Lectures on Elliptic Partial Differential Equations

(Method of Pseudodifferential Operators)

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Programme

I. Tempered distributions: differentiation, multiplication by smooth functions, Fourier transform.
II. Sobolev spaces, Sobolev embedding theorems.
III. Pseudodifferential operators: continuity in Sobolev spaces, composition.
IV. Elliptic partial differential operators: regulariser, the Fredholm properties, the Schauder a priori estimates.

Methods: Operations with distributions, calculation of their Fourier transforms, construction of fundamental solutions, continuity of integral and pseudodifferential operators in Hilbert spaces, application of pseudodifferential operators to elliptic partial differential operators, construction of regulariser.

Main Goals: To give an introduction to the applications of the methods of modern functional analysis to partial differential equations.

References

[1] A. Komech, Linear Partial Differential Equations with Constant Coefficients, p.127-260 in: Yu.V. Egorov, A.I. Komech, M.A. Shubin, Elements of the Modern Theory of Partial Differential Equations. Springer, Berlin, 1999.

[2] A. Komech, Book of Practical PDEs, http://www.math.tamu.edu/~comech/posobie

[3] I.M. Gel'fand, G.E. Shilov, Generalized functions. Vol. I: Properties and operations. New York and London: Academic Press, 1964.

[4] W. Rudin, Functional analysis. McGraw-Hill, NY, 1991.

[5] L. Schwartz, Méthodes Mathematiques pour les Sciences Physiques. Hermann, Paris, 1961.

[6] M. Shubin, Pseudodifferential operators and spectral theory. Springer, Berlin, 2001.

[7] M. Taylor, Pseudo differential operators. Springer, Berlin, 1974.

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Programme of exam

I. Fourier transform of test functions and tempered distributions: Lemmas 2.2, 2.7, 2.8, and Theorem 2.10.

II. Sobolev spaces: Lemmas 4.3, 4.4, 6.1.

III. First Sobolev's Embedding Theorem: Theorem 5.3.

IV. Sobolev's Compactness Embedding Theorem: Theorem 7.2.

V. Strongly elliptic PDE with constant coefficients: Theorem 8.5.

VI. Schur's lemma: Lemma 10.1.

VII. Boundedness of multiplication operator: Lemma 10.4.

VIII. Boundedness of PDO: Theorem 10.7.

IX. Composition of PDO: Theorem 11.1.

X. Regulariser for strongly elliptic PDE with variable coefficients: Theorem 12.4.

XI. Solvability of strongly elliptic PDE with variable coefficients: Theorem 13.3.

Contents

1 Tempered Distributions

1.1 Definitions and examples

Let us introduce the *Schwartz space* $\mathcal{S}(\mathbb{R})$ of test functions.

Definition 1.1 *i)* $\mathcal{S} = \mathcal{S}(\mathbb{R})$ *is the space of functions* $\varphi(x) \in C^\infty(\mathbb{R})$ *such that*

$$\|\varphi\|_{\alpha,N} := \sup_{x\in\mathbb{R}} (1+|x|)^N |\partial_x^\alpha \varphi(x)| < \infty \tag{1.1}$$

for any $\alpha, N = 1, 2, ...,$ *where* $\partial_x^\alpha \varphi(x) := \varphi^{(\alpha)}(x)$.
ii) The sequence $\varphi_n(x) \xrightarrow{\mathcal{S}} \varphi(x)$ *iff*

$$\|\varphi_n - \varphi\|_{\alpha,N} \to 0, \qquad n \to \infty \tag{1.2}$$

for any $\alpha, N = 1, 2, ...$

Let us note that (1.1) implies the bound

$$|\partial_x^\alpha \varphi(x)| \le \|\varphi\|_{\alpha,N}(1+|x|)^{-N}, \qquad x \in \mathbb{R}. \tag{1.3}$$

for any $\alpha, N = 1, 2, ...$ Hence, we have

Corollary 1.2 *For any* $\alpha = 0, 1, 2, ...$ *and* $q \in \mathbb{R}$ *the bound holds*

$$|\partial_x^\alpha \varphi(x)| \le \|\varphi\|_{\alpha,N}(1+|x|)^{-q}, \qquad x \in \mathbb{R} \tag{1.4}$$

for any $N \ge q$.

Example 1.3 *The function* $\varphi(x) = e^{-x^2}$ *belongs to the Schwartz space* $\mathcal{S}$ *(check this!).*

Definition 1.4 i) $\mathcal{S}'(\mathbb{R})$ *is the space of linear continuous functionals on* $\mathcal{S}(\mathbb{R})$*, i.e.* $f \in \mathcal{S}'(\mathbb{R})$ *if* f *is the map* $\mathcal{S} \to \mathcal{S}$*, and the following two conditions hold:*

$$\textbf{Linearity}: \quad f(\alpha\varphi + \beta\psi) = \alpha f(\varphi) + \beta f(\psi) \quad \textit{for any } \alpha, \beta \in \mathbb{C} \textit{ and } \varphi, \psi \in \mathcal{S}; \tag{1.5}$$

$$\textbf{Continuity}: \quad f(\varphi_n) \to f(\varphi) \textit{ if } \varphi_n \xrightarrow{\mathcal{S}} \varphi. \tag{1.6}$$

ii) *The functionals* $f \in \mathcal{S}'(\mathbb{R})$ *are called* **tempered distributions**.
iii) *The scalar product* $\langle f(x), \varphi(x)\rangle$ *denotes the value of a functional* $f \in \mathcal{S}'(\mathbb{R})$ *at the test function* $\varphi \in \mathcal{S}(\mathbb{R})$*, i.e.*

$$\langle f(x), \varphi(x)\rangle := f(\varphi), \qquad \varphi \in \mathcal{S}. \tag{1.7}$$

Example 1.5 *Let us consider a continuous function* $f(x) \in C(\mathbb{R})$ *satisfying the bounds*

$$|f(x)| \le C(1+|x|)^p, \qquad x \in \mathbb{R} \tag{1.8}$$

with some constants $C, p \in \mathbb{R}$*. Let us define the functional*

$$\langle f, \varphi\rangle := \int f(x)\varphi(x)dx, \qquad \varphi \in \mathcal{S}. \tag{1.9}$$

Lemma 1.6 *The integral (1.9) converges, and the functional is tempered distribution.*

Proof i) First we note that $|f(x)| \le \|\varphi\|_{0,N}(1+|x|)^{-p-2}$ by the bound (1.4) with an $q = p+2$, hence the integrand of (1.9) is bounded by $C\|\varphi\|_{0,N}(1+|x|)^{-2}$. Therefore, the integral converges:

$$|\langle f, \varphi\rangle| \le C\|\varphi\|_{0,N}\int(1+|x|)^{-2}dx < \infty. \tag{1.10}$$

ii) The linearity (1.5) of the functional follows from the properties of the integral.
iii) The continuity (1.6) of the functional follows from the estimate (1.10): if $\varphi_n \xrightarrow{\mathcal{S}} 0$ as $n \to \infty$, then $\|\varphi\|_{0,N} \to 0$, hence

$$|\langle f, \varphi_n\rangle| \le C\|\varphi_n\|_{0,N}\int(1+|x|)^{-2}dx \to 0, \qquad n \to \infty. \tag{1.11}$$

■

Remark 1.7 *The bounds (1.9) motivate the term "tempered distribution" for the corresponding functionals.*

Exercise 1.8 *Prove that functional of type (1.9) is tempered distribution if:*
i) $f(x) \in L^1(\mathbb{R})$,
ii) $f(x) \in L^2(\mathbb{R})$,
iii) $f(x) \in L^p(\mathbb{R})$ *with a* $p > 1$.
Hint: *Use the bound (1.4) with a suitable q depending on p. For ii) use the Cauchy-Schwarz inequality, and for iii) use the Hölder inequality.*

Exercise 1.9 *Let us check that* $e^x \notin \mathcal{S}'$. **Hint:** *Construct a sequence* $\varphi_n \in \mathcal{S}$ *such that* $\varphi_n \xrightarrow{\mathcal{S}} 0$ *as* $n \to \infty$ *while* $\langle f, \varphi_n\rangle \not\to 0$. *For example, take* $\varphi_n(x) = \phi(x-n)$, *where* ϕ *is an arbitrary nonnegative function* $\phi \in \mathcal{D}$ *with* $\int \phi(x)dx \ne 0$.

Remark 1.10 *We have the* **continuous** *inclusion* $\mathcal{D}(\mathbb{R}) \subset \mathcal{S}(R)$ *(check this!). Hence, each tempered distribution* $f \in \mathcal{S}'$ *belongs also to* $\mathcal{D}'$ *since the scalar product* $\langle f, \varphi\rangle$ *is linear and continuous for all test functions* $\varphi \in \mathcal{S}$, *hence also for all* $\varphi \in \mathcal{D}$. *Therefore, we have the map* $f \in \mathcal{S}' \mapsto f \in \mathcal{D}'$

Exercise 1.11 *Prove that the map* $\mathcal{D}' \mapsto \mathcal{S}'$ *is* **injective**, *so the map is* **the inclusion**: $\mathcal{D}' \subset \mathcal{S}'$.
Hint: *The space* $\mathcal{D}$ *is* **dense everywhere** *in* $\mathcal{S}$ *(check this!).*

Exercise 1.12 *Check that* $\delta(x) \in \mathcal{S}'$ *and* $pv\frac{1}{x} \in \mathcal{S}'$.

1.2 Differentiation of tempered distributions

Definition 1.13 *For* $f \in \mathcal{S}'$ *let us define the derivative* f' *as the following functional on* $\mathcal{S}$:

$$\langle f', \varphi\rangle := -\langle f, \varphi'\rangle, \qquad \varphi \in \mathcal{S}. \tag{1.12}$$

Lemma 1.14 $f' \in \mathcal{S}'$ *if* $f \in \mathcal{S}'$.

Exercise 1.15 *Prove this lemma.* **Hint:** $\varphi_n' \xrightarrow{\mathcal{S}} \varphi'$ *if* $\varphi_n \xrightarrow{\mathcal{S}} \varphi$ *(check this!).*

1.3 Convergence of tempered distributions

Definition 1.16 $f_n \xrightarrow{\mathcal{S}'} f$ *if*

(1.13) $$\langle f_n, \varphi\rangle \to \langle f, \varphi\rangle, \qquad \varphi \in \mathcal{S}.$$

Exercise 1.17 *Check that the differentiation is continuous operator in* $\mathcal{S}'$*, i.e.* $f_n' \xrightarrow{\mathcal{S}'} f'$ *if* $f_n \xrightarrow{\mathcal{S}'} f$*.*

1.4 Multiplication by smooth functions

Let us consider smooth functions $M(x)$ satisfying the bounds

(1.14) $$|M^{(k)}(x)| \le C(k)(1+|x|)^{p(k)}$$

with some constants $C(k), p(k) \in \mathbb{R}$ **for every** $k = 0, 1, 2...$

Example 1.18 *i) Any polynomial* $M(x) = \sum_{k=0}^{N} M_k x^k$ *satisfies the bounds (1.14) (find p(k) !).*
ii) The exponential function e^x **does not satisfy** *the bounds (1.14) (check this!).*

Lemma 1.19 *Let* $M(x)$ *satisfies the bounds (1.14). Then*
i) $M\varphi \in \mathcal{S}$ *for* $\varphi \in \mathcal{S}$*, and*
ii) Multiplication map $M : \varphi(x) \mapsto M(x)\varphi(x)$ *is linear continuous map* $\mathcal{S} \to \mathcal{S}$*.*

Proof i) By definition (1.1), we have to check that

(1.15) $$\sup_{x\in\mathbb{R}} (1+|x|)^N |\partial_x^\alpha (M(x)\varphi(x))| < \infty$$

for any $\alpha, N = 1, 2, ...$ This follows from the Leibniz formula

(1.16) $$\partial_x^\alpha (M(x)\varphi(x)) = \sum_{k=0}^{\alpha} C_\alpha^k M^{(k)}(x)\varphi^{(\alpha-k)}(x))$$

by the condition (1.14) since $\sup_{x\in\mathbb{R}} (1+|x|)^{N+p(k)} |\partial_x^\alpha \varphi(x)| < \infty$ by the bounds (1.4) with $q = N+p(k)$.
ii) The continuity of the map means that

(1.17) $$\sup_{x\in\mathbb{R}} (1+|x|)^N |\partial_x^\alpha (M(x)\psi_n(x))| \to 0, \qquad n \to \infty,$$

if $\sup_{x\in\mathbb{R}} (1+|x|)^N |\partial_x^\alpha \varphi_n(x)| \to 0$. This follows by arguments of previous step i). Finally, the linearity of the multiplication map is obvious. ■

Definition 1.20 *For* $f \in \mathcal{S}'$ *let us set*

(1.18) $$\langle M(x)f(x), \varphi(x)\rangle = \langle f(x), M(x)\varphi(x)\rangle, \qquad \varphi \in \mathcal{S}.$$

Here the right hand side is defined since $M(x)\varphi(x) \in \mathcal{S}$ *by previous lemma i).*

Lemma 1.21 *i) The functional (1.18) is tempered distribution, and*
ii) Multiplication operator $M : f(x) \mapsto M(x)f(x)$ *is linear continuous map* $\mathcal{S}' \to \mathcal{S}'$ *if* $M(x)$ *satisfies the bounds (1.14).*

Proof i) The functional (1.18) is obviously linear. Let us check its continuity: if $\varphi_n \xrightarrow{\mathcal{S}} 0$, then

(1.19) $$\langle M(x)f(x), \varphi_n(x)\rangle = \langle f(x), M(x)\varphi_n(x)\rangle \to 0, \qquad n \to \infty,$$

since $M(x)\varphi_n(x) \xrightarrow{\mathcal{S}} 0$ by previous lemma ii), and the functional f is continuous on $\mathcal{S}$.
ii) It remains to prove continuity of the map M in $\mathcal{S}'$: if $f_n \xrightarrow{\mathcal{S}'} f$, then

(1.20) $$\langle M(x)f_n(x), \varphi(x)\rangle := \langle f_n(x), M(x)\varphi(x)\rangle \to \langle f(x), M(x)\varphi(x)\rangle, \qquad \varphi \in \mathcal{S}.$$

Hence, $Mf_n \xrightarrow{\mathcal{S}'} Mf$ by definition of the convergence of tempered distributions. ■

2 Fourier Transform of Tempered Distributions

We are going to define the Fourier transform for tempered distributions. First we need to study the Fourier transform of the test functions.

2.1 Fourier Transform of Test Functions

Definition 2.1 *For $\varphi \in \mathcal{S}(\mathbb{R})$ the Fourier transform is defined by*

$$F\varphi(k) := \hat{\varphi}(k) := \int_{\mathbb{R}} e^{ikx}\varphi(x)dx, \quad k \in \mathbb{R}, \tag{2.1}$$

where $e^{ikx} := \cos kx + i \sin kx$.

It is well known that the inversion of the Fourier transform is given by

$$\varphi(x) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ikx}\hat{\varphi}(k)dk, \quad x \in \mathbb{R}. \tag{2.2}$$

Lemma 2.2
i) For any test function $\varphi \in \mathcal{S}$, its Fourier transform $\hat{\varphi}(k)$ also belongs to the Schwartz space $\mathcal{S}$.
ii) The operator $F : \varphi \mapsto \hat{\varphi}$ is linear and continuous in $\mathcal{S}$.

Proof i) We have to prove the bounds (1.1) for the Fourier transform $\varphi(k)$. i.e.

$$\|\hat{\varphi}\|_{\alpha,N} := \sup_{k\in\mathbb{R}}(1+|k|)^N|\partial_k^\alpha\hat{\varphi}(k)| < \infty, \qquad \alpha, N = 1, 2, ... \tag{2.3}$$

Equivalently

$$\sup_{k\in\mathbb{R}}|k|^N|\partial_k^\alpha\hat{\varphi}(k)| < \infty, \qquad \alpha, N = 1, 2, ... \tag{2.4}$$

Exercise 2.3 *Check that the bounds (2.4) imply (2.3).* **Hint:** *Prove the estimates*

$$\|\hat{\varphi}\|_{\alpha,N} \le C_N \sum_{M=0}^{N} \sup_{k\in\mathbb{R}}|k|^M|\partial_k^\alpha\hat{\varphi}(k)| \tag{2.5}$$

To prove bounds (2.4), note very important formulas of differentiation of the Fourier integral:

$$\partial_k\hat{\varphi}(k) = \int_{\mathbb{R}}(ix)e^{ikx}\varphi(x)dx = F[(ix)\psi(x)], \quad k \in \mathbb{R}. \tag{2.6}$$

Another important formula of multiplication we obtain integrating by parts:

$$k\hat{\varphi}(k) = \int_{\mathbb{R}}(-i\partial_x)e^{ikx}\varphi(x)dx = \int_{\mathbb{R}} e^{ikx}(i\partial_x)\varphi(x)dx = F[(i\partial_x)\varphi(x)],\ k \in \mathbb{R}. \tag{2.7}$$

In operator form the identities (2.6) and (2.7) can be written as

$$\partial_k\ F = F\ ix, \qquad k\ F = F\ i\partial_x \tag{2.8}$$

Applying α times the first formula and N times the second one, we obtain

$$k^N\partial_k^\alpha\hat{\varphi}(k) = F[(i\partial_x)^N\Big((ix)^\alpha\varphi(x)\Big)]. \tag{2.9}$$

This can be written as

$$k^N\partial_k^\alpha\hat\varphi(k)=\int e^{ikx}(i\partial_x)^N\Big((ix)^\alpha\varphi(x)\Big)dx, \tag{2.10}$$

that implies the bound

$$|k^N\partial_k^\alpha\hat\varphi(k)|\le\int|(i\partial_x)^N\Big((ix)^\alpha\varphi(x)\Big)|dx. \tag{2.11}$$

Now we apply the Leibniz formula for differentiation of the product and obtain

$$(i\partial_x)^N\Big((ix)^\alpha\varphi(x)\Big)=\sum_{M=0,\dots,N}C_N^{N-M}[(i\partial_x)^M(ix)^\alpha](i\partial_x)^{N-M}\varphi(x). \tag{2.12}$$

This formula implies that

$$|(i\partial_x)^N\Big((ix)^\alpha\varphi(x)\Big)|\le C_N\sum_{M=0,\dots,N}(1+|x|)^{\alpha-M}]|\varphi^{(N-M)}(x)|\le B_{\alpha,N}\|\varphi\|_{N,\alpha+2}(1+|x|)^{-2}. \tag{2.13}$$

Substituting into (2.11), we obtain that

$$|k^N\partial_k^\alpha\hat\varphi(k)|\le C_1\|\varphi\|_{N,\alpha+2}\int(1+|x|)^{-2}dx<\infty. \tag{2.14}$$

that implies the desired bound (2.4).

ii) The linearity of the operator F in the space $\mathcal{S}$ is obvious. To prove the continuity of the operator F in the space $\mathcal{S}$, we note that

$$\|\varphi\|_{\alpha,N}\le D_{\alpha,N}\|\varphi\|_{N,\alpha+2} \tag{2.15}$$

by the estimates (2.5) and bounds (2.14). Now the continuity follows from definition of the convergence in $\mathcal{S}$. ■

2.2 Definition

Lemma 2.4 *For any test functions $f(x),\varphi(k)\in\mathcal{S}$ the following identity holds*

$$\langle\hat f(k),\varphi(k)\rangle=\langle f(x),\hat\varphi(x)\rangle. \tag{2.16}$$

Proof The left hand side admits the following representation:

$$\langle\hat f(k),\varphi(k)\rangle=\int\Big(\frac{1}{2\pi}\int e^{ikx}f(x)dx\Big)\varphi(k)dk. \tag{2.17}$$

Applying here the Fubini theorem, we obtain

$$\langle\hat f(k),\varphi(k)\rangle=\int f(x)\Big(\frac{1}{2\pi}\int e^{ikx}\varphi(k)dk\Big)dx \tag{2.18}$$

that is the right hand side of (2.16). ■

Exercise 2.5 *Check that the Fubini theorem is applicable here.*

Now we are going to define the Fourier transform for tempered distributions. The definition can be done using the following identity:

Definition 2.6 *For any distribution $f \in \mathcal{S}'$ let us define the functional*

$$\langle \hat{f}(k), \varphi(k)\rangle = \langle f(x), \hat{\varphi}(x)\rangle, \qquad \varphi \in \mathcal{S}, \tag{2.19}$$

where the right hand side is well defined since $\hat{\varphi} \in \mathcal{S}$ by Lemma 2.2 i).

Lemma 2.7
i) For any tempered distribution $f \in \mathcal{S}'$, its Fourier transform $\hat{f}$ is also a tempered distribution.
ii) The operator $F : f \mapsto \hat{f}$ is linear and continuous in $\mathcal{S}'$.

Proof i) The linearity of the functional $\hat{f}$ follows easy: for any numbers $\alpha, \beta \in \mathbb{C}$ and test functions $\varphi, \psi \in \mathcal{S}$, we have

$$\begin{aligned}\langle \hat{f}(k), \alpha\varphi(k) + \beta\psi(k)\rangle = \langle f(x), \alpha\hat{\varphi}(x) + \beta\hat{\psi}(x)\rangle &= \alpha\langle f(x), \hat{\varphi}(x)\rangle + \beta\langle f(x), \hat{\psi}(x)\rangle \\ &= \alpha\langle \hat{f}(k), \varphi(k)\rangle + \beta\langle \hat{f}(k), \psi(k)\rangle,\end{aligned} \tag{2.20}$$

where the first and last identities follows by definition 2.6, while the middle one follows by the linearity of the functional f.

Similarly, the continuity of $\hat{f}$ follows from definition 2.6 and continuity of the functional f: if $\varphi_n(k) \xrightarrow{\mathcal{S}} \varphi(k)$ as $n \to \infty$, we have

$$\langle \hat{f}(k), \varphi_n(k)\rangle = \langle f(x), \hat{\varphi}_n(x)\rangle \to \langle f(x), \hat{\varphi}(x)\rangle = \langle \hat{f}(k), \varphi(k)\rangle \tag{2.21}$$

since $\hat{\varphi}_n(x) \xrightarrow{\mathcal{S}} \hat{\varphi}(x)$ by Lemma 2.2 ii).

ii) The continuity of the map $F : \mathcal{S}' \to \mathcal{S}'$ follows similarly: if $f_n(k) \xrightarrow{\mathcal{S}} f(k)$ as $n \to \infty$, we have

$$\langle \hat{f}_n(k), \varphi(k)\rangle = \langle f_n(x), \hat{\varphi}(x)\rangle \to \langle f(x), \hat{\varphi}(x)\rangle = \langle \hat{f}(k), \varphi(k)\rangle, \tag{2.22}$$

where the identities follow from definition 2.6. Finally, the linearity of the map F is easy to check (Please check!) ■

2.3 Parseval-Plancherel theory

Now the Fourier transform $\hat{f}(k)$ is defined for any tempered distribution $f(x)$. Let us study $\hat{f}(k)$ for the Lebesgue functions $f(x) \in L^p(\mathbb{R})$ with $p \geq 1$. This is possible since $L^p(\mathbb{R}) \subset \mathcal{S}'$ for every $p \geq 1$.

I. First consider $p = 1$.

Lemma 2.8 *For $f(x) \in L^1(\mathbb{R})$ the Fourier transform $\hat{f}(k)$ is a bounded continuous function, i.e. $\hat{f} \in C_b(\mathbb{R})$. It is given by the Fourier integral*

$$Ff(k) := \hat{f}(k) := \int_{\mathbb{R}} e^{ikx} f(x)dx, \quad k \in \mathbb{R}, \tag{2.23}$$

where identities hold in the sense of distributions though the right hand side is a classical continuous function.

Proof By definition 2.19 and the Fubini theorem, we have

$$\langle \hat{f}(k), \varphi(k)\rangle = \langle f(x), \hat{\varphi}(x)\rangle = \int_{\mathbb{R}} f(x)\Big(\int_R e^{ikx}\varphi(k)dk\Big)dx = \int_{\mathbb{R}}\Big(\int_R e^{ikx} f(x)dx\Big)\varphi(k)dk \tag{2.24}$$

for any $\varphi \in \mathcal{S}$ (check the conditions of the Fubini theorem!). The comparison of the first and last terms implies the formula (2.23) in the sense of the functionals.

Let us check that the integral (2.23) is bounded and continuous function of $k \in \mathbb{R}$. The boundedness follows from the estimate

$$|\hat{f}(k)| \le \int_{\mathbb{R}} |f(x)|dx < \infty, \quad k \in \mathbb{R}. \tag{2.25}$$

The continuity means that $\hat{f}(k_n) \to \hat{f}(k)$ if $k_n \to k$ as $n \to \infty$, i.e.

$$\int_{\mathbb{R}} e^{ik_n x} f(x)dx \to \int_{\mathbb{R}} e^{ikx} f(x)dx, \ \ k_n \to k. \tag{2.26}$$

This follows from the Lebesgue dominated convergence theorem since
i) The integrand converges:

$$e^{ik_n x} f(x) \to e^{ikx} f(x) \quad \text{for almost all} \ \ x \in \mathbb{R}. \tag{2.27}$$

ii) The integrand admits summable majorant which does not depend on n:

$$|e^{ik_n x} f(x)| \le |f(x)| \quad \text{for almost all} \ \ x \in \mathbb{R}. \tag{2.28}$$

The lemma is proved. ■

Exercise 2.9 *Check that for* $f(x) \in L^1(\mathbb{R})$ *we have*

$$\hat{f}(k) \to 0, \qquad |k| \to \infty. \tag{2.29}$$

Hints: i) Check (2.29) for characteristic functions of intervals.
ii) Approximate the function $f(x)$, in the norm of the space $L^1(\mathbb{R})$, by finite linear combinations of the characteristic functions.
iii) Use the bound (2.25).

II. Now consider $p = 2$.

Theorem 2.10 *(The Parseval Theorem) For* $f(x) \in L^2 = L^2(\mathbb{R})$ *the Fourier transform* $\hat{f}(k)$ *also belongs to* $L^2(\mathbb{R})$*, and the Parseval identity holds:*

$$\|\hat{f}\|^2 = 2\pi \|f\|^2. \tag{2.30}$$

Proof *Step i)* First let us prove (2.30) for the functions $f(x) \in \mathcal{S}$. For this purpose substitute $\varphi(k) = \overline{\hat{f}(k)}$ into (2.16): this is possible since $\overline{\hat{f}(k)} \in \mathcal{S}$ by Lemma 2.2 i). Let us note that $\hat{\varphi}(x) = \int e^{ikx} \overline{\hat{f}(k)} dk = \overline{\int e^{-ikx} \hat{f}(k) dk} = 2\pi \overline{f(x)}$ according to the inversion formula (2.2). Hence, (2.16) with $\varphi(k) = \overline{\hat{f}(k)}$ gives that

$$\langle \hat{f}(k), \overline{\hat{f}(k)} \rangle = 2\pi \langle f(x), \overline{f}(x) \rangle, \tag{2.31}$$

i.e. (2.30) and the Parseval Theorem 2.10 are proved for the functions $f(x) \in \mathcal{S}$.
Step ii) For general functions $f(x) \in L^2(\mathbb{R})$ we take an approximating sequence $f_n(x) \in \mathcal{S}(\mathbb{R})$ such that

$$\|f_n(x) - f(x)\|_{L^2} \to 0, \qquad n \to \infty. \tag{2.32}$$

Such sequence exists since $\mathcal{S}$ is dense in L^2 (Please check the density!). The convergence (2.32) implies that f_n is the Cauchy sequence, i.e.

$$\|f_n(x) - f_m(x)\|_{L^2} \to 0, \qquad n, m \to \infty. \tag{2.33}$$

Applying (2.30), we obtain from (2.33) that $\hat{f}_n$ is also the Cauchy sequence, i.e.

$$\|\hat{f}_n(k) - \hat{f}_m(k)\|_{L^2} \to 0, \qquad n, m \to \infty. \tag{2.34}$$

Now we use the completeness of the Hilbert space $L^2(\mathbb{R})$: every Cauchy sequence has limit function $g(k) \in L^2$, i.e.

$$\|\hat{f}_n(k) - g(k)\|_{L^2} \to 0, \qquad n \to \infty. \tag{2.35}$$

Step iii) Now let us prove that $g = \hat{f}$. Namely, the convergence (2.32) implies that (Please check this!)

$$f_n(x) \xrightarrow{\mathcal{S}'} f(x), \qquad n \to \infty. \tag{2.36}$$

Similarly, the convergence (2.35) implies that

$$\hat{f}_n(k) \xrightarrow{\mathcal{S}'} g(k), \qquad n \to \infty. \tag{2.37}$$

However, the convergence (2.36) implies that

$$\hat{f}_n(k) \xrightarrow{\mathcal{S}'} \hat{f}(k), \qquad n \to \infty. \tag{2.38}$$

by continuity of the Fourier transform in $\mathcal{S}'$ (Lemma 2.7 ii)). Finally, (2.37) and (2.38) imply that $\hat{f} = g \in L^2$.

Step iv) It remains to prove the Parseval identity (2.30). We have proved the identity for the test functions from $\mathcal{S}$, hence

$$\|\hat{f}_n\|^2 = 2\pi \|f_n\|^2. \tag{2.39}$$

Let us take here the limit $n \to \infty$. Then we get

$$\|g\|^2 = 2\pi \|f\|^2 \tag{2.40}$$

using (2.32) in the right hand side and (2.35) in the left hand side. Finally, this implies (2.30) taking into account that $g = \hat{f}$. ∎

Corollary 2.11 *The Parseval identity (2.30) means that the operator* $\frac{1}{\sqrt{2\pi}} F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *is an isometry, i.e.*

$$\|\frac{1}{\sqrt{2\pi}} \hat{f}(k)\|_{L^2(\mathbb{R})} = \|f(x)\|_{L^2(\mathbb{R})}. \tag{2.41}$$

3 Generalisation to n variables

3.1 Definitions and examples

Let us introduce test functions and tempered distributions of n variables $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $n > 1$. Let us denote $|x| = \sqrt{x_1^2 + ... + x_n^2}$, and $\partial^\alpha \varphi(x) := \partial_1^{\alpha_1} ... \partial_n^{\alpha_n} \varphi(x) = \varphi^{(\alpha)}(x)$ for any multiindex $\alpha = (\alpha_1, ..., \alpha_n)$ with $\alpha_j = 0, 1, 2, ...$ Here $\partial_1^{\alpha_1} \varphi(x) := \dfrac{\partial^{\alpha_1} \varphi(x)}{\partial x_1^{\alpha_1}}$, etc.

Definition 3.1 *i) $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is the space of functions $\varphi(x) \in C^\infty(\mathbb{R}^n)$ such that*

$$\|\varphi\|_{\alpha,N} := \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial_x^\alpha \varphi(x)| < \infty \tag{3.1}$$

for any $N = 1, 2, ...,$ and any multiindex $\alpha = (\alpha_1, ..., \alpha_n)$ with $\alpha_j = 0, 1, 2, ...$
ii) The sequence $\varphi_n(x) \xrightarrow{\mathcal{S}} \varphi(x)$ iff

$$\|\varphi_n - \varphi\|_{\alpha,N} \to 0, \qquad n \to \infty \tag{3.2}$$

for any $N = 1, 2, ...$ and $\alpha = (\alpha_1, ..., \alpha_n)$ with $\alpha_j = 0, 1, 2, ...$

Example 3.2 *Function $\varphi(x) = e^{-|x|^2}$ belongs to the space $\mathcal{S}(\mathbb{R}^n)$.*

Further, the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ is defined similarly to the case $n = 1$.

Example 3.3 *i) Let us consider a continuous function $f(x) \in C(\mathbb{R})$ satisfying the bounds*

$$|f(x)| \le C(1 + |x|)^p, \qquad x \in \mathbb{R} \tag{3.3}$$

with some constants $C, p \in \mathbb{R}$. Let us define the functional

$$\langle f, \varphi \rangle := \int f(x)\varphi(x)dx, \qquad \varphi \in \mathcal{S}. \tag{3.4}$$

Then the integral (3.4) converges, and the functional is tempered distribution.
ii) The functional (3.4) is tempered distribution if $f(x) \in L^p(\mathbb{R})$ with a $p \ge 1$.
iii) Dirac delta-function is defined as the distribution

$$\langle \delta(x), \varphi(x) \rangle = \varphi(0), \qquad \varphi \in \mathcal{S}. \tag{3.5}$$

It is also tempered distribution of n variables.

3.2 Differentiation

Differentiation of tempered distributions is defined similarly to the case $n = 1$: for any $k = 1, ..., n$,

$$\langle \partial_k f(x), \varphi(x) \rangle = -\langle f(x), \partial_k \varphi(x) \rangle, \qquad \varphi \in \mathcal{S}. \tag{3.6}$$

Then for any multiindex $\alpha = (\alpha_1, ..., \alpha_n)$, we get general formula

$$\langle \partial^\alpha f(x), \varphi(x) \rangle = (-1)^{\alpha_1 + ... + \alpha_n} \langle f(x), \partial^\alpha \varphi(x) \rangle. \tag{3.7}$$

3.3 Fourier transform

Further, the Fourier transform of test functions $\varphi \in \mathcal{S}(\mathbb{R})$ is defined by

$$F\varphi(k) := \hat{\varphi}(k) := \int_{\mathbb{R}^n} e^{ikx}\varphi(x)dx, \quad k \in \mathbb{R}^n, \tag{3.8}$$

where $kx = k_1x_1 + ... + k_nx_n$. The inversion of the Fourier transform is given by

$$\varphi(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ikx}\hat{\varphi}(k)dk, \quad x \in \mathbb{R}^n. \tag{3.9}$$

Similarly to the case $n = 1$, for any test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$, its Fourier transform $\hat{\varphi}(k)$ also belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, and the operator $F : \varphi \mapsto \hat{\varphi}$ is linear and continuous in $\mathcal{S}(\mathbb{R}^n)$.

Finally, the Fourier transform of tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by the Parseval identity:

$$\langle \hat{f}(k), \varphi(k)\rangle = \langle f(x), \hat{\varphi}(x)\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{3.10}$$

Lemma 2.8 and Theorem 2.10 generalises to any dimension $n > 1$:

Lemma 3.4 *For $f(x) \in L^1(\mathbb{R}^n)$ the Fourier transform $\hat{f}(k)$ is a bounded continuous function, i.e. $\hat{f} \in C_b(\mathbb{R}^n)$. It is given by the Fourier integral*

$$Ff(k) := \hat{f}(k) := \int_{\mathbb{R}^n} e^{ikx}f(x)dx, \quad k \in \mathbb{R}^n. \tag{3.11}$$

Theorem 3.5 *(The Parseval Theorem) For $f(x) \in L^2 = L^2(\mathbb{R}^n)$ the Fourier transform $\hat{f}(k)$ also belongs to $L^2(\mathbb{R}^n)$, and the Parseval identity holds (Cf. (2.30)):*

$$\|\hat{f}\|^2 = (2\pi)^n\|f\|^2. \tag{3.12}$$

3.4 Fourier transform of derivative

For any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, the following generalisations of the identities (2.8) hold:

$$k_j\hat{f}(k) = F[i\partial_j f(x)], \qquad \partial_j\hat{f}(k) = F[ix_jf(x)] \tag{3.13}$$

For the test function $f \in \mathcal{S}(\mathbb{R}^n)$ the identities follows similarly to (2.8). For tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ the proof of the first formula is the following:

$$\begin{aligned}\langle k_jFf(k), \varphi(k)\rangle = \langle Ff(k), k_j\varphi(k)\rangle &= \langle f(x), F(k_j\varphi(k))\rangle = \langle f(x), -i\partial_{x_j}F(\varphi(k))\rangle \\ &= \langle i\partial_{x_j}f(x), F(\varphi(k))\rangle = \langle F[i\partial_{x_j}f(x)], \varphi(k)\rangle.\end{aligned} \tag{3.14}$$

Here i) first identity follows by definition of distributions by smooth function k_j,
ii) second and last identities follow from definition of the Fourier transform for distributions,
iii) third identity follows by differentiation of the Fourier integral:

$$\begin{aligned}F(k_j\varphi(k)) = \int_{\mathbb{R}^n} e^{ikx}k_j\varphi(k)dk &= \textstyle\int_{\mathbb{R}^n}(-i\partial_{x_j})e^{ikx}\varphi(k)dk \\ &= -i\partial_{x_j}\int_{\mathbb{R}^n} e^{ikx}\varphi(k)dk = -i\partial_{x_j}F(\varphi).\end{aligned} \tag{3.15}$$

iv) The fourth identity in (3.14) follows by definition of the derivative for distributions.

Finally, second formula (3.13) follows similarly. ■

Exercise 3.6 *Prove the second formula (3.13).*

4 The Sobolev spaces

Here we introduce the Sobolev spaces and prove simplest properties. Let s be a real number.

Definition 4.1 *$H^s = H^s(\mathbb{R}^n)$ is the space of tempered distributions $f(x) \in \mathcal{S}'(\mathbb{R}^n)$ such that $(1+|k|)^s\hat{f}(k) \in L^2 = L^2(\mathbb{R}^n)$, and the corresponding Sobolev's norm is defined as*

$$\|f\|_s := \|(1+|k|)^s\hat{f}(k)\|_{L^2}. \tag{4.1}$$

Remark 4.2 *For $s = 0$ the Sobolev space $H^0(\mathbb{R}^n)$ coincides with the Hilbert space $L^2(\mathbb{R}^n)$ by the Parseval Lemma 3.5, and*

$$\|f\|_0 := \|\hat{f}(k)\|_{L^2} = (2\pi)^n\|f(x)\|_{L^2} \tag{4.2}$$

according to (3.12).

Lemma 4.3 *Every Sobolev space H^s is isomorphic to the Hilbert space $L^2 = L^2(\mathbb{R}^n)$. The map $(1+|k|)^sF : H^s \to L^2$ is continuous, and the inverse map $F^{-1}(1+|k|)^{-s} : L^2 \to H^s$ is also continuous.*

Proof i) By Definition 4.1, $g(k) = (1+|k|)^sFf(k) = (1+|k|)^s\hat{f}(k) \in L^2$ if $f(x) \in H^s$. Furthermore, $\|f(x)\|_s := \|g(k)\|_{L^2}$, hence

$$\|g(k)\|_{L^2} \le \|f(x)\|_s \tag{4.3}$$

that implies the continuity of the map $(1+|k|)^sF : f(x) \mapsto g(k)$ from H^s to L^2.
ii) We have $\hat{f}(k) = (1+|k|)^{-s}g(k)$, hence the inverse map is given by $f(x) = F^{-1}\hat{f} = F^{-1}[(1+|k|)^{-s}g(k)]$. It is important that this map is defined on the whole of $g(k) \in L^2$ since $(1+|k|)^{-s}g(k)$ is a tempered distribution for every $g(k) \in L^2$ (check this!), hence $f(x) = F^{-1}[(1+|k|)^{-s}g(k)]$ is also tempered distribution! Now we check that $\hat{f}(k) = (1+|k|)^{-s}g(k)$, hence $(1+|k|)^s\hat{f}(k) = g(k) \in L^2$, and therefore, by definition, $f \in H^s$. Furthermore, $\|f(x)\|_s := \|g(k)\|_{L^2}$, hence the map $F^{-1}(1+|k|)^{-s} : g(k) \mapsto f(x)$ is continuous from L^2 to H^s. ■

For a multiindex $\alpha = (\alpha_1, ..., \alpha_n)$ let us define monomial $k^\alpha := k_1^{\alpha_1}...k_n^{\alpha_n}$ of n complex variables $k_1, ..., k_n \in \mathbb{C}$. Consider polynomials $A(k) = \sum\limits_{|\alpha|\le m} a_\alpha k^\alpha$ of order $m = 0, 1, 2, ...$ and corresponding differential operators

$$A(\partial)u(x) = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha u(x). \tag{4.4}$$

For any tempered distribution $u(x) \in \mathcal{S}'$, we also have $A(\partial)u(x) \in \mathcal{S}'$ since any derivative of tempered distribution also is a tempered distribution. By definition, the Sobolev spaces H^s are subspaces of $\mathcal{S}'$, hence $A(\partial)u$ is a tempered distribution for any $u \in H^s$.

Lemma 4.4 *For any $s \in \mathbb{R}$ and $u \in H^s$ we have $A(\partial)u \in H^{s-m}$, and the operator $A(\partial) : H^s \to H^{s-m}$ is linear and continuous.*

Proof By definition, $A(\partial)u \in H^{s-m}$ if $(1+|k|)^{s-m}F[A(\partial)u](k) \in L^2$, and

$$\|A(\partial)u\|_{s-m} = \|(1+|k|)^{s-m}F[A(\partial)u](k)\|_{L^2}. \tag{4.5}$$

Let us calculate the Fourier transform: using first formulas of (3.13), we obtain that $F[\partial^\alpha u](k) = (-i\partial)^\alpha\hat{u}(k)$, hence $F[A(\partial)u](k) = A(-ik)\hat{u}(k)$, and (4.5) becomes

$$\|A(\partial)u\|_{s-m} = \|(1+|k|)^{s-m}A(-ik)\hat{u}(k)\|_{L^2}. \tag{4.6}$$

Note that the polynomial $A(k)$ is continuous function, hence the product $(1+|k|)^{s-m}A(-ik)\hat{u}(k)$ is measurable function of $k \in \mathbb{R}^n$. Furthermore, $|A(-ik)| \le C(1+|k|)^m$ for $k \in \mathbb{R}^n$, hence

$$(1+|k|)^{s-m}|A(-ik)| \le C(1+|k|)^s, \qquad \text{for} \quad k \in \mathbb{R}^n. \tag{4.7}$$

This implies that

$$\begin{aligned} \|(1+|k|)^{s-m}A(-ik)\hat{u}(k)\|_{L^2}^2 &= \int (1+|k|)^{2(s-m)}|A(-ik)\hat{u}(k)|^2 dk \le \int (1+|k|)^{2s}|\hat{u}(k)|^2 dk \\ &= C\|(1+|k|)^s\hat{u}(k)\|_{L^2}^2 = C\|u\|_s^2. \end{aligned} \tag{4.8}$$

Therefore (4.6) implies that

$$\|A(\partial)u\|_{s-m} \le C_1\|u\|_s < \infty \tag{4.9}$$

that proves the lemma. ■

5 First Sobolev's embedding theorem

First we state the following generalisation of Lemma 2.8 to n variables:

Lemma 5.1 *For $f(x) \in L^1(\mathbb{R}^n)$ the Fourier transform $\hat{f}(k)$ is a bounded continuous function, i.e. $\hat{f} \in C_b(\mathbb{R}^n)$. It is given by the Fourier integral*

$$Ff(k) := \hat{f}(k) := \int_{\mathbb{R}^n} e^{ikx} f(x)dx, \quad k \in \mathbb{R}^n. \tag{5.1}$$

The proof of the lemma coincides with the proof of Lemma 2.8.

The lemma can be reformulated for the inverse Fourier transform as follows:

Lemma 5.2 *For $g(k) \in L^1(\mathbb{R}^n)$ the inverse Fourier transform $F^{-1}g(x)$ is a bounded continuous function, i.e. $F^{-1}g \in C_b(\mathbb{R}^n)$. It is given by the Fourier integral*

$$F^{-1}g(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ikx} g(k)dk, \quad x \in \mathbb{R}^n. \tag{5.2}$$

Now we can prove first Sobolev's embedding theorem is the following.

Theorem 5.3 *(First Sobolev's embedding theorem) Let a tempered distribution $u(x) \in H^s(\mathbb{R}^n)$ with $s > n/2$. Then $u(x)$ is a continuous function in $\mathbb{R}^n$, and*

$$\sup_{x\in\mathbb{R}^n} |u(x)| \le C(s)\|u\|_s, \tag{5.3}$$

where the constant $C(s) < \infty$ does not depend on $u(x)$. In other words, we have continuous embedding $H^s(\mathbb{R}^n) \subset C_b(\mathbb{R}^n)$ if $s > n/2$.

Proof *Step i)* First we will prove that $\hat{u}(k) \in L^1(\mathbb{R}^n)$. Namely, by definition, $u(x) \in H^s(\mathbb{R}^n)$ means that $(1 + |k|)^s\hat{u}(k) \in L^2$, i.e.

$$\int |(1 + |k|)^s \hat{u}(k)|^2 dk < \infty \tag{5.4}$$

Then by the Cauchy-Schwartz inequality, we get

$$\begin{aligned} \int |\hat{u}(k)|dk &= \int (1 + |k|)^{-s} |(1 + |k|)^s \hat{u}(k)|dk \\ &\le \Big(\int (1 + |k|)^{-2s} dk\Big)^{1/2} \Big(\int |(1 + |k|)^s \hat{u}(k)|^2 dk\Big)^{1/2} < I(s)\|u\|_s < \infty \end{aligned} \tag{5.5}$$

since $I(s) := \Big(\int (1 + |k|)^{-2s} dk\Big)^{1/2} < \infty$ for $s > n/2$ (Check this!).

Step ii) We have proved that $\hat{u}(k) \in L^1(\mathbb{R}^n)$. Therefore, $u(x) = F^{-1}\hat{u}(k)$ is bounded continuous function by Lemma 5.2. To prove the bound (5.3), let us write (5.2) with $g(k) = \hat{u}(k)$: then $F^{-1}g(k) = u(x)$, hence (5.2) becomes

$$u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ikx} \hat{u}(k)dk, \quad x \in \mathbb{R}^n. \tag{5.6}$$

Therefore,

$$\sup_{x\in\mathbb{R}^n} |u(x)| \le \frac{1}{(2\pi)^n} \int |\hat{u}(k)|dk, \quad x \in \mathbb{R}^n. \tag{5.7}$$

Substituting here (5.5), we obtain (5.3) with the constant $C(s) = I(s)/(2\pi)^n$. ■

Exercise 5.4 *Check that* $C(s) \to \infty$ *as* $s \to n/2$.

Corollary 5.5 *Let a tempered distribution* $u(x) \in H^s(\mathbb{R}^n)$ *with* $s > n/2 + k$, $k \geq 0$. *Then for any multiindex* $\alpha = (\alpha_1, ..., \alpha_n)$, *the derivative* $\partial^\alpha u(x)$ *is bounded continuous function in* $\mathbb{R}^n$ *if* $|\alpha| \leq k$.

In other words, we have continuous embedding $H^s(\mathbb{R}) \subset C_b^k(\mathbb{R})$ *if* $s > n/2 + k$.

Example 5.6 *I. For one variable,* $n = 1$*:* $H^1(\mathbb{R}) \subset C_b(\mathbb{R})$ *since* $1 > 1/2$, $H^2(\mathbb{R}) \subset C_b^1(\mathbb{R})$ *since* $2 > 1/2 + 1$,...
II. For three variables, $n = 3$*:* $H^2(\mathbb{R}) \subset C_b(\mathbb{R})$ *since* $2 > 3/2$, $H^3(\mathbb{R}) \subset C_b^1(\mathbb{R})$ *since* $3 > 1/2 + 2$,...

6 Sobolev's spaces for integer $s \geq 0$

First we give an equivalent characterisation to the Sobolev spaces with integer $s = 0, 1, ...$

Lemma 6.1 *i) For $s = 0, 1, ...$ the Sobolev space $H^s(\mathbb{R}^n)$ coincides with the space of functions $u(x) \in L^2(\mathbb{R}^n)$ such that $\partial^\alpha u(x) \in L^2(\mathbb{R}^n)$ for any multiindex $\alpha = (\alpha_1, ..., \alpha_n)$ with $|\alpha| \leq s$.*
ii) The Sobolev norm $\|u\|_s$ is equivalent to the norm $|||u|||_s$ defined by

$$|||u|||_s^2 = \sum_{|\alpha| \leq s} \|\partial^\alpha u\|_{L^2}^2 \tag{6.1}$$

Remark 6.2 *i) In this lemma the derivatives $\partial^\alpha u(x)$ of a function $u(x) \in L^2(\mathbb{R}^n)$ are understood in the sense of distributions: if one treat the derivatives in the classical sense, then the lemma would fail.*
ii) The equivalence of the norms means that

$$|||u|||_s \leq C_1 \|u\|_s \qquad for \quad u \in H^s, \tag{6.2}$$

for a constant $C_1 < \infty$, and also

$$\|u\|_s \leq C_2 |||u|||_s \qquad for \quad u \in H^s. \tag{6.3}$$

Proof of Lemma 6.1 We have to check that i) $\partial^\alpha u(x) \in L^2(\mathbb{R}^n)$ for $u \in H^s$, and ii) vice versa, $u \in H^s$ if $\partial^\alpha u(x) \in L^2(\mathbb{R}^n)$ for any $|\alpha| \leq s$.

Step i) Lemma 4.4 implies that $\partial^\alpha u(x) \in L^2(\mathbb{R}^n)$ for $u \in H^s$, and the bound (6.2) holds.

Step ii) Vice versa, let us consider a function $u(x) \in L^2(\mathbb{R}^n)$ such that $\partial^\alpha u(x) \in L^2(\mathbb{R}^n)$ for any multiindex $\alpha = (\alpha_1, ..., \alpha_n)$ with $|\alpha| \leq s$. We have to prove that $u \in H^s$, and the bound (6.3) holds. By the Parseval Theorem 3.5, we have that $F[\partial^\alpha u] \in L^2(\mathbb{R}^n)$. On the other hand, $F[\partial^\alpha u] = (-ik)^\alpha \hat{u}(k)$ by the formula (3.13), hence $(-ik)^\alpha \hat{u}(k) \in L^2(\mathbb{R}^n)$. Furthermore, by the Parseval identity (3.12), we have

$$\int |(-ik)^\alpha \hat{u}(k)|^2 dk = \|F[\partial^\alpha u]\|_{L^2}^2 = (2\pi)^n \|\partial^\alpha u\|_{L^2}^2 < \infty. \tag{6.4}$$

Summing up the identities with $|\alpha| \leq s$, we obtain that

$$\int [\sum_{|\alpha| \leq s} |(-ik)^\alpha|^2] \; |\hat{u}(k)|^2 dk = C \sum_{|\alpha| \leq s} \|\partial^\alpha u\|_{L^2}^2 = C |||u|||_s^2 < \infty. \tag{6.5}$$

Let us note that the function $S(k) = \sum_{|\alpha| \leq s} |(-ik)^\alpha|^2 \neq 0$ for each $k \in \mathbb{R}^n$, hence (6.5) implies that the tempered distribution $\hat{u}(k)$ is measurable Lebesgue function. Moreover, the function $S(k)$ admits the following bound from below:

$$B(1 + |k|)^{2s} \leq S(k) := \sum_{|\alpha| \leq s} |(-ik)^\alpha|^2, \qquad k \in \mathbb{R}^n. \tag{6.6}$$

Exercise 6.3 *Check the bound (6.6).*

The bound (6.6) implies that

$$B \int (1 + |k|)^{2s} \; |\hat{u}(k)|^2 dk \leq \int [\sum_{|\alpha| \leq s} |(-ik)^\alpha|^2] \; |\hat{u}(k)|^2 dk < \infty \tag{6.7}$$

by (6.5). Therefore,

$$B \|u\|_s^2 \leq C |||u|||_s^2 < \infty \tag{6.8}$$

by definitions of the norms. Hence, $u \in H^s$, and the bound (6.3) holds. ■

7 Sobolev's theorem on compact embedding

Definition 4.1 implies that $H^{s_1}(\mathbb{R}^n) \subset H^{s_2}(\mathbb{R}^n)$ if $s_1 > s_2$ (Check this!). We will prove that some bounded sets in $H^{s_1}(\mathbb{R}^n)$ are precompact in $H^{s_2}(\mathbb{R}^n)$.

We will consider the case $s_1 \geq 0$ for the simplicity of exposition, though all statements below are valid also for $s_1 < 0$.

Let us consider an open region $\Omega \subset \mathbb{R}^n$.

Definition 7.1 *$\overset{o}{H}{}^{s_1}(\Omega)$ is the subspace of tempered distributions $u(x) \in H^{s_1}(\mathbb{R}^n)$ such that $u(x)$ vanishes in the complement of Ω, i.e. $u(x) = 0, \ \ x \in \mathbb{R}^n \setminus \Omega$. The norm in $\overset{o}{H}{}^{s_1}(\Omega)$ coincides with the norm in $H^{s_1}(\mathbb{R}^n)$.*

By this definition, $\overset{o}{H}{}^{s_1}(\Omega) \subset H^{s_1}(\mathbb{R}^n)$, and hence $\overset{o}{H}{}^{s_1}(\Omega) \subset H^{s_2}(\mathbb{R}^n)$.

Theorem 7.2 *Let the region Ω be bounded in $\mathbb{R}^n$, and $s_1 > s_2$. Then the embedding $\overset{o}{H}{}^{s_1}(\Omega) \subset H^{s_2}(\mathbb{R}^n)$ is compact, i.e. for any bounded sequence of the functions $u_j(x) \in \overset{o}{H}{}^{s_1}(\Omega)$ there exists a subsequence $u_{j'}(x)$ converging in $H^{s_2}(\mathbb{R}^n)$, i.e.*

$$\|u_{j'}(x) - u(x)\|_{s_2} \to 0 \qquad as \quad j' \to \infty, \tag{7.1}$$

where $u(x) \in H^{s_2}(\mathbb{R}^n)$.

Proof We will prove the theorem for the case $s_1 \geq 0$. The proof for general $s_1 < 0$ is very similar.

We shall check that $u_j(x)$ is the Cauchy sequence in $H^{s_2}(\mathbb{R}^n)$, i.e.

$$\|u_{j'}(x) - u_{m'}(x)\|_{s_2} \to 0 \qquad as \quad j', m' \to \infty. \tag{7.2}$$

Then (7.1) would follow since $H^{s_1}(\mathbb{R}^n)$ is complete Hilbert space.

Step i) First we prove that the sequence $u_j(x)$ is bounded in $L^1(\mathbb{R}^n)$, i.e.

$$\int\limits_\Omega |u_j(x)|dx \leq B_1 < \infty, \qquad j = 1, 2, ... \tag{7.3}$$

Namely, by definition of bounded sequence in $\overset{o}{H}{}^{s_1}(\Omega)$,

$$\sup_{j \in \mathbf{N}} \|u_j(x)\|_{s_1} \leq B < \infty. \tag{7.4}$$

Since $s_1 \geq 0$, we have $H^{s_1}(\mathbb{R}^n) \subset H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. Therefore,

$$\int |u_j(x)|^2 dx = \|u_j(x)\|_0^2 \leq \|u_j(x)\|_{s_1}^2 \leq B^2. \tag{7.5}$$

Moreover, we have

$$u_j(x) = 0 \quad \text{ for } \quad x \in \mathbb{R}^n \setminus \Omega \tag{7.6}$$

since $u_j(x) \in \overset{o}{H}{}^{s_1}(\Omega)$. Finally, by the Cauchy-Schwartz inequality and (7.5), we obtain

$$\int\limits_\Omega |u_j(x)|dx \leq \Big(\int\limits_\Omega 1 dx\Big)^{1/2} \Big(\int\limits_\Omega |u_j(x)|^2 dx\Big)^{1/2} \leq |\Omega|^{1/2} B. \tag{7.7}$$

Here $|\Omega|$ is the volume of the region Ω. The volume is finite since the region is bounded by our assumption, hence (7.7) implies (7.3) with $B_1 = |\Omega|^{1/2}B$.

Step ii) Further let us study the Fourier transform of the functions $u_j(x)$. First, we have

$$\hat{u}_j(k) = \int\limits_\Omega e^{ikx} u_j(x)dx \tag{7.8}$$

by Lemma 3.4 and (7.6) since we have proved that $u_j(x) \in L^1$. Let us prove that the derivatives of the functions are bounded, i.e.

$$\sup_{k\in\mathbb{R}^n} |\frac{\partial}{\partial k_l}\hat{u}_j(k)| < B_l < \infty, \qquad j = 1, 2, ... \tag{7.9}$$

for any $l = 1, ..., n$. For the proof we write definition of partial derivative:

$$\frac{\partial}{\partial k_l}\hat{u}_j(k) = \lim_{\varepsilon\to 0}\int\limits_\Omega \frac{e^{i(k+\varepsilon e_l)x} - e^{ikx}}{\varepsilon} u_j(x)dx, \tag{7.10}$$

where e_l is the unit vector $e_l = (0, ..., 1_l, ..., 0)$. The limit exists and is given by

$$\frac{\partial}{\partial k_l}\hat{u}_j(k) = \int\limits_\Omega \frac{\partial}{\partial k_l} e^{ikx} u_j(x)dx. \tag{7.11}$$

This follows by the Lebesgue Dominated Convergence Theorem since

I. the integrands converge for almost all points $x \in \Omega$

II. There exists summable majorant:

$$|\frac{e^{i(k+\varepsilon e_l)x} - e^{ikx}}{\varepsilon} u_j(x)| \le M(\Omega)|u_j(x)|, \qquad x \in \Omega \tag{7.12}$$

since $\frac{e^{i(k+\varepsilon e_l)x} - e^{ikx}}{\varepsilon}$ is bounded function for $x \in \Omega$:

$$\frac{e^{i(k+\varepsilon e_l)x} - e^{ikx}}{\varepsilon} \le M(\Omega) < \infty, \qquad x \in \Omega, \quad \varepsilon \in (0, 1). \tag{7.13}$$

Exercise 7.3 *Check the bound (7.13).* **Hints:**

$$i) \quad \frac{|e^{i(k+\varepsilon e_l)x} - e^{ikx}|}{\varepsilon} = \frac{|e^{i\varepsilon e_l x} - 1|}{\varepsilon} = \frac{|e^{i\varepsilon x_l} - 1|}{\varepsilon}$$

$$ii) \quad \frac{|e^{i\varepsilon\phi} - 1|}{\varepsilon} = \frac{|e^{i\varepsilon\phi/2} - e^{-i\varepsilon\phi/2}|}{\varepsilon} = \frac{2|\sin\frac{\varepsilon\phi}{2}|}{\varepsilon} \to |\phi| \qquad \text{as} \qquad \varepsilon \to 0.$$

Step iii) Now we consider the norms in (7.2):

$$\Delta_{jm}(R) := \|u_j(x) - u_m(x)\|^2_{s_2} = \int (1 + |k|)^{2s_2}|\hat{u}_j(k) - \hat{u}_m(k)|^2 dk. \tag{7.14}$$

Let us split the region of integration into the ball $|k| < R$ and its complement $|k| > R$, where the radius R will be chosen later:

$$\begin{aligned} \Delta_{jm}(R) = &= \int\limits_{|k|<R} (1 + |k|)^{2s_2}|\hat{u}_j(k) - \hat{u}_m(k)|^2 dk + \int\limits_{|k|>R} (1 + |k|)^{2s_2}|\hat{u}_j(k) - \hat{u}_m(k)|^2 dk \\ &= I_{jm}(R) + J_{jm}(R). \end{aligned} \tag{7.15}$$

First we estimate the second integral:

$$J_{jm}(R) = \int\limits_{|k|>R} (1+|k|)^{2s_1}(1+|k|)^{2s_2-2s_1}|\hat{u}_j(k)-\hat{u}_m(k)|^2 dk$$

$$\le (1+|R|)^{2s_2-2s_1}\int\limits_{|k|>R}(1+|k|)^{2s_1}|\hat{u}_j(k)-\hat{u}_m(k)|^2 dk \tag{7.16}$$

since $s_2 - s_1 < 0$. Therefore,

$$J_{jm}(R) \le (1+|R|)^{2s_2-2s_1}\int\limits_{|k|>R}(1+|k|)^{2s_1}2(|\hat{u}_j(k)|^2+|\hat{u}_m(k)|^2)dk$$

$$\le (1+|R|)^{2s_2-2s_1}\int\limits_{\mathbb{R}^n}(1+|k|)^{2s_1}2(|\hat{u}_j(k)|^2+|\hat{u}_m(k)|^2)dk$$

$$\le \frac{2B^2}{(1+|R|)^{2(s_1-s_2)}} \tag{7.17}$$

according to (7.3).

Step iv) Finally we can prove (7.2). First we use the bounds (7.9): they imply that the functions $\hat{u}_j(k)$ are *equicontinuous* in $\mathbb{R}^n$. On the other hand, the bound (7.3) implies that the functions are *uniformly bounded* in $\mathbb{R}^n$. Hence, by the Arzela-Ascoli Theorem, there exists a subsequence $\hat{u}_{j'}(k)$ converging at every point of $\mathbb{R}^n$. Therefore,

$$|\hat{u}_{j'}(k)-\hat{u}_{m'}(k)| \to 0 \quad as \quad j', m' \to \infty, \tag{7.18}$$

and this convergence is uniform in every compact set of k. In particular, for any fixed $R > 0$,

$$\max_{|k|\le R}|\hat{u}_{j'}(k)-\hat{u}_{m'}(k)| \to 0 \quad as \quad j', m' \to \infty. \tag{7.19}$$

This implies that first integral in (7.15) converges to zero: for any fixed $R > 0$,

$$I_{j'm'}(R) \to 0 \quad as \quad j', m' \to \infty. \tag{7.20}$$

It remains to note that $J_{j'm'}(R_\varepsilon)$ is small for large R by (7.17) uniformly in j' and m'. This proves (7.2). With detail: for any $\varepsilon > 0$

a) (7.17) implies that there exists $R_\varepsilon > 0$ such that

$$|J_{j'm'}(R_\varepsilon)| < \varepsilon/2 \qquad \text{for all} \qquad j', m' \tag{7.21}$$

since $s_1 - s_2 > 0$.

b) (7.20) implies that for the fixed R_ε there exists a number N_ε such that

$$|I_{j'm'}(R_\varepsilon)| < \varepsilon/2 \qquad \text{for} \qquad j', m' \ge N_\varepsilon. \tag{7.22}$$

On the other hand, $\Delta_{j'm'} = I_{j'm'}(R_\varepsilon) + J_{j'm'}(R_\varepsilon)$ by (7.15). Therefore, (7.21) and (7.22) imply that

$$|\Delta_{j'm'}| \le |I_{j'm'}(R_\varepsilon)| + |J_{j'm'}(R_\varepsilon)| < \varepsilon \qquad \text{for} \qquad j', m' \ge N_\varepsilon. \tag{7.23}$$

■

8 Elliptic partial differential equations with constant coefficients

We will consider partial differential equations in $\mathbb{R}^n$ with variable coefficients

$$\sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha u(x) = f(x), \qquad x\in\mathbb{R}^n. \tag{8.1}$$

We plan to study in the future the questions on existence, uniqueness and smoothness of solutions $u(x)$.

Example 8.1 *For example, we will consider the stationary Schrödinger equation*

$$(H-E)u(x) := (-\Delta + V(x) - E)u(x) = f(x) = f(x), \qquad x\in\mathbb{R}^n, \tag{8.2}$$

where $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ *is the* **Laplace operator**, $H = -\Delta + V(x)$ *is the* **Schrödinger operator**, $V(x)$ *is the potential energy, and* $E\in\mathbb{C}$ *is a complex (energy) parameter.*

The investigation of the inverse operator $(H-E)^{-1} = (-\Delta+V(x)-E)^{-1}$ *is one of main goals of* **quantum scattering theory**.

Here we will study the equations with constant coefficients

$$Au(x) := \sum_{|\alpha|\le m} a_\alpha \partial^\alpha u(x) = f(x), \qquad x\in\mathbb{R}^n. \tag{8.3}$$

We assume that the right hand side $f(x)$ is tempered distribution, $f(x)\in\mathcal{S}'(\mathbb{R}^n)$, and seek for the solution $u(x)$ also in the space of tempered distributions, $u(x)\in\mathcal{S}'(\mathbb{R}^n)$.

Definition 8.2 *The polynomial* $a(k) = \sum_{|\alpha|\le m} a_\alpha(-ik)^\alpha$ *is the* **symbol** *of differential operator* A.

In the Fourier transform, the equation (8.3) becomes

$$a(k)\hat{u}(k) = \hat{f}(k), \qquad k\in\mathbb{R}^n. \tag{8.4}$$

due to formulas (3.13).

Definition 8.3 *i) Differential operator* $A(\partial)$ *is* **elliptic of order** m *if*

$$|a(k)| > c(1+|k|)^m, \qquad |k|>R, \quad k\in\mathbb{R}^n \tag{8.5}$$

for some $R>0$ *and* $c>0$.
ii) Differential operator $A(\partial)$ *is* **strongly elliptic of order** m *if*

$$|a(k)| > c(1+|k|)^m, \qquad k\in\mathbb{R}^n, \tag{8.6}$$

where $c>0$.

Example 8.4 *i) The Laplace operator* Δ *(and* $-\Delta$*) is elliptic (check this!).* **Hint:** *Symbol of the Laplace operator is* $-\sum_{j=1}^n k_j^2 = -|k|^2$.
ii) The Schrödinger operator $-\Delta - E$ *is elliptic for every* $E\in\mathbb{C}$ *(check this!).*
iii) The Schrödinger operator $-\Delta - E$ *is strongly elliptic*
a) for every $E<0$ *(check this!);*
b) for every $E\in\mathbb{C}\setminus\mathbb{R}^+$ *where* $\mathbb{R}^+ = \{E\in\mathbb{R} : E\ge 0\}$ *(check this!).*

Theorem 8.5 *Let us consider equation with constant coefficients (8.3). Let us assume that A is strongly elliptic operator of order m, and $f(x) \in H^{s-m}(\mathbb{R}^n)$. Then for any $s \in \mathbb{R}$, we have*
i) Equation (8.3) admits unique solution $u(x) \in H^s(\mathbb{R}^n)$.
ii) The bound holds,

$$\|u\|_s \leq C\|f\|_{s-m}. \tag{8.7}$$

Proof In Fourier transform, equation (8.3) becomes (8.3) where $\hat{u}(k)$ and $\hat{f}(k)$ are measurable Lebesgue functions in $\mathbb{R}^n$. Therefore, the solution is given by

$$\hat{u}(k) = \frac{\hat{f}(k)}{a(k)} \qquad \text{for almost all} \qquad k \in \mathbb{R}^n \tag{8.8}$$

since $a(k) \neq 0$ for $k \in \mathbb{R}^n$. It remains to set $u(x) := F^{-1}\hat{u}(k)$ and check that $u(x)$ belongs to the Sobolev space $H^s(\mathbb{R}^n)$. By definition,

$$\|u\|_s^2 = \int (1+|k|)^{2s}|\hat{u}(k)|^2 dk = \int (1+|k|)^{2s}\Big|\frac{\hat{f}(k)}{a(k)}\Big|^2 dk = \int \frac{(1+|k|)^{2s}}{|a(k)|^2}|\hat{f}(k)|^2 dk. \tag{8.9}$$

Note that

$$\frac{(1+|k|)^{2s}}{|a(k)|^2} \leq C(1+|k|)^{2(s-m)}, \qquad k \in \mathbb{R}^n \tag{8.10}$$

by (8.6). Hence, (8.9) implies that

$$\|u\|_s^2 \leq C\int (1+|k|)^{2(s-m)}|\hat{f}(k)|^2 dk = C\|f\|_{s-m}^2 < \infty \tag{8.11}$$

since $f \in H^{s-m}$. This proves the bound (8.7). ■

Corollary 8.6 *Strongly elliptic operator A of order m is isomorphism $H^s \to H^{s-m}$. The continuity of the direct operator $A : H^s \to H^{s-m}$ follows from Lemma4.4, and the continuity of the inverse operator $A^{-1} : H^{s-m} \to H^s$ follows from the bound (8.7).*

Exercise 8.7 *Check the inequality (8.10).* **Hints:** *Due to (8.6), i) The function*

$$R(k) := \frac{(1+|k|)^{2s}}{|a(k)|^2(1+|k|)^{2(s-m)}} \tag{8.12}$$

is continuous in $\mathbb{R}^n$, and ii)

$$\lim_{|k|\to\infty} R(k) < \infty. \tag{8.13}$$

In next lectures we will extend Theorem 8.5 and Corollary 8.6 to strongly elliptic partial differential equations with **variable coefficients**.

9 Pseudodifferential operators

9.1 Fourier representation for differential operators

Let us consider the Fourier representation for partial differential operators in $\mathbb{R}^n$ with variable coefficients

(9.1) $$Au(x) = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha u(x), \qquad x \in \mathbb{R}^n,$$

for test functions $u(x) \in \mathcal{S}(\mathbb{R}^n)$.

First let us consider the case of constant coefficients: $a_\alpha(x) = a_\alpha$. Then $Au(x) = \sum\limits_{|\alpha|\le m} a_\alpha \partial^\alpha u(x)$, and in the Fourier representation we have $FAu(k) = a(k)\hat{u}(k)$, where $a(k)$ is the **symbol** of the operator A:

(9.2) $$a(k) := \sum_{|\alpha|\le m} a_\alpha(-ik)^\alpha.$$

Since $u(x) \in \mathcal{S}(\mathbb{R}^n)$, we have also $\hat{u}(k) \in \mathcal{S}(\mathbb{R}^n)$, and hence $a(-ik)\hat{u}(k) \in L^1(\mathbb{R}^n)$ (Check this!). Then the inverse Fourier transform $Au(x) = F^{-1}a(k)\hat{u}(k)$ can be written as the standard Fourier integral

(9.3) $$Au(x) = \frac{1}{(2\pi)^n}\int e^{-ikx}a(k)\hat{u}(k)dk.$$

Now consider the case of variable coefficients (8.1). Applying (9.3) to the operator ∂^α, we write

(9.4) $$\partial^\alpha u(x) = \frac{1}{(2\pi)^n}\int e^{-ikx}(-ik)^\alpha\hat{u}(k)dk.$$

Multiplying by $a_\alpha(x)$ and summing up, we obtain that

(9.5) $$Au(x) = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha u(x) = \frac{1}{(2\pi)^n}\int e^{-ikx}\sum_{|\alpha|\le m} a_\alpha(x)(-ik)^\alpha\hat{u}(k)dk.$$

This can be written as

(9.6) $$Au(x) = \frac{1}{(2\pi)^n}\int e^{-ikx}a(x,k)\hat{u}(k)dk,$$

where the polynomial

(9.7) $$a(x,k) := \sum_{|\alpha|\le m} a_\alpha(x)(-ik)^\alpha$$

is the **symbol** of the differential operator A (Cf. Definition 8.2).

For general, not necessarily polynomial, functions $a(x,k)$, the operators of type (9.6) are called **pseudodifferential operators** if $a(x,k)$ satisfies appropriate estimates that we are going to discuss.

9.2 Classes of symbols and pseudodifferential operators

Definition 9.1 *For a real number $m \in \mathbb{R}$, class of symbols $\mathcal{S}^m$ consists of the functions $a(x,k) \in C^\infty(\mathbb{R}^n)$ such that $a(x,k) = a^0(k) + a'(x,k)$, where the terms $a^0(k)$ and $a'(x,k)$ satisfy the following estimates*

I. For any multiindex α, the bound holds,

(9.8) $$|\partial_k^\alpha a^0(k)| \le C(\alpha)(1+|k|)^{m-|\alpha|}, \qquad k \in \mathbb{R}^n.$$

II. For any multiindexes α, β, and number $N > 0$, the bound holds,

(9.9) $$(1+|x|)^N|\partial_k^\alpha\partial_x^\beta a'(x,k)| \le C(\alpha,\beta,N)(1+|k|)^{m-|\alpha|}, \qquad x,k \in \mathbb{R}^n.$$

Exercise 9.2 *i) Check that* $a(k) = (1+|k|^2)^{m/2} \in \mathcal{S}^m$ *for any* $m \in \mathbb{R}$.
ii) Check that the symbol $a(x,k) = \sum\limits_{|\alpha|\le m} a_\alpha(x)(-ik)^\alpha$ *belongs to the class* $\mathcal{S}^m$ *if for any multiindex* α *we have* $a_\alpha(x) = a^0_\alpha(k) + a'_\alpha(x)$, *where* $a'_\alpha(x) \in \mathcal{S}(\mathbb{R}^n)$.

Exercise 9.3 *Check that* $a_1(x,k)a_2(x,k) \in \mathcal{S}^{m_1+m_2}$ *if* $a_1(x,k) \in \mathcal{S}^{m_1}$ *and* $a_2(x,k) \in \mathcal{S}^{m_2}$, *i.e. the union of the classes* $\cup_{m\in\mathbb{R}}\mathcal{S}^m$ *is an* **algebra**.

Now consider an operator A of type (9.6 with the symbol $a(x,k)$ from the class $\mathcal{S}^m$ with an $m \in \mathbb{R}$. The operator is defined for the test functions $u(x) \in \mathcal{S}(\mathbb{R}^n)$ since the integral (9.6) converges then (Check this!).

Definition 9.4 *An operator* A *of type (9.6) is called* **pseudodifferential operator** *of class* $\mathcal{A}^m$ *if the corresponding symbol* $a(x,k)$ *belongs to the class* $\mathcal{S}^m$ *with* $m \in \mathbb{R}$.

10 Boundedness of pseudodifferential operators

Here we discuss boundedness of pseudodifferential operators in the Sobolev spaces.

10.1 Schur's Lemma

Let us consider an integral operator defined for test functions $v(k) \in \mathcal{S}(\mathbb{R}^n)$ by the following formula

$$Sv(k) = \int_{\mathbb{R}^n} S(k,k')v(k')dk', \qquad k \in \mathbb{R}^n. \tag{10.1}$$

Lemma 10.1 *Let the integral kernel $S(k,k')$ satisfies the estimates*

$$\int_{\mathbb{R}^n} |S(k,k')|dk' \le C < \infty, \qquad k \in \mathbb{R}^n, \tag{10.2}$$

$$\int_{\mathbb{R}^n} |S(k,k')|dk \le C < \infty, \qquad k' \in \mathbb{R}^n. \tag{10.3}$$

Then $Sv(k) \in L^2(\mathbb{R}^n)$ for $v(k) \in \mathcal{S}(\mathbb{R}^n)$, and

$$\|Sv\|_{L^2} \le C\|v\|_{L^2}. \tag{10.4}$$

Proof Definition (10.1) implies the inequality

$$|Sv(k)| \le \int_{\mathbb{R}^n} |S(k,k')v(k')|dk' = \int_{\mathbb{R}^n} |S(k,k')|^{1/2}|S(k,k')|^{1/2}|v(k')|dk', \qquad k \in \mathbb{R}^n. \tag{10.5}$$

Applying here the Cauchy-Schwartz inequality, we obtain the inequality

$$\begin{aligned} |Sv(k)|^2 \le \int_{\mathbb{R}^n} |S(k,k')v(k')|dk' &= \int_{\mathbb{R}^n} |S(k,k')|dk' \int_{\mathbb{R}^n} |S(k,k')||v(k')|^2dk' \\ &\le C\int_{\mathbb{R}^n} |S(k,k')||v(k')|^2dk', \qquad k \in \mathbb{R}^n \end{aligned} \tag{10.6}$$

by condition (10.2). Integrating (10.6) over $k \in \mathbb{R}^n$, we obtain that

$$\int |Sv(k)|^2dk \le C\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} |S(k,k')||v(k')|^2dk'\Big)dk. \tag{10.7}$$

This implies by the Fubini theorem, that

$$\int |Sv(k)|^2dk \le C\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} |S(k,k')|dk\Big)|v(k')|^2dk'. \tag{10.8}$$

Now we use second condition (10.3) and obtain

$$\int |Sv(k)|^2dk \le C^2\int_{\mathbb{R}^n} |v(k')|^2dk' \tag{10.9}$$

that implies (10.4). ■

Corollary 10.2 *Under the conditions (10.2), (10.3), the integral operator S admits unique extension from test functions $v(k) \in \mathcal{S}(\mathbb{R}^n)$ to the functions $v(k) \in L^2(\mathbb{R}^n)$ as bounded operator $S : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.*

Exercise 10.3 *Justify the application of the Fubini theorem in (10.8).* **Hint:** *Use the condition (10.3).*

10.2 Application to operator of multiplication

Let us consider the operator of multiplication, $M : u(x) \mapsto a(x)u(x)$, by a test function $a(x) \in \mathcal{S}(\mathbb{R}^n)$.

Lemma 10.4 *Let $a(x) \in \mathcal{S}(\mathbb{R}^n)$. Then the operator M is continuous $H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ for any $s \in \mathbb{R}$.*

Proof We have to prove the bound

$$\|a(x)u(x)\|_s \le C(s)\|u(x)\|_s, \qquad u \in H^s. \tag{10.10}$$

By definition,

$$\|a(x)u(x)\|_s^2 = \|(1+|k|)^s[F(au)](k)\|_{L^2}. \tag{10.11}$$

Let us calculate the Fourier transform $F(au)$:

$$F(au)(k) = \int e^{ikx}a(x)u(x)dx. \tag{10.12}$$

Substituting the Fourier representation $u(x) = \dfrac{1}{(2\pi)^n}\displaystyle\int e^{-ik'x}\hat{u}(k')dk'$, we obtain by the Fubini theorem that

$$\begin{aligned} F(au)(k) &= \int e^{ikx}a(x)\Big(\frac{1}{(2\pi)^n}\int e^{-ik'x}\hat{u}(k')dk'\Big)dx \\ &= \frac{1}{(2\pi)^n}\int\Big(\int e^{i(k-k')x}a(x)dx\Big)\hat{u}(k')dk' = \frac{1}{(2\pi)^n}\int \hat{a}(k-k')\hat{u}(k')dk'. \end{aligned} \tag{10.13}$$

Therefore, (10.11) becomes

$$\|a(x)u(x)\|_s^2 = \frac{1}{(2\pi)^n}\|(1+|k|)^s\int \hat{a}(k-k')\hat{u}(k')dk'\|_{L^2} \tag{10.14}$$

Hence, the bound (10.10) can be written in the form

$$\|(1+|k|)^s\int \hat{a}(k-k')\hat{u}(k')dk'\|_{L^2} \le C(s)\|(1+|k'|)^s\hat{u}(k')\|_{L^2} \tag{10.15}$$

Finally, denoting $v(k') := (1+|k'|)^s\hat{u}(k')$, rewrite (10.15) as

$$\|(1+|k|)^s\int \hat{a}(k-k')\frac{v(k')}{(1+|k'|)^s}dk'\|_{L^2} \le C(s)\|\hat{v}(k')\|_{L^2} \tag{10.16}$$

This bound is equivalent to boundedness in L^2 of integral operator with integral kernel

$$S(k,k') = (1+|k|)^s\frac{\hat{a}(k-k')}{(1+|k'|)^s} = \hat{a}(k-k')\frac{(1+|k|)^s}{(1+|k'|)^s} \tag{10.17}$$

It remains to prove the estimates (10.2) and (10.3) for the kernel (10.17). To prove the estimates, we first note that

$$\frac{(1+|k|)^s}{(1+|k'|)^s} \le C(s)(1+|k-k'|)^{|s|}, \qquad k,k' \in \mathbb{R}^n. \tag{10.18}$$

(this is known as the *Peetre inequality*). Further, $|\hat{a}(k-k')| \le C(N)(1+|k-k'|)^{-N}$ for any $N > 0$. Therefore, we obtain the bound

$$S(k,k') \le C(s,N))(1+|k-k'|)^{|s|-N} \tag{10.19}$$

Taking $|s| - N > n$, we obtain (10.2) and (10.3). ∎

Exercise 10.5 *Prove the Peetre inequality (10.18).* **Hints:** *i) Consider first $s = 1$, then $s > 0$. ii) Reduce the case $s < 0$ to $-s > 0$.*

Exercise 10.6 *Check that (10.19) with $|s| - N > n$ implies (10.2) and (10.3).*

10.3 Boundedness of pseudodifferential operators

Theorem 10.7 *Let A be pseudodifferential operator of class $\mathcal{A}^m$ with an $m \in \mathbb{R}$. Then $Au(x)$ is defined by formula (9.6) for test functions $u \in \mathcal{S}(\mathbb{R}^n)$, and for any $s \in \mathbb{R}$ the bound holds*

$$\|Au(x)\|_{s-m} \le C(s)\|u(x)\|_s \tag{10.20}$$

where $C(s) < \infty$ does not depend on $u(x)$.

Example 10.8 For example, let us prove the theorem for particular case of differential operators of class $\mathcal{A}^m$ from Exercise 9.2 ii) with integer $m = 0, 1, 2$.

I. First we consider integer $s = m, m+1, \ldots$. In this case the Sobolev norms $\|Au(x)\|_{s-m}$ and $\|u(x)\|_s$ admit equivalent characterisation (6.1), hence (10.20) is equivalent to the estimate

$$|||Au(x)|||_{s-m} \le C(s)|||u(x)|||_s. \tag{10.21}$$

By (9.1) and triangle inequality for the norm $|||\cdot|||$, we have

$$|||Au(x)|||_{s-m} \le \sum_{|\alpha|\le m} |||a_\alpha(x)\partial^\alpha u(x)|||_{s-m}. \tag{10.22}$$

Hence, it suffices to prove the estimate for every summand in (10.22), i.e.

$$|||a_\alpha(x)\partial^\alpha u(x)|||_{s-m} \le C(s)|||u(x)|||_s \tag{10.23}$$

for $|\alpha| \le m$: then summing up in $|\alpha|$, we obtain (10.21). Further, (10.23) means by definition that

$$\sum_{|\beta|\le s-m} \|\partial^\beta\Big[a_\alpha(x)\partial^\alpha u(x)\Big]\|_{L^2} \le C(s)\sum_{|\gamma|\le s}\|\partial^\gamma u(x)\|_{L^2}. \tag{10.24}$$

Finally, this inequality holds since every derivative $\partial^\beta\Big[a_\alpha(x)\partial^\alpha u(x)\Big]$ is the sum of products of derivatives of the coefficients $a_\alpha(x)$ and derivatives of $\partial^\gamma u(x)$ with $|\gamma| \le s$ since $|\beta| \le s-m$ and $|\alpha| \le m$. It remains to note that every derivative of $a_\alpha(x)$ is bounded function in $\mathbb{R}^n$ since $a_\alpha(x) = a^0_\alpha(k) + a'_\alpha(x)$ with $a'_\alpha(x) \in \mathcal{S}(\mathbb{R}^n)$. ■

II. Now let us prove the bound (10.20) for the same class of differential operators and for all $s \in \mathbb{R}$. Similarly to (10.22), we have by triangle inequality for the norm $\|\cdot\|$ that

$$\|Au(x)\|_{s-m} \le \sum_{|\alpha|\le m} \|a_\alpha(x)\partial^\alpha u(x)\|_{s-m}. \tag{10.25}$$

Hence, again it suffices to prove the estimate for every summand in (10.25), i.e.

$$\|a_\alpha(x)\partial^\alpha u(x)\|_{s-m} \le C(s)\|u(x)\|_s \tag{10.26}$$

for $|\alpha| \le m$. Since $a_\alpha(x) = a^0_\alpha(k) + a'_\alpha(x)$, we have

$$\|a_\alpha(x)\partial^\alpha u(x)\|_{s-m} \le \|a^0_\alpha\partial^\alpha u(x)\|_{s-m} + \|a'_\alpha(x)\partial^\alpha u(x)\|_{s-m}. \tag{10.27}$$

The first term in the right hand side is bounded by $C(s)\|u(x)\|_s$ by Lemma 4.4. Finally, for the second term we obtain from Lemma 10.4 that

$$\|a'_\alpha(x)\partial^\alpha u(x)\|_{s-m} \le C(s-m)\|\partial^\alpha u(x)\|_{s-m} \tag{10.28}$$

since $a'_\alpha(x) \in \mathcal{S}(\mathbb{R}^n)$. ■

Proof of Theorem 10.7 for general case *Step i)* By definition 9.1, the formula (9.6) can be rewritten as

(10.29) $$Au(x) = A^0u(x) + A'u(x) = \frac{1}{(2\pi)^n}\int e^{-ikx}a^0(k)\hat{u}(k)dk + \frac{1}{(2\pi)^n}\int e^{-ikx}a'(x,k)\hat{u}(k)dk.$$

It suffices to prove the bound of type (10.20) for each of two terms.

Step ii) For the first term the bound follows easily. Namely, by definition of the Sobolev norm,

(10.30) $$\|A^0u(x)\|_{s-m} = \|(1+|k|)^{s-m}\widehat{A^0u}(k)\|_{L^2}.$$

Further, definition of $A^0u(x)$ in (10.29) implies that $F\Big[A^0u\Big](k) = a^0(k)\hat{u}(k)$, hence

(10.31) $$\|(1+|k|)^{s-m}\widehat{A^0u}(k)\|_{L^2} = \|(1+|k|)^{s-m}a^0(k)\hat{u}(k)\|_{L^2} \le C\|(1+|k|)^s\hat{u}(k)\|_{L^2}$$

since $a^0(k) \le C(1+|k|)^m$ by condition (9.8) with $\alpha = 0$. Combining the bounds (10.30) and (10.31), we obtain the desired estimate

(10.32) $$\|A^0u(x)\|_{s-m} \le C\|u(x)\|_s.$$

Step iii) It remains to prove similar bound for second term,

(10.33) $$A'u(x) = \frac{1}{(2\pi)^n}\int e^{-ikx}a'(x,k)\hat{u}(k)dk.$$

As in (10.30)

(10.34) $$\|A'u(x)\|_{s-m} = \|(1+|k|)^{s-m}\widehat{A'u}(k)\|_{L^2}.$$

Let us calculate the Fourier transform $\widehat{A'u}(k)$. First let us prove that $A'u(x) \in L^1(\mathbb{R}^n)$. Indeed, condition (9.8) with $\alpha = \beta = 0$ implies that $|a'(x,k)| \le C(N)(1+|x|)^{-N}(1+|k|)^m$ for any $N > 0$. On the other hand, $|\hat{u}(k)| \le C(M)(1+|k|)^{-M}$ for any $M > 0$ since $u(x) \in \mathcal{S}$ and also $\hat{u}(k) \in \mathcal{S}$. Therefore,

(10.35) $$|a'(x,k)\hat{u}(k)| \le C(N,M)(1+|x|)^{-N}(1+|k|)^{m-M}$$

Therefore, (10.33) implies that

(10.36) $$|A'u(x)| \le C'(N,M)(1+|x|)^{-N}\int(1+|k|)^{m-M}dk \le C''(N,M)(1+|x|)^{-N}$$

if we take M sufficiently large, so that $m - M < -n$. Finally, this inequality implies that $A'u(x) \in L^1(\mathbb{R}^n)$ if we take $N > n$. Therefore, Lemma 5.1 implies that the Fourier transform $F\Big[A'u\Big](k)$ is given by standard Fourier integral:

(10.37) $$\widehat{A'u}(k) = \int e^{ikx}A'u(x)dx = \int e^{ikx}\Big(\frac{1}{(2\pi)^n}\int e^{-ik'x}a'(x,k')\hat{u}(k')dk'\Big)dx.$$

Applying here the Fubini theorem (it is possible by (10.35) with $m - M < -n$ and $N > n$), we obtain

(10.38) $$\widehat{A'u}(k) = \frac{1}{(2\pi)^n}\int\Big(\int e^{i(k-k')x}a'(x,k')dx\Big)\hat{u}(k')dk' = \frac{1}{(2\pi)^n}\int \hat{a}'(k-k',k')\hat{u}(k')dk'.$$

Step iv) Now we substitute (10.38) into (10.39) and obtain that

$$\begin{aligned}
\|A'u(x)\|_{s-m} &= C\|(1+|k|)^{s-m}\int \hat{a}'(k-k',k')\hat{u}(k')dk'\|_{L^2} \\
&= C\|\int \frac{(1+|k|)^{s-m}}{(1+|k'|)^s}\hat{a}'(k-k',k')(1+|k'|)^s\hat{u}(k')dk'\|_{L^2} \\
&= C\|\int S(k,k')v(k')dk'\|_{L^2}, \qquad (10.39)
\end{aligned}$$

where $S(k,k') = \dfrac{(1+|k|)^{s-m}}{(1+|k'|)^s}\hat{a}'(k-k',k')$ and $v(k') = (1+|k'|)^s\hat{u}(k')$.

Lemma 10.9 *The kernel $S(k,k')$ satisfies conditions (10.2) and (10.3) of Schur's lemma.*

Proof The crucial observation is that condition (9.9) with $\alpha = 0$ implies the bound

$$|\hat{a}'(k-k',k')| \le C(N)(1+|k-k'|)^{-N}(1+|k'|)^m \tag{10.40}$$

for any $N > 0$. This follows by arguments similar to the proof of Lemma 2.2. The bound implies that

$$\begin{aligned} |S(k,k')| &\le C\frac{(1+|k|)^{s-m}}{(1+|k'|)^s}(1+|k-k'|)^{-N}(1+|k'|)^m \\ &= C\frac{(1+|k|)^{s-m}}{(1+|k'|)^{s-m}}(1+|k-k'|)^{-N} \le C(1+|k-k'|)^{-N+|s-m|}. \end{aligned} \tag{10.41}$$

by the Peetre inequality (10.18) with $s-m$ instead of s. Now conditions (10.2) and (10.3) follow if we take N sufficiently large so that $-N+|s-m| < -n$. ■

Finally, applying Schur's lemma in (10.39), we obtain that

$$\|A'u(x)\|_{s-m} = \|\int S(k,k')v(k')dk'\|_{L^2} \le C\|v(k)\|_{L^2} = C\|u(x)\|_s, \tag{10.42}$$

that proves bound of type (10.20) for operator (10.39). ■

11 Composition of pseudodifferential operators

Here we consider composition of two pseudodifferential operators A_1 and A_2 classes $\mathcal{A}^{m_1}$ and $\mathcal{A}^{m_2}$ respectively.

Theorem 11.1 *Let the symbols $a_1(x,k)$ and $a_2(x,k)$ of pseudodifferential operators A_1 and A_2, belong to the classes $\mathcal{S}^{m_1}$ and $\mathcal{S}^{m_2}$ respectively. Then the operator A_1A_2 belongs to the class $\mathcal{A}^{m_1+m_2}$, and its symbol $a(x,k)$ admits asymptotic expansion (generalised Leibniz Formula)*

$$a(x,k) \sim \sum_{|\gamma|\geq 0} \frac{1}{\gamma!}(i\partial_k)^\gamma a_1(x,k)\partial_x^\gamma a_2(x,k). \tag{11.1}$$

First let us discuss the meaning of asymptotic expansion (11.1): by definition, it means that for any $N = 0,1,2,...$ the finite expansion holds

$$a(x,k) = \sum_{|\gamma|\geq 0}^{N-1} \frac{1}{\gamma!}(i\partial_k)^\gamma a_1(x,k)\partial_x^\gamma a_2(x,k) + R_N(x,k), \tag{11.2}$$

where the remainder $R_N(x,k)$ belongs to class $\mathcal{S}^{m_1+m_2-N}$.

Next let us consider the properties of all terms in the asymptotic expansion. Definition 9.1 implies that

i) the symbol $(i\partial_k)^\alpha a_1(x,k)$ belongs to class $\mathcal{S}^{m_1-|\alpha|}$ (**Exercise:** Check this!), and
ii) the symbol $\partial_x^\alpha a_2(x,k)$ belongs to class $\mathcal{S}^{m_2}$ (**Exercise:** Check this!).

Therefore, their product $(i\partial_k)^\alpha a_1(x,k)\partial_x^\alpha a_2(x,k)$ belongs to class $\mathcal{S}^{m_1+m_2-|\alpha|}$ by Exercise 9.3.

11.1 Composition of differential operators

I. First consider simplest example of differential operators $A_1 = \frac{d}{dx}$ and $A_2 = a(x)$ in dimension $n = 1$. Their composition is

$$Au(x) := A_1A_2u(x) = \frac{d}{dx}\Big[a(x)u(x)\Big] = a'(x)u(x) + a(x)u'(x). \tag{11.3}$$

This means that $A = A_1A_2 = a(x)\frac{d}{dx} + a'(x)$, hence the symbol of the composition is

$$a(x,k) = a(x)(-ik) + a'(x). \tag{11.4}$$

This formula coincides with (11.2) since the symbols $a_1(x,k) = -ik$ and $a_1(x,k) = a(x)$, hence
i) the term in (11.2) with $\gamma = 0$ is the product $(-ik)a(x)$,
ii) next term with $\gamma = 1$ is $a(x)$, and
iii) all terms with $|\gamma| > 1$ vanish.

II. Second, let us consider general differential operators A_1 and A_2 in dimension $n = 1$:

$$A_l u(x) = \sum_{\alpha\leq m_l} a_{l\alpha}(x)\partial^\alpha u(x), \qquad l = 1,2. \tag{11.5}$$

Their composition is also a differential operator

$$Au(x) := A_1A_2u(x) = \sum_{\substack{\alpha_1 \leq m_1 \\ \alpha_2 \leq m_2}} a_{1\alpha_1}(x)\partial^{\alpha_1}\Big[a_{2\alpha_2}(x)\partial^{\alpha_2}u(x)\Big]. \tag{11.6}$$

To define the symbol of this operator, we have to rewrite it in the standard form (9.1), where all coefficients are on the left of the operators of differentiation. It suffices to rewrite each term in the sum (11.6). Moreover, it suffices to consider the composition $\partial^{\alpha_1}\Big[a_{2\alpha_2}(x)\partial^{\alpha_2}u(x)\Big]$. Using the Leibniz formula for the product, we obtain that

$$\partial^{\alpha_1}\Big[a_{2\alpha_2}(x)\partial^{\alpha_2}u(x)\Big] = \sum_{\gamma\le\alpha_1} C^{\gamma}_{\alpha_1}\partial^{\gamma}_x a_{2\alpha_2}(x)\partial^{\alpha_1-\gamma+\alpha_2}u(x), \tag{11.7}$$

where $C^{\gamma}_{\alpha_1} = \dfrac{\alpha_1!}{\gamma!(\alpha_1-\gamma)!}$. Hence we can rewrite (11.6) in the standard form as

$$Au(x) = \sum_{\substack{\alpha_1\le m_1\\ \alpha_2\le m_2}} a_{1\alpha_1}(x)\sum_{\gamma\le\alpha_1} C^{\gamma}_{\alpha_1}\partial^{\gamma}_x a_{2\alpha_2}(x)\partial^{\alpha_1-\gamma+\alpha_2}u(x). \tag{11.8}$$

Now the symbol of the composition is

$$a(x,k) = \sum_{\substack{\alpha_1\le m_1\\ \alpha_2\le m_2}} a_{1\alpha_1}(x)\sum_{\gamma\le\alpha_1} C^{\gamma}_{\alpha_1}\partial^{\gamma}_x a_{2\alpha_2}(x)(-ik)^{\alpha_1-\gamma+\alpha_2} \tag{11.9}$$

according to formula (9.7).

Let us check that the symbol coincides with general formula (11.2) for the particular case of differential operators A_1 and A_2. Indeed, we can rewrite (11.9) as

$$a(x,k) = \sum_{\gamma\le\alpha_1}\sum_{\alpha_1\le m_1}\frac{1}{\gamma!}\frac{\alpha_1!}{(\alpha_1-\gamma)!}(-ik)^{\alpha_1-\gamma}a_{1\alpha_1}(x)\partial^{\gamma}_x\sum_{\alpha_2\le m_2}a_{2\alpha_2}(x)(-ik)^{\alpha_2}. \tag{11.10}$$

It remains to note that $\dfrac{\alpha_1!}{(\alpha_1-\gamma)!}(-ik)^{\alpha_1-\gamma} = (i\partial)^{\gamma}(-ik)^{\alpha_1}$ (**Exercise:** Check this!). Hence (11.10) becomes

$$a(x,k) = \sum_{\gamma\le\alpha_1}\sum_{\alpha_1\le m_1}\frac{1}{\gamma!}(i\partial)^{\gamma}(-ik)^{\alpha_1}a_{1\alpha_1}(x)\partial^{\gamma}_x\sum_{\alpha_2\le m_2}a_{2\alpha_2}(x)(-ik)^{\alpha_2} = \sum_{\gamma\le m_1}\frac{1}{\gamma!}(i\partial_k)^{\gamma}a_1(x,k)\partial^{\gamma}_x a_2(x,k). \tag{11.11}$$

Finally, this formula coincides with (11.2) since $(i\partial_k)^{\gamma}a_1(x,k) = 0$ for $\gamma > m_1$.

Now we are prepared to prove formula (11.2) for general pseudodifferential operators A_1 and A_2.

11.2 Composition of PDO

Step i) Let us take any test function $u(x)\in\mathcal{S}(\mathbb{R}^n)$. By definition of pseudodifferential operator A_1, we have

$$A_1A_2u(x) = \frac{1}{(2\pi)^n}\int e^{-ikx}a_1(x,k)\widehat{A_2u}(k)dk, \tag{11.12}$$

By definition of pseudodifferential operator A_2, we have

$$A_2u(x) = \frac{1}{(2\pi)^n}\int e^{-ikx}a_2(x,k)\hat u(k)dk = \frac{1}{(2\pi)^n}\int e^{-ikx}a_2^0(k)\hat u(k)dk + \frac{1}{(2\pi)^n}\int e^{-ikx}a_2'(x,k)\hat u(k)dk$$

$$= A_2^0u(x) + A_2'u(x). \tag{11.13}$$

Respectively, $A_1A_2 = A_1A_2^0 + A_1A_2'$, and it suffices to prove formula (11.2) for each term separately since the formula is linear in symbol $a_2(x,k)$.

Step ii) First let us consider term $A_1A_2^0$:

$$A_1A_2^0u(x) = \frac{1}{(2\pi)^n}\int e^{-ikx}a_1(x,k)\widehat{A_2^0u}(k)dk \tag{11.14}$$

Definition of A_2^0u in (11.13) implies that $\widehat{A_2^0u}(k) = a_2^0(k)\hat{u}(k)$, hence (11.14) becomes

$$A_1A_2^0u(x) = \frac{1}{(2\pi)^n}\int e^{-ikx}a_1(x,k)a_2^0(k)\hat{u}(k)dk. \tag{11.15}$$

By definition 9.4 (see (9.6)), this means that $A_1A_2^0$ is pseudodifferential operator with symbol $a(x,k) = a_1(x,k)a_2^0(k)$. It remains to note that this product coincides with formula (11.2) for the composition $A_1A_2^0$ since

i) first term with $\gamma = 0$ is just product of the symbols, and

ii) all derivatives $\partial_x^\gamma a_2^0(k)$ vanish for $|\gamma| > 0$.

Step iii) It remains to study term A_1A_2':

$$A_1A_2'u(x) = \frac{1}{(2\pi)^n}\int e^{-ikx}a_1(x,k)\widehat{A_2'u}(k)dk \tag{11.16}$$

Let us calculate symbol of the composition A_1A_2'. First, let us evaluate the Fourier transform $\widehat{A_2'u}$. Definition of A_2^0u in (11.13) implies that

$$A_2'u(x) = \frac{1}{(2\pi)^n}\int e^{-ik'x}a_2'(x,k')\hat{u}(k')dk', \tag{11.17}$$

Hence by the Fubini theorem, we obtain, similarly to (10.38), that

$$\widehat{A_2'u}(k) = \frac{1}{(2\pi)^n}\int \hat{a}_2'(k-k',k')\hat{u}(k')dk', \tag{11.18}$$

where integral kernel $\hat{a}_2'(k-k',k')$ satisfies the estimate of type (10.40):

$$|\hat{a}_2'(k-k',k')| \le C(M_2)(1+|k-k'|)^{-M_2}(1+|k'|)^{m_2} \tag{11.19}$$

for any $M_2 > 0$. Substituting expression (11.18) into (11.16), we obtain

$$A_1A_2'u(x) = \frac{1}{(2\pi)^n}\int e^{-ikx}a_1(x,k)\Big(\frac{1}{(2\pi)^n}\int \hat{a}_2'(k-k',k')\hat{u}(k')dk'\Big)dk. \tag{11.20}$$

We are going to apply the Fubini theorem to change the order of integrations, and rewrite (11.20) as

$$A_1A_2'u(x) = \frac{1}{(2\pi)^n}\int\Big(\frac{1}{(2\pi)^n}\int e^{-ikx}a_1(x,k)\hat{a}_2'(k-k',k')dk\Big)\hat{u}(k')dk'. \tag{11.21}$$

Exercise 11.2 Justify the applicability of the Fubini theorem. **Hint:** By 9.1, we have that $|a_1(x,k)| \le C(1+|k|)^{m_1}$. Hence, (11.19) implies that the integrand of (11.20) admits the estimate

$$|e^{-ikx}a_1(x,k)\hat{a}_2'(k-k',k')\hat{u}(k')| \le C(M_2,M_3)(1+|k|)^{m_1}(1+|k-k'|)^{-M_2}(1+|k'|)^{m_2-M_3} \tag{11.22}$$

for any $M_2, M_3 > 0$, since $|\hat{u}(k')| \le C(1+|k'|)^{-M_3}$ for any $M_3 > 0$. Applying the Peetre inequality $(1+|k|)^{m_1} \le (1+|k'|)^{m_1}(1+|k-k'|)^{|m_1|}$, we obtain

$$|e^{-ikx}a_1(x,k)\hat{a}_2'(k-k',k')\hat{u}(k')| \le C(M_2,M_3)(1+|k-k'|)^{|m_1|-M_2}(1+|k'|)^{m_1+m_2-M_3} \tag{11.23}$$

where the right hand side is summable function of $(k,k') \in \mathbb{R}^{2n}$ if $|m_1|-M_2 < -n$ and $m_1+m_2-M_3 < -n$.

Finally, we can write (11.20) in the form

$$A_1A_2'u(x) = \frac{1}{(2\pi)^n}\int e^{-ik'x}\Big(\frac{1}{(2\pi)^n}\int e^{-i(k-k')x}a_1(x,k)\hat a_2'(k-k',k')dk\Big)\hat u(k')dk'. \tag{11.24}$$

By definition 9.4 (see (9.6)) this means that A_1A_2' is pseudodifferential operator with symbol

$$a(x,k') = \frac{1}{(2\pi)^n}\int e^{-i(k-k')x}a_1(x,k)\hat a_2'(k-k',k')dk. \tag{11.25}$$

Step iv) It remains to prove the asymptotic expansion (11.1) for this symbol. First we expand the symbol $a_1(x,k)$ in the Taylor series in k with the centre at k':

$$a_1(x,k) = \sum_{|\gamma|=0}^{N-1}\frac{1}{\gamma!}\partial_k^\gamma a_1(x,k')(k-k')^\gamma + r_N(x,k,k'), \tag{11.26}$$

where the remainder $r_N(x,k,k')$ can be represented in the Cauchy form

$$r_N(x,k,k') = \sum_{|\gamma|=N}\frac{1}{\gamma!}\Big[\int_0^1(1-t)^{N-1}\partial_k^\gamma a_1(x,k'+t(k-k'))dt\Big](k-k')^\gamma. \tag{11.27}$$

Substituting (11.26) into (11.25), we obtain that

$$\begin{aligned} a(x,k') &= \frac{1}{(2\pi)^n}\sum_{|\gamma|=0}^{N-1}\frac{1}{\gamma!}\partial_k^\gamma a_1(x,k')\int e^{-i(k-k')x}(k-k')^\gamma\hat a_2'(k-k',k')dk \\ &+ \frac{1}{(2\pi)^n}\int e^{-i(k-k')x}r_N(x,k,k')\hat a_2'(k-k',k')dk. \end{aligned} \tag{11.28}$$

We note that the sum in (11.28) coincides with the sum in finite asymptotic expansion (11.2) since

$$\frac{1}{(2\pi)^n}\int e^{-i(k-k')x}(k-k')^\gamma\hat a_2'(k-k',k')dk = (i\partial_x)^\gamma a_2(x,k') \tag{11.29}$$

by formula (5.1) for the inversion of the Fourier transform. Note that the formula is applicable since the integral converges by the bound (11.19) with $N > n+|\gamma|$.

Step v) It remains to prove necessary bounds for the remainder

$$R_N(x,k') = \frac{1}{(2\pi)^n}\int e^{-i(k-k')x}r_N(x,k,k')\hat a_2'(k-k',k')dk. \tag{11.30}$$

We have to prove that $R_N(x,k')\in\mathcal{S}^{m_1+m_2-N}$. More precisely, we will prove the bounds of type (9.9):

$$(1+|x|)^M|\partial_k^\alpha\partial_x^\beta R_N(x,k')| \le C(\alpha,\beta,M)(1+|k'|)^{m_1+m_2-N-|\alpha|}, \qquad x,k'\in\mathbb{R}^n \tag{11.31}$$

for any $M>0$ and any multiindexes α and β.

Remark 11.3 *Bounds (11.31) mean that the first component $R_N^0(k')$ for this symbol vanishes.*

We will prove bounds (11.31) for $M = 0$ and $\alpha = \beta = 0$. First, (11.30) implies that

$$|R_N(x,k')| \le C \int |r_N(x,k,k')\hat{a}_2'(k-k',k')| dk. \tag{11.32}$$

The Cauchy formula (11.27) implies that

$$\begin{aligned} |r_N(x,k,k')| &\le \sum_{|\gamma|=N} \frac{1}{\gamma!}\Big[\int_0^1 (1-t)^{N-1}|\partial_k^\gamma a_1(x,k'+t(k-k'))|dt\Big]|(k-k')^\gamma| \\ &\le C(1+|k'+t(k-k')|)^{m_1-N}(1+|k-k'|)^N. \end{aligned} \tag{11.33}$$

Substituting the estimate to (11.32), we obtain that

$$\begin{aligned} &|R_N(x,k')| \\ &\le C\int\Big[\int_0^1 (1-t)^{N-1}(1+|k'+t(k-k')|)^{m_1-N}dt\Big](1+|k-k'|)^N \hat{a}_2'(k-k',k')|dk. \end{aligned} \tag{11.34}$$

Now we use the estimate (11.19) for $\hat{a}_2'(k-k',k')$ and obtain

$$\begin{aligned} &|R_N(x,k')| \\ &\le C(M_2)(1+|k'|)^{m_2}\int\Big[\int_0^1 (1-t)^{N-1}(1+|k'+t(k-k')|)^{m_1-N}dt\Big](1+|k-k'|)^{N-M_2}dk. \end{aligned} \tag{11.35}$$

Therefore, the bound (11.31) with $M = 0$ and $\alpha = \beta = 0$ follows from next lemma.

Lemma 11.4 *For sufficiently large $M_2 > 0$, we have*

$$\int_{\mathbb{R}^n}\Big[\int_0^1 (1-t)^{N-1}(1+|k'+t(k-k')|)^{m_1-N}dt\Big](1+|k-k'|)^{N-M_2}dk \le C(1+|k'|)^{m_1-N}. \tag{11.36}$$

Proof Let us split the integration over $\mathbb{R}^n$ into two regions:
i) over $|k-k'| < |k'|/2$ and
ii) over $|k-k'| > |k'|/2$.

I. In first region, we have $k' + t(k-k') \approx k'$, or more precisely,

$$\frac{1}{2}|k'| \le |k'+t(k-k')| \le \frac{3}{2}|k'|, \qquad 0 \le t \le 1. \tag{11.37}$$

Hence,

$$C_1(1+|k'|)^{m_1-N} \le (1+|k'+t(k-k')|)^{m_1-N} \le C_2(1+|k'|)^{m_1-N}, \tag{11.38}$$

where $C_1, C_2 > 0$. Therefore, for large M_2 the first integral is bounded by

$$\begin{aligned} C(1+|k'|)^{m_1-N}\int_{|k-k'|<|k'|/2}(1+|k-k'|)^{N-M_2}dk &\le C'(1+|k'|)^{m_1-N}\int_{\mathbb{R}^n}(1+|k-k'|)^{N-M_2}dk \\ &\le C''(1+|k'|)^{m_1-N} \end{aligned} \tag{11.39}$$

since the integral is finite for $M_2 > N + n$.

II. It remains to bound the integral over second region $|k-k'|>|k'|/2$. Then $|k'+t(k-k')|\le |k'|+|t(k-k')|\le 3|k-k'|$, hence

$$(1+|k'+t(k-k')|)^{m_1-N}\le C(1+|k-k'|)^{|m_1-N|} \tag{11.40}$$

Hence, the integral over second region is bounded by

$$\begin{aligned} C\int\limits_{|k-k'|>|k'|/2}(1+|k-k'|)^{|m_1-N|+N-M_2}dk &= C\int_{|k'|/2}^{\infty}(1+r)^{|m_1-N|+N-M_2}r^{n-1}dr \\ &\le C''(1+r)^{|m_1-N|+N-M_2+n}\Big|_{|k'|/2}^{\infty} \\ &= C''(1+|k'|/2)^{|m_1-N|+N-M_2+n} \qquad (11.41) \end{aligned}$$

if $|m_1-N|+N-M_2+n<0$. Therefore, the integral is bounded by $C(1+|k'|)^{m_1-N}$ if M_2 is sufficiently large so that $|m_1-N|+N-M_2+n<m_1-N$. ■

We have proved bounds (11.31) for $M=0$ and $\alpha=\beta=0$. For all other values of the parameters the proof is similar. Theorem 11.1 is proved. ■

12 Regulariser of elliptic equations

We consider partial differential operator

$$Au(x) = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha u(x) \tag{12.1}$$

with symbol

$$a(x,k) = \sum_{|\alpha|\le m} a_\alpha(x)(-ik)^\alpha. \tag{12.2}$$

In next section we will study the questions on existence, uniqueness and smoothness of solutions $u(x)$ to partial differential equation in $\mathbb{R}^n$

$$Au(x) = f(x), \qquad x \in \mathbb{R}^n \tag{12.3}$$

for **elliptic** and **strongly elliptic** operators A.

Definition 12.1 *(Cf. Definition 8.3) i) Differential operator A is* **elliptic of order** *m if*

$$|a(x,k)| > c(1+|k|)^m, \qquad |k| > R, \quad x,k \in \mathbb{R}^n \tag{12.4}$$

for some $R > 0$ and $c > 0$.
iii) Differential operator A is **strongly elliptic of order** *m if*

$$|a(x,k)| > c(1+|k|)^m, \qquad x,k \in \mathbb{R}^n, \tag{12.5}$$

where $c > 0$.

Example 12.2 *i) Let $V(x) \in \mathcal{S}(\mathbb{R}^n)$. Then the Schrödinger operator $-\Delta + V(x) - E$ is* **elliptic** *for every $E \in \mathbb{C}$ (check this!).*
ii) Let $V(x) \in \mathcal{S}(\mathbb{R}^n)$, and $V(x) \ge 0$ for $x \in \mathbb{R}^n$. Then the Schrödinger operator $-\Delta + V(x) - E$ is **strongly elliptic**
a) for every $E < 0$ (check this!);
b) for every $E \in \mathbb{C} \setminus \mathbb{R}^+$ where $\mathbb{R}^+ = \{E \in \mathbb{R} : E \ge 0\}$ (check this!).

We start with definition of "regulariser" R of the operator A which is "almost inverse" to the operator A in the following sense:

Definition 12.3 *An operator $R : H^{s-m} \to H^s$ is called* **regulariser** *of the operator A if*
I. R is PDO of class $\mathcal{A}^{-m}$, hence continuous $H^{s-m} \to H^s$ for any $s \in \mathbb{R}$;

II. $RA = 1 + T$ and $AR = 1 + \mathcal{T}$, where T and $\mathcal{T}$ are PDO of class $\mathcal{A}^{-1}$, hence continuous $H^s \to H^{s+1}$ and $H^{s-m} \to H^{s-m+1}$ respectively;

III. T and $\mathcal{T}$ are **compact operators** *in H^s and H^{s-m} respectively.*

We will construct the regulariser for **strongly elliptic** operators A by the following formula:

$$Rf(x) = \frac{1}{(2\pi)^n}\int e^{-ikx} r(x,k)\hat{f}(k)dk, \qquad \text{where} \quad r(x,k) = \frac{1}{a(x,k)} \tag{12.6}$$

Theorem 12.4 *Let A be strongly elliptic operator. Then the operator (12.6) is regulariser of A.*

Proof

Exercise 12.5 Check property I. **Hint:** Definition (12.5) implies that the inverse symbol $r(x,k) = 1/a(x,k)$ belongs to class $\mathcal{S}^{-m}$. Therefore, $R : H^{s-m} \to H^s$ is continuous by Theorem 10.7.

Exercise 12.6 Check property II. **Hint:** Apply Theorem 11.1 to the compositions RA and RA: in generalised Leibniz formula (11.2) with $N = 1$, the term with $\gamma = 0$ gives unit operator 1, while the symbols of remainders T and $\mathcal{T}$ belong to class $\mathcal{S}^{-1}$. Therefore, $T : H^s \to H^{s+1}$ and $\mathcal{T} : H^{s-m} \to H^{s-m+1}$ are continuous by Theorem 10.7.

It remains to prove property III:

Proposition 12.7 *T and $\mathcal{T}$ are* **compact operators** *in H^s and H^{s-m} respectively, for any $s \in \mathbb{R}$.*

Proof We will prove the theorem with an additional assumption on the coefficients $a_\alpha(x) = a^0_\alpha + a'_\alpha(x)$ of the operator (12.1). Namely, let all terms $a'_\alpha(x)$ vanish for large $|x|$, i.e.

$$a'_\alpha(x) = 0, \qquad |x| \ge R, \quad |\alpha| \le m. \tag{12.7}$$

Then in the corresponding decomposition of the symbol $a(x,k) = a^0(k) + a'(x,k)$, the term $a'(x,k)$ also vanishes for large $|x|$, hence we have

$$a(x,k) = a^0(k), \qquad |x| \ge R. \tag{12.8}$$

In other words, the symbol $a(x,k)$ does not depend on x for $|x| \ge R$. Therefore, the same is true for inverse symbol $r(x,k) = 1/a(x,k)$. Hence, $r(x,k) = r^0(k) + r'(x,k)$, and

$$r'(x,k) = r^0(k), \qquad |x| \ge R. \tag{12.9}$$

For concreteness, we will prove the compactness for the operator $T : H^s \to H^s$ since the compactness of operator $\mathcal{T} : H^{s-m} \to H^{s-m}$ follows similarly.

Let us denote by $c(x,k)$ symbol of the composition $C = RA$. Applying generalised Leibniz formula (11.2) with sufficiently large $N > 0$, we obtain the expansion

$$c(x,k) = \sum_{|\gamma|\ge 0}^{N-1} \frac{1}{\gamma!}(i\partial_k)^\gamma r(x,k)\partial_x^\gamma a(x,k) + R_N(x,k), \tag{12.10}$$

Then the corresponding operator C admits the decomposition

$$C = \sum_{|\gamma|\ge 0}^{N-1} C_\gamma + D_N, \tag{12.11}$$

where C_γ stands for PDO with symbol $\frac{1}{\gamma!}(i\partial_k)^\gamma r(x,k)\partial_x^\gamma a(x,k)$, and D_N stands for PDO with symbol $R_N(x,k)$.

We have $C_0 = r(x,k)a(x,k) = 1$, so it remains to prove compactness of each term in (12.11) with $|\gamma| > 0$.

Lemma 12.8 *Each operator $C_\gamma : H^s \to H^s$ is compact if $|\gamma| > 0$.*

Proof We have to prove that for any bounded sequence $u_j \in H^s$, the sequence $C_\gamma u_j$ has a limit point, i.e. there exists a converging in H^s subsequence $C_\gamma u_{j'}$:

$$\|C_\gamma u_{j'} - v\|_s \to 0, \qquad j' \to \infty \tag{12.12}$$

for some $v \in H^s$. By Theorem 10.7 operator $C_\gamma : H^s \to H^{s+|\gamma|}$ is bounded since C_γ is PDO of class $\mathcal{A}^{-|\gamma|}$ by Theorem 11.1. Therefore, the sequence $C_\gamma u_j$ is bounded in $H^{s+|\gamma|}(\mathbb{R}^n)$, hence bounded in $H^{s+1}(\mathbb{R}^n)$ since $|\gamma| \geq 1$. Furthermore,

$$C_\gamma u_j(x) = \frac{1}{(2\pi)^n}\int e^{-ikx}\frac{1}{\gamma!}(i\partial_k)^\gamma r(x,k)\partial_x^\gamma a(x,k)\hat{u}_j(k)dk = 0, \qquad |x| \geq R \tag{12.13}$$

since $\partial_x^\gamma a(x,k) = 0$ for $|x| > R$ by (12.8) and condition $|\gamma| > 0$. Therefore, $C_\gamma u_j(x) \in \overset{o}{H}{}^{s+1}(\Omega)$, where Ω stands for the ball $\{x \in \mathbb{R}^n : |x| < R\}$.

So, functions $C_\gamma u_j$ belong to a bounded set in $\overset{o}{H}{}^{s+1}(\Omega)$. Therefore, the sequence $C_\gamma u_j$ contains a converging subsequence in $H^s(\mathbb{R}^n)$ by Sobolev's theorem on compact embedding $\overset{o}{H}{}^{s+1}(\Omega) \subset H^s(\mathbb{R}^n)$ (Theorem 7.2). ■

It remains to prove compactness of operator $D_N : H^s \to H^s$ for sufficiently large $N > 0$.

Proposition 12.9 *Operator $D_N : H^s \to H^s$ is compact if $N > 0$ is sufficiently large.*

Proof Let us introduce PDO Λ^s with symbol $(1+|k|)^s$: by definition 9.1, we have

$$\Lambda^s u = \frac{1}{(2\pi)^n}F^{-1}\Big[(1+|k|)^s\hat{u}(k)\Big]. \tag{12.14}$$

Then $\Lambda^s : H^s \to L^2$ is isomorphism by definition of the Sobolev space H^s. Indeed, $u(x) \in H^s$ is equivalent to the fact that $(1+|k|)^s\hat{u}(k) \in L^2$. Finally, last relation is equivalent to $F^{-1}\Big[(1+|k|)^s\hat{u}(k)\Big] \in L^2$ by the Parseval Theorem. Hence, $\Lambda^s u \in L^2$ by (12.14).

Obviously, Λ^{-s} is inverse operator to Λ^s. Hence, we can write $D_N = \Lambda^{-s}\Lambda^s D_N \Lambda^{-s}\Lambda^s$. It suffices to prove that operator $K := \Lambda^s D_N \Lambda^{-s} : L^2 \to L^2$ is compact in L^2. Then the composition $D_N = \Lambda^{-s}K\Lambda^s : H^s \to H^s$ will be compact. Indeed:

i) $\Lambda^s u_j$ is bounded sequence in L^2 since $\Lambda^s : H^s \to L^2$ is bounded operator;

ii) Then the sequence $K\Lambda^s u_j$ contains converging in L^2 subsequence $K\Lambda^s u_{j'}$;

iii) Hence the subsequence $D_N u_j = \Lambda^{-s}K\Lambda^s u_{j'}$ is converging in H^s since $\Lambda^{-s} : L^2 \to H^s$ is continuous operator.

So, it remains to prove

Lemma 12.10 *Operator $K := \Lambda^s D_N \Lambda^{-s} : L^2 \to L^2$ is compact in L^2 if $N > 0$ is sufficiently large.*

Proof By Theorem 11.1, the composition $K := \Lambda^s D_N \Lambda^{-s}$ is PDO of class $\mathcal{A}^{-N}$, and the Fourier representation $\widehat{Ku}(k)$ admits integral representation

$$\widehat{Ku}(k) = (1+|k|)^s\int_{\mathbb{R}^n}\hat{R}_N(k-k',k')(1+|k'|)^{-s}\hat{u}(k')dk', \tag{12.15}$$

similarly to (11.18). The function $\hat{R}_N(k-k',k')$ admits the bound

$$|\hat{R}_N(k-k',k')| \leq C(M)(1+|k-k'|)^{-M}(1+|k'|)^{-N} \tag{12.16}$$

for any $M > 0$. The bounds follow from (11.31) similarly to (11.19) and (10.40). Therefore, the integral kernel $K(k,k') := (1+|k|)^s \hat{R}_N(k-k',k')(1+|k'|)^{-s}$ of the operator K admits the bound

$$
\begin{aligned}
|K(k,k')| &\le C(M)(1+|k|)^s(1+|k-k'|)^{-M}(1+|k'|)^{-N} \\
&\le C'(M)(1+|k-k'|)^{-M+|s|}(1+|k'|)^{-N+s}, \qquad (12.17)
\end{aligned}
$$

where we used the Peetre inequality $(1+|k|)^s \le (1+|k-k'|)^{|s|}(1+|k'|)^s$ as in (11.23). This estimate implies that the integral operator K belongs to the *Hilbert-Schmidt class* of operators, i.e.

$$
\int |K(k,k')|^2 dk dk' < \infty \qquad (12.18)
$$

if $2(-M+|s|) < -n$ and $2(-N+s) < -n$. This property provides that the operator K is compact in $L^2(\mathbb{R}^n)$ (see e.g. Exercise 15 (c) in [4]). ■

13 Applications of regulariser

13.1 Smoothness of solutions

Lemma 13.1 *("Schauder's Lemma") Let A be strongly elliptic operator, and consider a solution $u \in H^s$ of equation (12.3). Let us assume that $f \in H^t$, where $t \in \mathbb{R}$. Then $u \in H^{t+m}$.*

Proof The lemma is trivial if $t + m \leq s$, since $H^s \subset H^{t+m}$ in this case. Further we consider the case $t + m > s$, so $j := t + m - s > 0$, and additionally assume that j is integer, i.e. $j = 1, 2, ...$ for simplicity of exposition. Then $t + m = s + j$, and we have to check that $\|u\|_{s+j} < \infty$.

Applying R to both sides of (12.3), we obtain $RAu = u+Tu = Rf$. Rewriting this as $u = Rf - Tu$, we obtain that

$$\|u\|_s \leq \|Rf\|_s + \|Tu\|_s \leq C\|f\|_{s-m} + C_1\|u\|_{s-1} \tag{13.1}$$

since $R \in \mathcal{A}^{-m}$ and $T \in \mathcal{A}^{-1}$. Hence, the following **Schauder a priori estimate** holds for the solutions to (12.3):

$$\|u\|_{s+1} \leq C\|f\|_{s+1-m} + C_1\|u\|_s < \infty. \tag{13.2}$$

Hence, $\|u\|_{s+1} < \infty$ since $\|f\|_{s+1-m} < \infty$ due to $s + 1 - m \leq t$. By induction, $\|u\|_{s+l} < \infty$ until $s + l - m \leq t$. Taking $l = j$, we obtain that $\|u\|_{s+j} < \infty$. ■

Corollary 13.2 *("Weyl's lemma") Let A be strongly elliptic operator. Then for any $s \in \mathbb{R}$, all solutions $u(x) \in H^s(\mathbb{R}^n)$ to (homogeneous) equation (12.3) with $f = 0$ are smooth functions.*

Proof Applying Schauder's lemma, we obtain that $u \in H^{t+m}$ with any $t \in \mathbb{R}$ since $f = 0 \in H^t$ with any $t \in \mathbb{R}$. Hence, $u \in H^s$ with any $s \in \mathbb{R}$. Finally, this implies that $u(x) \in C^\infty(\mathbb{R}^n)$ by Corollary 5.5.

13.2 Solvability of elliptic equations

Next theorem is main result of our course.

Theorem 13.3 *(Cf. Theorem 8.5) Let us consider equation with variable coefficients (12.3). Let us assume that A is strongly elliptic operator of order m. Then for any $s \in \mathbb{R}$,*
i) The solutions to corresponding homogeneous equation

$$Au(x) = 0, \qquad x \in \mathbb{R}^n, \tag{13.3}$$

constitute finite dimensional space in $H^s(\mathbb{R}^n)$.
ii) Nonhomogeneous equation (12.3) with $f(x) \in H^{s-m}(\mathbb{R}^n)$, admits solution $u(x) \in H^s(\mathbb{R}^n)$ if $f(x)$ satisfies finite number of "orthogonality conditions" of type

$$L_j(f) = 0, \qquad j = 1, ..., M, \tag{13.4}$$

where L_j are linear continuous functionals on the space $H^{s-m}(\mathbb{R}^n)$.

Proof i) Applying the regulariser R to both sides of (13.3), we obtain

$$RAu = u + Tu = 0, \tag{13.5}$$

where T is compact operator in H^s. Hence, the linear space of the solutions $u(x)$ is finite dimensional by First Fredholm Theorem [4].

ii) Let us seek the solution to (12.3) in the form $u = Rg$. Substituting to (12.3), we obtain the equation

$$ARg = g + \mathcal{T}g = f, \tag{13.6}$$

where $\mathcal{T}$ is compact operator in H^{s-m}. Hence, Second Fredholm Theorem [4] guaranties that the equation $g + \mathcal{T}g = f$ admits a solution $g \in H^{s-m}$ if $f(x)$ satisfies finite number of "orthogonality conditions" of type (13.4). Then $u = Rg$ is solution to (12.3) by (13.6). ■