\end{aligned} \tag{9.3}$$

where $\overline{d}_1:=\sup_{0\le t\le t_1}|d_1(t)|$. Since $\|Z_1(t)\|_{\pm\beta}\le C(\overline{d}_1)\|Z(t)\|_{\pm\beta}$ by Lemma 8.2, we can rewrite (9.3) as

$$\begin{aligned}&\|\boldsymbol{P}_1Z_1(t)\|_{-2-\delta}\le\frac{C(\overline{d}_1)}{(1+t)^{1+\delta}}\|Z(0)\|_\beta\\&+C(\overline{d}_1)\int_0^t\frac{1}{(1+|t-s|)^{1+\delta}}\left[\|Z(s)\|_{-\beta}\int_s^{t_1}\|Z(\tau)\|^2_{-\beta}d\tau+\|Z(s)\|^2_{-\beta}\right]ds,\ \ 0\le t\le t_1,\end{aligned} \tag{9.4}$$

Let us introduce the *majorant*

$$m(t) := \sup_{s\in[0,t]} (1+s)^{1+\delta}\|Z(s)\|_{-\beta}\ , \qquad t\in[0,t_*). \tag{9.5}$$

To estimate $d_1(t)$ by $m(t_1)$ we note that

$$w(s)-v(t_1) = w(s)-v(s)+v(s)-v(t_1) = \dot c(s) + \int_s^{t_1} \dot v(\tau)d\tau \tag{9.6}$$

by (6.9). Hence, (8.2), Lemma 6.2 and the definition (9.5) imply

$$\begin{aligned}|d_1(t)| &= |\int_{t_1}^t (w(s)-v(t_1))ds| \le \int_t^{t_1}\Big(|\dot c(s)| + \int_s^{t_1}|\dot v(\tau)|d\tau\Big)ds \\ &\le Cm^2(t_1)\int_t^{t_1}\Big(\frac{1}{(1+s)^{2+2\delta}} + \int_s^{t_1}\frac{d\tau}{(1+\tau)^{2+2\delta}}\Big)ds \le Cm^2(t_1), \quad 0\le t\le t_1.\end{aligned} \tag{9.7}$$

We can replace in (9.4) the constants $C(\overline{d}_1)$ by C if $m(t_1)$ is bounded for $t_1\ge 0$. In order to do this replacement, we reduce the exit time. Let us denote by ε a fixed positive number which we will specify below.

Definition 9.1 t_*' *is the exit time*

$$t_*' = \sup\{t\in[0,t_*) : m(s)\le\varepsilon,\quad 0\le s\le t\}. \tag{9.8}$$

Now (9.4) and (9.7) imply that for $t_1 < t_*'$

$$\begin{aligned}&\|\boldsymbol{P}_1 Z_1(t)\|_{-2-\delta} \le \frac{C}{(1+t)^{1+\delta}}\|Z(0)\|_\beta \\ &+C\int_0^t \frac{1}{(1+|t-s|)^{1+\delta}}\Big[\|Z(s)\|_{-\beta}\int_s^{t_1}\|Z(\tau)\|^2_{-\beta}d\tau + \|Z(s)\|^2_{-\beta}\Big]ds,\ \ 0\le t\le t_1.\end{aligned} \tag{9.9}$$

10 Symplectic Orthogonality

Finally, we are going to change $\boldsymbol{P}_1 Z_1(t)$ by $Z(t)$ in the left hand side of (9.9). We will prove that it is possible using again that $d_\beta \ll 1$ in (2.15) and due to the following important bound:

Lemma 10.1 *For sufficiently small $\varepsilon > 0$, we have for $t_1 < t_*'$:*

$$\|Z(t)\|_{-2-\delta} \le C\|\boldsymbol{P}_1 Z_1(t)\|_{-2-\delta}, \quad 0\le t\le t_1, \tag{10.1}$$

where C depends only on ρ and $\tilde v$.

Proof The proof is based on the symplectic orthogonality (6.21), i.e.

$$\mathbf{\Pi}_{v(t)} Z(t) = 0, \quad t\in[0,t_1], \tag{10.2}$$

and on the fact that all the spaces $\mathcal{Z}(t) := \boldsymbol{P}_{v(t)}\mathcal{E}$ are almost parallel for all t.

Namely, we first note that $\|Z(t)\|_{-2-\delta} \le C(\varepsilon)\|Z_1(t)\|_{-2-\delta}$ by Lemma 8.2, since $|d_1(t)|\le C\varepsilon^2$ for $t\le t_1 < t_*'$ by (9.7). Therefore, it suffices to prove that

$$\|Z_1(t)\|_{-2-\delta} \le 2\|\boldsymbol{P}_1 Z_1(t)\|_{-2-\delta}, \qquad 0\le t\le t_1. \tag{10.3}$$

This estimate will follow from

$$\|\mathbf{\Pi}_{v_1} Z_1(t)\|_{-2-\delta} \le \frac{1}{2}\|Z_1(t)\|_{-2-\delta},\ \ 0\le t\le t_1\,, \tag{10.4}$$

since $\boldsymbol{P}_1 Z_1(t) = Z_1(t) - \mathbf{\Pi}_{v_1} Z_1(t)$, where $v_1 = v(t_1)$. To prove (10.4), we write (10.2) as

$$\mathbf{\Pi}_{v(t),1} Z_1(t) = 0, \quad t\in[0,t_1], \tag{10.5}$$

where $\mathbf{\Pi}_{v(t),1}Z_1(t)$ is $\mathbf{\Pi}_{v(t)}Z(t)$ expressed in terms of the variable $y_1 = y + d_1(t)$. Hence, (10.4) follows from (10.5) if the difference $\mathbf{\Pi}_{v_1} - \mathbf{\Pi}_{v(t),1}$ is small uniformly in t, i.e.

$$\|(\mathbf{\Pi}_{v_1} - \mathbf{\Pi}_{v(t),1})Z_1(t)\|_{-2-\delta} < \frac{1}{2}\|Z_1(t)\|_{-2-\delta}, \qquad 0 \le t \le t_1. \tag{10.6}$$

It remains to justify (10.6) for a sufficiently small $\varepsilon > 0$. In order to prove the bound (10.6), we will need the formula (6.19) and the following relation which follows from (6.19):

$$\mathbf{\Pi}_{v(t),1}Z_1(t) = \sum \mathbf{\Pi}_{jl}(v(t))\tau_{j,1}(v(t))\Omega(\tau_{l,1}(v(t)), Z_1(t)), \tag{10.7}$$

where $\tau_{j,1}(v(t))$ are the vectors $\tau_j(v(t))$ expressed in the variables y_1. In detail (cf. (3.3)),

$$\left.\begin{array}{rcl} \tau_{j,1}(v) & := & (-\partial_j\psi_v(y_1 - d_1(t)), -\partial_j\pi_v(y_1 - d_1(t)), e_j, 0), \\ \tau_{j+3,1}(v) & := & (\partial_{v_j}\psi_v(y_1 - d_1(t)), \partial_{v_j}\pi_v(y_1 - d_1(t)), 0, \partial_{v_j}p_v), \end{array}\right| \quad j = 1, 2, 3, \tag{10.8}$$

where $v = v(t)$. Thus, we have to estimate the difference of

$$\mathbf{\Pi}_{v_1}Z_1(t) = \sum \mathbf{\Pi}_{jl}(v_1)\tau_j(v_1, y_1)\Omega(\tau_l(v_1, y_1), Z_1(t, y_1))$$

and

$$\mathbf{\Pi}_{v(t),1}Z_1(t) = \sum \mathbf{\Pi}_{jl}(v(t))\tau_j(v(t), y_1 - d_1(t))\Omega(\tau_l(v(t), y_1 - d_1(t)), Z_1(t, y_1)).$$

The estimate is based on the following bounds. First,

$$|\mathbf{\Pi}_{jl}(v(t)) - \mathbf{\Pi}_{jl}(v(t_1))| = |\int_t^{t_1} \dot v(s)\cdot\nabla_v\mathbf{\Pi}_{jl}(v(s))ds| \le C\int_t^{t_1}|\dot v(s)|ds, \qquad 0 \le t \le t_1, \tag{10.9}$$

since $|\nabla_v\mathbf{\Pi}_{jl}(v(s))|$ is uniformly bounded by (6.8). Second,

$$|\Omega(\tau_l(v_1, y_1) - \tau_l(v(t), y_1 - d_1(t)), Z_1(t, y_1))| \le \|\tau_l(v_1, y_1) - \tau_l(v(t), y_1 - d_1(t))\|_{2+\delta}\|Z_1(t, y_1)\|_{-2-\delta}.$$

Further, since $|d_1(t)| \le C\varepsilon^2$ and $\nabla\tau_j$ are smooth and sufficiently fast decaying at infinity functions, Lemma 8.2 implies

$$\|\tau_{j,1}(v(t)) - \tau_j(v(t))\|_{2+\delta} \le C|d_1(t)| \le C\varepsilon^2, \qquad 0 \le t \le t_1 \tag{10.10}$$

for all $j = 1, 2, \ldots, 6$, where C depends only on δ and $\tilde v$. Finally,

$$\tau_j(v(t)) - \tau_j(v(t_1)) = \int_t^{t_1} \dot v(s)\cdot\nabla_v\tau_j(v(s))ds,$$

and therefore

$$\|\tau_j(v(t)) - \tau_j(v(t_1))\|_{2+\delta} \le C\int_t^{t_1}|\dot v(s)|ds, \qquad 0 \le t \le t_1. \tag{10.11}$$

At last, the bounds (10.6) will follow from (6.19), (10.7) and (10.10)-(10.9) if we establish that the integral in the right hand side of (10.11) and (10.9) can be made as small as we please by choosing $\varepsilon > 0$ sufficiently small. Indeed,

$$\int_t^{t_1}|\dot v(s)|ds \le Cm^2(t_1)\int_t^{t_1}\frac{ds}{(1+s)^{2+2\delta}} \le C\varepsilon^2, \qquad 0 \le t \le t_1. \tag{10.12}$$

The proof is complete. □

11 Decay of Transversal Component

Here we complete the proof of Proposition 6.3.

Step i) We fix an ε, $0 < \varepsilon \le r_{-\beta}(\tilde v)$ and $t_*' = t_*'(\varepsilon)$ for which Lemma 10.1 holds. Then the bound of type (9.9) holds with $\|\boldsymbol{P}_1Z_1(t)\|_{-2-\delta}$ in the left hand side replaced by $\|Z(t)\|_{-\beta}$:

$$\begin{aligned} \|Z(t)\|_{-\beta} &\le \|Z(t)\|_{-2-\delta} \le C\|\boldsymbol{P}_1Z_1(t)\|_{-2-\delta} \le \frac{C}{(1+t)^{1+\delta}}\|Z(0)\|_\beta \\ &+C\int_0^t \frac{1}{(1+|t-s|)^{1+\delta}}\left[\|Z(s)\|_{-\beta}\int_s^{t_1}\|Z(\tau)\|_{-\beta}^2d\tau + \|Z(s)\|_{-\beta}^2\right]ds, \ 0 \le t \le t_1 \end{aligned} \tag{11.1}$$

for $t_1 < t'_*$. This implies an integral inequality for the majorant $m(t)$ introduced by (9.5). Namely, multiplying both sides of (11.1) by $(1+t)^{1+\delta}$, and taking the supremum in $t \in [0, t_1]$, we get

$$m(t_1) \le C\|Z(0)\|_\beta + C \sup_{t\in[0,t_1]} \int_0^t \frac{(1+t)^{1+\delta}}{(1+|t-s|)^{1+\delta}} \left[\frac{m(s)}{(1+s)^{1+\delta}} \int_s^{t_1} \frac{m^2(\tau)d\tau}{(1+\tau)^{2+2\delta}} + \frac{m^2(s)}{(1+s)^{2+2\delta}}\right] ds$$

for $t_1 \le t'_*$. Taking into account that $m(t)$ is a monotone increasing function, we get

$$m(t_1) \le C\|Z(0)\|_\beta + C[m^3(t_1) + m^2(t_1)]I(t_1), \qquad t_1 \le t'_*. \tag{11.2}$$

where

$$I(t_1) = \sup_{t\in[0,t_1]} \int_0^t \frac{(1+t)^{1+\delta}}{(1+|t-s|)^{1+\delta}} \left[\frac{1}{(1+s)^{1+\delta}} \int_s^{t_1} \frac{d\tau}{(1+\tau)^{2+2\delta}} + \frac{1}{(1+s)^{2+2\delta}}\right] ds \le \overline{I} < \infty, \qquad t_1 \ge 0.$$

Therefore, (11.2) becomes

$$m(t_1) \le C\|Z(0)\|_\beta + C\overline{I}[m^3(t_1) + m^2(t_1)], \qquad t_1 < t'_*. \tag{11.3}$$

This inequality implies that $m(t_1)$ is bounded for $t_1 < t'_*$, and moreover,

$$m(t_1) \le C_1\|Z(0)\|_\beta, \qquad t_1 < t'_*, \tag{11.4}$$

since $m(0) = \|Z(0)\|_\beta$ is sufficiently small by (3.7).

Step ii) The constant C_1 in the estimate (11.4) does not depend on t_* and t'_* by Lemma 10.1. We choose d_β in (2.15) so small that $\|Z(0)\|_\beta < \varepsilon/(2C_1)$. It is possible due to (3.7). Then the estimate (11.4) implies that $t'_* = t_*$ and therefore (11.4) holds for all $t_1 < t_*$. Then the bound (9.7) holds for all $t < t_*$. Therefore, (6.4) also holds for all $t < t_*$. Finally, this implies that $t_* = \infty$, hence also $t'_* = \infty$ and (11.4) holds for all $t_1 > 0$ if d_β is small enough. □

12 Soliton Asymptotics

Here we prove our main Theorem 2.5 relying on the decay (6.13). First we will prove the asymptotics (2.16) for the vector components, and afterwards the asymptotics (2.17) for the fields.

Asymptotics for the vector components From (4.3) we have $\dot q = \dot b + \dot Q$, and from (6.15), (6.16) with $\beta = 4+\delta$, and (4.9) it follows that $\dot Q = B_{v(t)}P + \mathcal{O}(\|Z\|^2_{-\beta})$. Thus,

$$\dot q = \dot b + \dot Q = v(t) + \dot c(t) + B_{v(t)}P(t) + \mathcal{O}(\|Z\|^2_{-\beta}). \tag{12.1}$$

The equation (6.10) and the estimates (6.11), (6.13) imply

$$|\dot c(t)| + |\dot v(t)| \le \frac{C_1(\rho, \overline{v}, d_\beta)}{(1+t)^{2+2\delta}}, \qquad t \ge 0. \tag{12.2}$$

Therefore, $c(t) = c_+ + \mathcal{O}(t^{-(1+2\delta)})$ and $v(t) = v_+ + \mathcal{O}(t^{-(1+2\delta)})$, $t \to \infty$. Since $|P| \le \|Z\|_{-\beta}$, the estimate (6.13), and (12.2), (12.1) imply that

$$\dot q(t) = v_+ + \mathcal{O}(t^{-1-\delta}). \tag{12.3}$$

Similarly,

$$b(t) = c(t) + \int_0^t v(s)ds = v_+t + a_+ + \mathcal{O}(t^{-2\delta}), \tag{12.4}$$

hence the second part of (2.16) follows:

$$q(t) = b(t) + Q(t) = v_+t + a_+ + \mathcal{O}(t^{-2\delta}), \tag{12.5}$$

since $Q(t) = \mathcal{O}(t^{-1-\delta})$ by (6.13).

Asymptotics for the fields We apply the approach developed in [15], see also [13, 14, 16, 19]. For the field part of the solution, $F(t) = (\psi(x,t), \pi(x,t))$ let us define the *accompanying soliton field* as

$$F_{\mathrm{v(t)}}(t) = (\psi_{\mathrm{v(t)}}(x - q(t)), \pi_{\mathrm{v(t)}}(x - q(t))),$$

where we define now $\mathrm{v}(t) = \dot{q}(t)$, cf. (12.1). Then for the difference $Z(t) = F(t) - F_{\mathrm{v(t)}}(t)$ we obtain easily the equation [19], Eq. (2.5),

$$\dot{Z}(t) = AZ(t) - \dot{\mathrm{v}} \cdot \nabla_{\mathrm{v}} F_{\mathrm{v}(t)}(t), \quad A(\psi, \pi) = (\pi, \Delta\psi).$$

Then

$$Z(t) = W_0(t)Z(0) - \int_0^t W_0(t-s)[\dot{\mathrm{v}}(s) \cdot \nabla_{\mathrm{v}} F_{\mathrm{v}(s)}(s)]ds. \tag{12.6}$$

Since $\|(\psi_{v_+}, \pi_{v_+})(x - v_+t - a_+) - F_{\mathrm{v}(t)}(t)\|_{\mathcal{F}} = \mathcal{O}(t^{-2\delta})$ by (12.3) and (12.5), to obtain the asymptotics (2.17) it suffices to prove that $Z(t) = W_0(t)\Psi_+ + r_+(t)$ with some $\Psi_+ \in \mathcal{F}$ and $\|r_+(t)\|_{\mathcal{F}} = \mathcal{O}(t^{-\delta})$. This is equivalent to

$$W_0(-t)Z(t) = \mathbf{\Psi}_+ + r_+'(t), \tag{12.7}$$

where $\|r_+'(t)\|_{\mathcal{F}} = \mathcal{O}(t^{-\delta})$ since $W_0(t)$ is a unitary group in the Sobolev space $\mathcal{F}$ by the energy conservation for the free wave equation. Finally, (12.7) holds since (12.6) implies that

$$W_0(-t)Z(t) = Z(0) + \int_0^t W_0(-s)R(s)ds, \quad R(s) = \dot{\mathrm{v}}(s) \cdot \nabla_{\mathrm{v}} F_{\mathrm{v}(s)}(s), \tag{12.8}$$

where the integral in the right hand side of (12.8) converges in the Hilbert space $\mathcal{F}$ with the rate $\mathcal{O}(t^{-\delta})$. The latter holds since $\|W_0(-s)R(s)\|_{\mathcal{F}} = \mathcal{O}(s^{-1-\delta})$ by the unitarity of $W_0(-s)$ and the decay rate $\|R(s)\|_{\mathcal{F}} = \mathcal{O}(s^{-1-\delta})$. Let us prove this rate of decay. It suffices to prove that $|\dot{\mathrm{v}}(s)| = \mathcal{O}(s^{-1-\delta})$, or equivalently $|\dot{p}(s)| = \mathcal{O}(s^{-1-\delta})$. Substitute (4.2) to the last equation of (1.2) and obtain

$$\begin{aligned} \dot{p}(t) &= \int \left[\psi_{v(t)}(x - b(t)) + \Psi(x - b(t), t)\right] \nabla\rho(x - b(t) - Q(t))dx \qquad (12.9)\\ &= \int \psi_{v(t)}(y)\nabla\rho(y)dy + \int \psi_{v(t)}(y)\left[\nabla\rho(y - Q(t)) - \nabla\rho(y)\right]dy + \int \Psi(y,t)\nabla\rho(y - Q(t))dy. \end{aligned}$$

The first integral in the right hand side is zero by the stationary equations (2.8). The second integral is $\mathcal{O}(t^{-1-\delta})$, since $Q(t) = \mathcal{O}(t^{-1-\delta})$, and by the conditions (1.9) on ρ. Finally, the third integral is $\mathcal{O}(t^{-1-\delta})$ by the estimate (6.13). The proof is complete. □

A Appendix: Structure of the matrix $M^{-1}(i\omega)$

We prove Lemmas 7.10 and 7.12. Recall that for $\omega \in \mathbb{R}$

$$M(i\omega) = \begin{pmatrix} i\omega I_3 & -B_v \\ -F(i\omega) & i\omega I_3 \end{pmatrix},$$

where

$$B_v = \begin{pmatrix} \nu^3 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad F(i\omega) = \begin{pmatrix} f_1(\omega) & 0 & 0 \\ 0 & f(\omega) & 0 \\ 0 & 0 & f(\omega) \end{pmatrix}.$$

Here $f_1(\omega) = F_{11}(i\omega + 0)$, $f(\omega) = F_{22}(i\omega + 0) = F_{33}(i\omega + 0)$ with

$$F_{jj}(\lambda) = \int dk \frac{|\hat{\rho}|^2 k_j^2}{k^2 + (\lambda + ik_1 v)^2} - \int dk \frac{|\hat{\rho}|^2 k_j^2}{k^2 - (k_1 v)^2}. \tag{A.1}$$

$F_{jj}(\lambda)$ are analytic functions in $\mathbb{C}$ by Lemma 7.4. Thus,

$$F_{jj}(\lambda) = F_{jj}(0) + F_{jj}'(0)\lambda + \frac{F_{jj}''(0)}{2}\lambda^2 + \ldots$$

Here $F_{jj}(0) = 0$ by (A.1). Further, by (A.1)

$$F'_{jj}(\lambda) = -2\int dk\, k_j^2|\hat\rho|^2 \frac{\lambda + ivk_1}{(k^2 + (\lambda + ivk_1)^2)^2} \tag{A.2}$$

and

$$F'_{jj}(0) = -2iv\int dk\, k_j^2|\hat\rho|^2 \frac{k_1}{(k^2 - (vk_1)^2)^2} = 0,$$

since the integrand function is odd in k_1. Hence, we obtain $F_{jj}(\lambda) = \lambda^2 r_j(\lambda)$, where $r_j(\lambda)$ is analytic in $\mathbb{C}$. Note that $r_j(0) = F''_{jj}(0)/2$. By (A.2) we have

$$F''_{jj}(\lambda) = -2\int dk\, k_j^2|\hat\rho|^2 \frac{k^2 - 3(\lambda + ivk_1)^2}{(k^2 + (\lambda + ivk_1)^2)^3}$$

and finally,

$$F_{jj}(i\omega) = -\omega^2 r_j(\omega), \quad r_j(0) = -\int dk\, k_j^2|\hat\rho|^2 \frac{k^2 + 3(vk_1)^2}{(k^2 - (vk_1)^2)^3}. \tag{A.3}$$

Let us denote $r(\omega) = r_2(\omega) = r_3(\omega)$. Then

$$L(\omega) := M^{-1}(i\omega)\begin{pmatrix} L_{11}(\omega) & L_{12}(\omega) \\ L_{21}(\omega) & L_{22}(\omega) \end{pmatrix}, \tag{A.4}$$

where

$$L_{11}(\omega) = \begin{pmatrix} \frac{-i\omega}{\omega^2 + \nu^3 f_1(\omega)} & 0 & 0 \\ 0 & \frac{-i\omega}{\omega^2 + \nu f(\omega)} & 0 \\ 0 & 0 & \frac{-i\omega}{\omega^2 + \nu f(\omega)} \end{pmatrix} = \frac{1}{\omega}\begin{pmatrix} \frac{-i}{1 - \nu^3 r_1(\omega)} & 0 & 0 \\ 0 & \frac{-i}{1 - \nu r(\omega)} & 0 \\ 0 & 0 & \frac{-i}{1 - \nu r(\omega)} \end{pmatrix} \tag{A.5}$$

by (A.3); we denote the last matrix $\mathcal{L}_{11}(\omega)$. Similarly,

$$L_{12}(\omega) = \begin{pmatrix} \frac{-\nu^3}{\omega^2 + \nu^3 f_1(\omega)} & 0 & 0 \\ 0 & \frac{-\nu}{\omega^2 + \nu f(\omega)} & 0 \\ 0 & 0 & \frac{-\nu}{\omega^2 + \nu f(\omega)} \end{pmatrix} = \frac{1}{\omega^2}\begin{pmatrix} \frac{-\nu^3}{1 - \nu^3 r_1(\omega)} & 0 & 0 \\ 0 & \frac{-\nu}{1 - \nu r(\omega)} & 0 \\ 0 & 0 & \frac{-\nu}{1 - \nu r(\omega)} \end{pmatrix}, \tag{A.6}$$

we denote the last matrix $\mathcal{L}_{12}(\omega)$. Note that

$$\mathcal{L}_{11}(\omega) = i\mathcal{L}_{12}(\omega)B_v^{-1}.$$

Further,

$$L_{21} = \begin{pmatrix} \frac{-f_1(\omega)}{\omega^2 + \nu^3 f_1(\omega)} & 0 & 0 \\ 0 & \frac{-f(\omega)}{\omega^2 + \nu f(\omega)} & 0 \\ 0 & 0 & \frac{-f(\omega)}{\omega^2 + \nu f(\omega)} \end{pmatrix} = \begin{pmatrix} \frac{r_1(\omega)}{1 - \nu^3 r_1(\omega)} & 0 & 0 \\ 0 & \frac{r(\omega)}{1 - \nu r(\omega)} & 0 \\ 0 & 0 & \frac{r(\omega)}{1 - \nu r(\omega)} \end{pmatrix}, \tag{A.7}$$

so we put $\mathcal{L}_{21} = L_{21}$. Finally,

$$L_{22}(\omega) = L_{11}(\omega) = \frac{1}{\omega}\mathcal{L}_{11}(\omega), \tag{A.8}$$

and thus, $\mathcal{L}_{22}(\omega) = \mathcal{L}_{11}(\omega)$. Note that the denominators of the matrix elements of each matrix $\mathcal{L}_{11}$ to $\mathcal{L}_{22}$ are nonzero at $\omega = 0$, since $r_1(0) < 0$ and $r(0) < 0$ by (A.3). For $\omega \neq 0$ the denominators are nonzero by Lemma 7.6. This completes the proof of Lemma 7.12. Finally, (A.4) to (A.8) imply Lemma 7.10, since $r_j(\omega) = -F_{jj}(i\omega)/\omega^2$, where $F_{jj}(i\omega)$ are bounded functions by (7.25).

B Appendix: Symplectic orthogonality conditions

Let us check that the symplectic orthogonality conditions (7.41) with $j = 1, 2, 3$ read the first equation of (7.47). By (7.8),

$$\Phi(\Psi_0, \Pi_0)(0) = i\langle ikv\hat{\Psi}_0 + \hat{\Pi}_0, \frac{k\hat{\rho}}{\hat{D}(0)}\rangle, \qquad \hat{D}(0) = k^2 - (kv)^2.$$

On the other hand, by (7.41) with $j = 1, 2, 3$, and (3.3), (2.10), we have

$$0 = \Omega(X_0, \tau_j) = -\langle \Psi_0, \partial_j \pi_v\rangle + \langle \Pi_0, \partial_j \psi_v\rangle - P_0 \cdot e_j = \langle \hat{\Psi}_0, \frac{k_j kv\hat{\rho}}{\hat{D}(0)}\rangle + \langle \hat{\Pi}_0, \frac{ik_j\hat{\rho}}{\hat{D}(0)}\rangle - P_0 \cdot e_j$$

$$= \langle kv\hat{\Psi}_0, \frac{k_j\hat{\rho}}{\hat{D}(0)}\rangle - i\langle \hat{\Pi}_0, \frac{k_j\hat{\rho}}{\hat{D}(0)}\rangle - (P_0)_j = -\Phi_j(\Psi_0, \Pi_0)(0) - (P_0)_j.$$

Now let us check that the conditions (7.41) with $j = 4, 5, 6$ read the second equation of (7.47). By (7.8)

$$\Phi'(0) = i\langle \frac{\hat{\Psi}_0}{\hat{D}(0)}, k\hat{\rho}\rangle - i\langle 2ikv\frac{ikv\hat{\Psi}_0 + \hat{\Pi}_0}{\hat{D}^2(0)}, k\hat{\rho}\rangle = \langle \frac{(k^2 + (kv)^2)i\hat{\Psi}_0 + 2kv\Pi_0}{(k^2 - (kv)^2)^2}, k\hat{\rho}\rangle,$$

where the integral converges by the condition (1.11). On the other hand, by (7.41) with $j = 4, 5, 6$, and (3.3), (2.10), we have for $j = 1, 2, 3$

$$0 = \Omega(X_0, \tau_{3+j}) = \langle \hat{\Psi}_0, \partial_{v_j}\pi_v\rangle - \langle \hat{\Pi}_0, \partial_{v_j}\psi_v\rangle + Q_0 \cdot \partial_{v_j} p_v$$

$$= \langle \hat{\Psi}_0, -ik_j\frac{k^2 + (kv)^2}{\hat{D}^2(0)}\hat{\rho}\rangle + \langle \hat{\Pi}_0, 2k_j\frac{kv\hat{\rho}}{\hat{D}^2(0)}\rangle + Q_0 \cdot \partial_{v_j} p_v = \Phi'_j(0) + (B_v^{-1}Q_0)_j,$$

since $Q_0 \cdot \partial_{v_j} p_v = Q_0 \cdot B_v^{-1} e_j = B_v^{-1} Q_0 \cdot e_j$.

References

[1] M. Abraham, Theorie der Elektrizitat, Band 2: Elektromagnetische Theorie der Strahlung, Teubner, Leipzig (1905).

[2] D. Bambusi and S. Cuccagna, *On dispersion of small energy solutions of the nonlinear Klein Gordon equation with a potential*, http://arxiv.org/abs/0908.4548.

[3] H. Beresticky and P.L. Lions, Nonlinear scalar field equations, *Arch. Rat. Mech. and Anal.* **82** (1983), no.4, 313-375.

[4] V.S. Buslaev and G.S. Perelman, *On nonlinear scattering of states which are close to a soliton*, in "Méthodes Semi-Classiques, Vol.2 Colloque International (Nantes, juin 1991)", Asterisque, **208** (1992) 49-63.

[5] V.S. Buslaev and G.S. Perelman, *Scattering for the nonlinear Schrödinger equation: states close to a soliton*, St. Petersburg Math. J., **4** (1993), 1111-1142.

[6] V.S. Buslaev and G.S. Perelman, *On the stability of solitary waves for nonlinear Schrödinger equations*, Nonlinear evolution equations (N.N.Uraltseva, eds.), Transl. Ser. 2, 164, Amer. Math. Soc., Providence, RI, 1995, pp. 75-98.

[7] S. Cuccagna, *Stabilization of solutions to nonlinear Schrödinger equations*, Comm. Pure Appl. Math., **54** (2001), 1110-1145.

[8] S. Cuccagna, *On asymptotic stability of ground states of NLS*, Rev. Math. Phys., **15** (2003), 877-903.

[9] S. Cuccagna, *The Hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states*, http:// arxiv.org/abs/0910.3797.

[10] S. Cuccagna and T. Mizumachi, *On asymptotic stability in energy space of ground states for Nonlinear Schrödinger equations*, Comm. Math. Phys., **284** (2008), 51-87.

[11] W. Eckhaus and A. van Harten, “The Inverse Scattering Transformation and the Theory of Solitons. An Introduction”, North-Holland, Amsterdam, 1981.

[12] M. Esteban, V. Georgiev, and E. Sere, Stationary solutions of the Maxwell-Dirac and the Klein-Gordon-Dirac equations, *Calc. Var. Partial Differ. Equ.* **4** (1996), no.3, 265-281.

[13] V. Imaikin, A. Komech, and P. Markowich, *Scattering of solitons of the Klein-Gordon equation coupled to a classical particle*, J. Math. Phys., **44** (2003), 1202-1217.

[14] V. Imaikin, A. Komech, and H. Spohn, *Soliton-like asymptotics and scattering for a particle coupled to Maxwell field*, Russ. J. Math. Phys., **9** (2002), 428-436.

[15] V. Imaikin, A. Komech, and H. Spohn, *Scattering theory for a particle coupled to a scalar field*, Discrete Contin. Dyn. Syst., **10** (2003), 387-396.

[16] V. Imaikin, A. Komech, and H. Spohn, *Rotating charge coupled to the Maxwell field: scattering theory and adiabatic limit*, Monatsh. Math., **142** (2004), 143-156.

[17] V. Imaikin, A. Komech, and B. Vainberg, *On scattering of solitons for the Klein-Gordon equation coupled to a particle*, Comm. Math. Phys., **268** (2006), 321-367.

[18] V. Imaikin, A. Komech, and B. Vainberg, *On scattering of solitons for wave equation coupled to a particle*, pp. 249-273 in: “CRM Proceedings and Lecture Notes”, **42** (2007).

[19] A. Komech, M. Kunze, and H. Spohn, *Effective dynamics for a mechanical particle coupled to a wave field*, Comm. Math. Phys., **203** (1999), 1-19.

[20] A. Komech, M. Kunze, and H. Spohn, *Long-time asymptotics for a classical particle interacting with a scalar wave field*, Comm. Partial Differential Equations, **22** (1997), 307-335.

[21] A.I. Komech and H. Spohn, *Soliton-like asymptotics for a classical particle interacting with a scalar wave field*, Nonlinear Anal., **33** (1998), 13-24.

[22] A.I. Komech and H. Spohn, Long-time asymptotics for the coupled Maxwell-Lorentz equations, *Comm. Partial Diff. Eqns.* **25** (2000), no. 3/4, 558-585.

[23] V. Imaikin, A.I. Komech and N. Mauser. Soliton-type asymptotics for the coupled Maxwell-Lorentz equations, *Ann. Inst. Poincaré, Phys. Theor.* **5** (2004), 1117-1135.

[24] E.A. Kopylova, *Weighted energy decay for 3D wave equation*, Asymptotic Anal., **65** (2009), 1-16.

[25] Y. Martel and F. Merle, *Asymptotic stability of solitons of the gKdV equations with general nonlinearity*, Math. Ann., **341** (2008), 391-427.

[26] J. Miller and M. Weinstein, *Asymptotic stability of solitary waves for the regularized long-wave equation*, Comm. Pure Appl. Math., **49** (1996), 399-441.

[27] R.L. Pego and M.I. Weinstein, *On asymptotic stability of solitary waves*, Phys. Lett. A, **162** (1992), 263-268.

[28] R.L. Pego and M.I. Weinstein, *Asymptotic stability of solitary waves*, Comm. Math. Phys., **164** (1994), 305-349.

[29] M. Reed and B. Simon, “Methods of modern mathematical physics. Vol. IV, Analysis of operators”, Acad. Press, N.-Y., 1978.

[30] I.M. Sigal, *Nonlinear wave and Schrödinger equations. I. Instability of periodic and quasiperiodic solutions*, Comm. Math. Phys., **153** (1993), 297-320.

[31] A. Soffer and M.I. Weinstein, *Multichannel nonlinear scattering for nonintegrable systems*, in “Proceedings of Conference on an Integrable and Nonintegrable Systems, June, 1988, Oleron, France, Integrable Systems and Applications”, Springer Lecture Notes in Physics, Volume 342.

[32] A. Soffer and M.I. Weinstein, *Multichannel nonlinear scattering in nonintegrable systems*, Comm. Math. Phys., **133** (1990), 119-146.

[33] A. Soffer and M.I. Weinstein, *Multichannel nonlinear scattering and stability II. The case of Anisotropic and potential and data*, J. Differential Equations, **98** (1992), 376-390.

[34] A. Soffer and M.I. Weinstein, *Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations*, Invent. Math., **136** (1999), 9-74.

[35] A. Soffer and M.I. Weinstein, *Selection of the ground state for nonlinear Schrödinger equations*, ArXiv:nlin.PS/0308020, 2003.

[36] H. Spohn, “Dynamics of Charged Particles and Their Radiation Field”, Cambridge University Press, Cambridge, 2004.

[37] B. Vainberg, “Asymptotic methods in equations of mathematical physics”, Gordon and Breach Publishers, New-York-London, 1989.

Email adresses of the authors:

ivm61@mail.ru

alexander.komech@univie.ac.at

brvainbe@uncc.edu