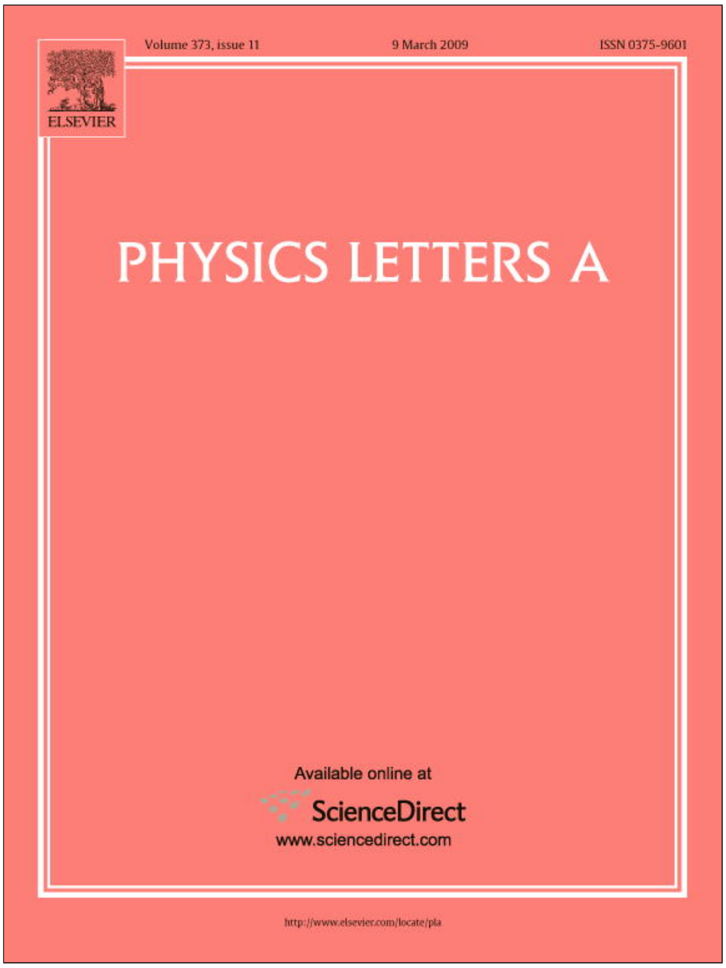
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# Scattering in the nonlinear Lamb system

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A B S T R A C T

We obtain long time asymptotics for the solutions to a string coupled to a nonlinear oscillator: each finite energy solution decays to a sum of a stationary state and a dispersive wave. The asymptotics hold in global energy norm. The dispersive waves are expressed via initial data and solution to an ordinary differential equation. The asymptotics give a mathematical model for the Bohr's transitions between quantum stationary states.



## 1. Introduction

In this Letter we consider the scattering in the nonlinear Lamb system

$$\begin{cases} \ddot{u}(x,t) = u''(x,t), \quad x \in \mathbb{R} \setminus \{0\}, \\ m\ddot{y}(t) = F(y(t)) + u'(+0,t) - u'(-0,t); \quad y(t) \equiv u(0,t), \end{cases} \tag{1.1}$$

where $m > 0$. Here $\dot{u} := \frac{\partial u}{\partial t}$, $u' := \frac{\partial u}{\partial x}$. The case $m = 0$ was considered in [22]. The solutions $u(x,t)$ take the values in $\mathbb{R}^d$ with $d \geqslant 1$.

Physically, the problem (1.1) describes small crosswise oscillations of an infinite string stretched parallel to the $0x$-axis; a particle of mass $m > 0$ is attached to the string at the point $x = 0$; $F(y)$ is an external (nonlinear) force field perpendicular to $0x$, the field subjects the particle (see Fig. 1).

The system (1.1) has been introduced originally by H. Lamb [21] in the linear case when $F(y) = -\omega^2 y$. The Lamb system with general nonlinear $F(y)$ and the oscillator mass $m \geqslant 0$ has been considered in [13] where the questions of irreversibility and nonrecurrence were discussed. The system was studied further in [14–17] where the global attraction to stationary states has been established for the first time, and in [6] where metastable regimes were studied for the stochastic Lamb system.

The Lamb system (1.1) is used in all the papers cited above as an example of simplest nontrivial nonlinear time reversible conservative system allowing an effective analysis of various questions. In present Letter, we study the nonlinear scattering for the Lamb system with a nontrivial attractor. We consider the Cauchy problem for the system (1.1) with the initial conditions

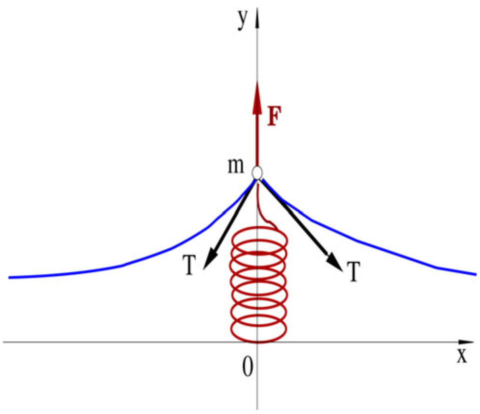


**Fig. 1.** String coupled to an oscillator.

$$u|_{t=0} = u_0(x), \qquad \dot{u}|_{t=0} = v_0(x), \qquad \dot{y}|_{t=0} = p_0 \tag{1.2}$$

where $y(t) := u(0,t)$. Let us denote $Y(t) = (u(x,t), \dot{u}(x,t), \dot{y}(t))$. Then the Cauchy problem (1.1), (1.2) *formally* reads

$$\dot{Y}(t) = \mathbf{F}\big(Y(t)\big) \quad \text{for } t \in \mathbb{R}, \qquad Y(0) = Y_0, \tag{1.3}$$

where $Y_0 = (u_0, v_0, p_0)$, and

$$\mathbf{F}\big(Y(t)\big) = \big(\dot{u}(\cdot,t), u''(x,t)|_{x\neq 0}, F\big(u(0,t)\big) + u'(+0,t) - u'(-0,t)\big).$$

An exact statement of the Cauchy problem will be formulated in next section.

We will establish the scattering asymptotics

$$Y(t) \sim S_\pm + W(t)\Psi_\pm, \quad t \to \pm\infty, \tag{1.4}$$

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where $S_\pm = (s_\pm, 0, 0)$ are the limit stationary states with $s_\pm \in Z := \{s \in \mathbb{R}:\ F(s) = 0\}$, $W(t)$ is the dynamical group of the free wave equation, and $\Psi_\pm \in \mathcal{E}$ are the corresponding asymptotic states. The asymptotics (1.4) hold in the *global energy norm* of a Hilbert phase if the following limits exist:

$$u_0^+ := \lim_{x\to+\infty} u_0(x), \qquad u_0^- := \lim_{x\to-\infty} u_0(x),$$
$$I_0 := \int_{-\infty}^{\infty} v_0(y)\,dy. \tag{1.5}$$

Let us comment on related works. The scattering asymptotics (1.4) are inspired by the Niels Bohr postulates (1913) on transitions between quantum stationary states:

I. The first postulate states the "transitions" $S_- \mapsto S_+$ between the stationary states which suggests the attraction $Y(t) \to S_\pm$ as $t \to \pm\infty$ in the *local energy norms* that has been proved in [14–17].
II. The second postulate states that the transition is followed by "radiation" which suggests to include (in [18,22]) the dispersive wave $W(t)\Psi_\pm$ into the asymptotics (1.4) which holds now in the *global energy norm*.

The asymptotics of type (1.4) with $S_\pm = 0$ were studied in scattering theory for the linear and nonlinear wave, Schrödinger and Klein–Gordon equations by many authors (see e.g. [24] and [23, 27]).

First scattering asymptotics (1.4) with a nontrivial set of stationary states is established for the first time in present paper.

The scattering asymptotics similar to (1.4) were proved in [1–5, 20,25,26] for the nonlinear Schrödinger and Klein–Gordon equations, and in [10–12,19] for the Schrödinger, Klein–Gordon and Maxwell equations coupled to a particle. However, all the results concern the solutions with the initial states *sufficiently close to the solitary manifold*. In [7–9] similar asymptotics were proved for all finite energy solutions to 3D wave, Klein–Gordon and Maxwell equations coupled to a particle. In these papers, $S_\pm$ in the asymptotics of type (1.4) stand for the *solitons* of the coupled systems.

The paper is organized as follows. In Section 2 we introduce the phase space and formulate well posedness. In Section 3 we describe stationary states, formulate the main results and prove the existence of dynamics and basic lemma on relaxation. In Section 4 we give some examples. In Sections 5, 6 we prove the main result on the scattering asymptotics.

## 2. Phase space and dynamics

Let us introduce a phase space $\mathcal{E}$ of finite energy states for the system (1.1). Denote by $\|\cdot\|$ resp. $\|\cdot\|_R$ the norm in the Hilbert space $L^2 := L^2(\mathbb{R}, \mathbb{R}^d)$ resp. $L^2((-R, R); \mathbb{R}^d)$.

**Definition 2.1.** (i) $\mathcal{E}$ is the Hilbert space of the triples $(u(x), v(x), p) \in C(\mathbb{R}, \mathbb{R}^d) \oplus L^2 \oplus \mathbb{R}^d$ with $u'(x) \in L^2$ and the *global energy norm*

$$\|(u, v, p)\|_{\mathcal{E}} = \|u'\| + |u(0)| + \|v\| + |p|. \tag{2.1}$$

(ii) $\mathcal{E}_F$ is the space $\mathcal{E}$ endowed with the topology defined by the *local energy seminorms*

$$\|(u, v, p)\|_{\mathcal{E},R} \equiv \|u'\|_R + |u(0)| + \|v\|_R + |p|, \quad R > 0. \tag{2.2}$$

We assume that

$$F(u) \in C^1(\mathbb{R}^d, \mathbb{R}^d), \qquad F(u) = -\nabla V(u), \tag{2.3}$$
$$V(u) \to +\infty, \quad |u| \to \infty. \tag{2.4}$$

Then the system (1.1) is formally Hamiltonian with the phase space $\mathcal{E}$ and the Hamilton functional

$$\mathcal{H}(u, v, p) = \frac{1}{2}\int \left[|v(x)|^2 + |u'(x)|^2\right] dx + m\frac{|p|^2}{2} + V(u(0)) \tag{2.5}$$

for $(u, v, p) \in \mathcal{E}$. We consider solutions $u(x,t)$ such that $Y(t) = (u(\cdot,t), \dot u(\cdot,t), \dot y(t)) \in C(\mathbb{R}, \mathcal{E})$, where $y(t) \equiv u(0,t)$.

Let us discuss the definition of the Cauchy problem (1.1), (1.2) for the trajectories $Y(t) \in C(\mathbb{R}, \mathcal{E})$. At first, $u \in C(\mathbb{R}^2, \mathbb{R}^d)$ due to $Y(t) \in C(\mathbb{R}, \mathcal{E})$. Then the first equation in (1.1) is equivalent to the d'Alembert decomposition

$$u(x,t) = f_\pm(x-t) + g_\pm(x+t), \quad \pm x > 0, \tag{2.6}$$

where

$$f_\pm, g_\pm \in C(\mathbb{R}, \mathbb{R}^d). \tag{2.7}$$

Therefore,

$$\dot u(x,t) = -f'_\pm(x-t) + g'_\pm(x+t),$$
$$u'(x,t) = f'_\pm(x-t) + g'_\pm(x+t) \quad \text{for } \pm x > 0, \tag{2.8}$$

where all the derivatives are understood in the sense of distributions. The assumption $Y(t) \in C(\mathbb{R}, \mathcal{E})$ implies

$$f'_\pm, g'_\pm \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^d). \tag{2.9}$$

We now explain the second equation of (1.1).

**Definition 2.2.** In the second equation of (1.1) we set

$$u'(0\pm, t) := f'_\pm(-t) + g'_\pm(t) \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^d), \tag{2.10}$$

while the derivative $\ddot y(t)$ of $y(t) \equiv u(0,t) \in C(\mathbb{R}, \mathbb{R}^d)$ is understood in the sense of distributions.

Note that the functions $f_\pm$ and $g_\pm$ in (2.6) are unique up to an additive constant. Hence definition (2.10) is unambiguous.

**Proposition 2.3.** *(Cf. [15].) Let the conditions (2.3), (2.4) hold, $m > 0$, and $d \geqslant 1$. Then*

(i) *For every $Y_0 \in \mathcal{E}$ the Cauchy problem (1.3) admits a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$.*
(ii) *The map $U(t): Y_0 \mapsto Y(t)$ is continuous in $\mathcal{E}$.*
(iii) *The energy is conserved,*

$$\mathcal{H}(Y(t)) = const, \quad t \in \mathbb{R}. \tag{2.11}$$

(iv) *The a priori bounds hold,*

$$\sup_{t\in\mathbb{R}} \|Y(t)\|_{\mathcal{E}} < \infty.$$

## 3. Main results

The stationary states $S = (s(x), 0, 0) \in \mathcal{E}$ for (1.3) are evidently determined: the set $\mathcal{S}$ of all stationary states $S \in \mathcal{E}$ is given by

$$\mathcal{S} = \{S_z = (z, 0, 0):\ z \in Z\}, \tag{3.1}$$

where $Z = \{z \in \mathbb{R}^d:\ F(z) = 0\}$. The main result means that the set $\mathcal{S}$ is the minimal global point attractor of the system (1.1) in the space $\mathcal{E}_F$. Let us denote $\mathcal{E}_0 = \{(u, v, 0) \in \mathcal{E}\}$, and $\tilde W(t)(u, v, 0) := (W(t)(u, v), 0)$, where $W(t)$ is the dynamical group of free wave equation corresponding to $F(u) \equiv 0$.

**Theorem 3.1.** *(Cf. [15].) Let all assumptions of Proposition 2.3 hold and an initial state $Y_0 \in \mathcal{E}$. Then*

(i) *The corresponding solution* $Y(t) \in C(\mathbb{R}, \mathcal{E})$ *to the Cauchy problem* (1.3) *converges to the set* $\mathcal{S}$ *in the local energy seminorms:*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S} \quad \text{as } t \to \pm\infty.$$

(ii) *Let additionally the set* $Z$ *be a discrete subset in* $\mathbb{R}^d$. *Then there exist the limit stationary states* $S_\pm \in \mathcal{S}$ *depending on the solution* $Y(t)$*:*

$$Y(t) \xrightarrow{\mathcal{E}_F} S_\pm, \quad t \to \pm\infty. \tag{3.2}$$

**Theorem 3.2.** *Let all assumptions of* Proposition 2.3 *hold, and additionally, the finite limits* (1.5) *there exist. Then the scattering asymptotics hold*

$$Y(t) = S_\pm + \tilde{W}(t)\Psi_\pm + r_\pm(t) \tag{3.3}$$

*with* $S_\pm \in \mathcal{S}$, *and some asymptotic states* $\Psi_\pm \in \mathcal{E}_0$; *the remainder is small in the global energy norm*:

$$\|r_\pm(t)\|_{\mathcal{E}} \to 0, \quad t \to \pm\infty. \tag{3.4}$$

**Remark 3.3.**

(i) It suffices to prove both Theorems 3.1 and 3.2 for $t \to \infty$ since the Lamb system (1.1) is time reversible.
(ii) The "weak" convergence (3.2) and (2.4), (2.5) imply that

$$\mathcal{H}(S_\pm) \leqslant \mathcal{H}\big(Y(t)\big) \equiv \mathcal{H}(Y_0), \quad t \in \mathbb{R} \tag{3.5}$$

by the Fatou theorem.

Let us note that Proposition 2.3 and Theorem 3.1 are proved in [15] for one-dimensional oscillator with $d = 1$ for initial conditions $u_0'(x) = v_0(x) = 0$ for $|x| > \text{const}$. Here we consider more general initial conditions of finite energy, and arbitrary finite-dimensional oscillator with $d \geqslant 1$. The proof of Proposition 2.3 in this general case is similar to the corresponding one-dimensional Theorem 1.1 in [15]. We give a sketch of the proof since we need the constructions for the proof of Theorems 3.1 and 3.2.

### 3.1. Existence of dynamics

We recall briefly the construction [15] of the solution to the Cauchy problem (1.3) with the initial conditions $Y_0 = (u_0, v_0, p_0) \in \mathcal{E}$. We construct unique solution $u(x,t)$ such that $Y(t) = (u(\cdot,t), \dot{u}(\cdot,t)) \in C(\mathbb{R}, \mathcal{E})$. The solution admits the d'Alembert representation (2.6) where $f_\pm(z), g_\pm(z)$ for $\pm z > 0$ are defined by the d'Alembert formulas

$$f_\pm(z) := \frac{u_0(z)}{2} - \frac{1}{2}\int_0^z v_0(y)\,dy,$$

$$g_\pm(z) := \frac{u_0(z)}{2} + \frac{1}{2}\int_0^z v_0(y)\,dy, \quad \pm z > 0. \tag{3.6}$$

These formulas imply that

$$f_\pm'(z), g_\pm'(z) \in L^2\big(\mathbb{R}^\pm, \mathbb{R}^d\big) \tag{3.7}$$

since $(u_0, v_0) \in \mathcal{E}$. The "outgoing waves" $f_+(z)$ for $z < 0$ and $g_-(z)$ for $z > 0$ are given by

$$f_+(-t) := y(t) - g_+(t), \qquad g_-(t) := y(t) - f_-(-t), \quad t > 0 \tag{3.8}$$

since $y(t) := u(0,t) = f_+(-t) + g_+(t) = f_-(-t) + g_-(t)$. Hence,

$$u(x,t) = \begin{cases} y(t-x) + g_+(x+t) - g_+(t-x), \\ \quad 0 < x < t, \\ y(t+x) + f_-(x-t) - f_-(-x-t), \\ \quad -t < x < 0, \end{cases} \quad t > 0. \tag{3.9}$$

Finally, the function $y(t)$ can be determined from the Cauchy problem for the "reduced equation" (see [15])

$$m\ddot{y}(t) = F\big(y(t)\big) - 2\dot{y}(t) + 2\dot{w}_{\text{in}}(t), \quad t > 0,$$
$$y(0) = u_0(0), \qquad \dot{y}(0) = p_0, \tag{3.10}$$

where

$$w_{\text{in}}(t) = g_+(t) + f_-(-t) \tag{3.11}$$

for $t > 0$ is the "incident wave". Note that

$$\dot{w}_{\text{in}} \in L^2\big(\mathbb{R}^+, \mathbb{R}^d\big) \tag{3.12}$$

by (3.7), hence the Cauchy problem (3.10) admits a unique solution for all $t > 0$, and the a priori bound holds:

$$\sup_{t>0}|y(t)| + \sup_{t>0}\big|\dot{y}(t)\big| + \int_0^\infty \big|\dot{y}(t)\big|^2\,dt \leqslant B < \infty, \tag{3.13}$$

where $B$ is bounded for bounded $\|(u_0, v_0, p_0)\|_{\mathcal{E}}$. These arguments imply (see [15]) that the Cauchy problem (1.3) admits a unique solution $Y(t) = (u(x,t), \dot{u}(x,t), \dot{y}(t)) \in C(\mathbb{R}, \mathcal{E})$ for any $Y_0 \in \mathcal{E}$, where $u(x,t)$ is defined by (2.6), (3.6), and (3.9).

The a priori bound (3.13) implies that $y(t) \in C(\overline{\mathbb{R}^+})$. Hence $f_+(-0) = f_+(0)$ and $g_-(-0) = g_-(+0)$ since

$$f_+(-0) = y(0) - g_+(0) = \frac{u_0(0)}{2}, \qquad f_+(+0) = \frac{u_0(0)}{2} \tag{3.14}$$

and

$$g_-(-0) = \frac{u_0(0)}{2}, \qquad g_-(+0) = y(0) - f_-(-0) = \frac{u_0(0)}{2} \tag{3.15}$$

by (3.8) and (3.6).

**Corollary 3.4.** (3.13) *and* (3.8) *imply that*

$$f_+' \in L^2\big(\mathbb{R}^-, \mathbb{R}^d\big), \qquad g_-' \in L^2\big(\mathbb{R}^+, \mathbb{R}^d\big) \tag{3.16}$$

*by* (3.7). *Hence,* (3.14) *and* (3.15) *imply that*

$$f_+', g_-' \in L^2\big(\mathbb{R}, \mathbb{R}^d\big). \tag{3.17}$$

### 3.2. Relaxation for reduced equation

The following lemma on relaxation for the reduced equation plays a crucial role in the proofs of Theorem 3.1 and Theorem 3.2. The lemma extends to all $d \geqslant 1$ the [15, Lemma 2.1] corresponding to $d = 1$. Let us note that the proof of the Lemma 2.1 in [15] is essentially one-dimensional.

Let us denote $\mathcal{Z} = \{(z, 0) \in \mathbb{R}^{2d}\colon\ z \in Z\}$.

**Lemma 3.5.** *Let all assumptions of* Theorem 3.1 *hold. Then*

(i) *For every solution* $y(t)$ *to the equation* (3.10)

$$\big(y(t), \dot{y}(t)\big) \to \mathcal{Z}, \quad t \to \infty. \tag{3.18}$$

(ii) *Let, additionally,* $Z$ *be a discrete subset in* $\mathbb{R}^d$. *Then there exists a* $(z, 0) \in \mathcal{Z}$ *such that*

$$\big(y(t), \dot{y}(t)\big) \to (z, 0), \quad t \to \infty.$$

**Proof.** Obviously, (ii) follows from (i). Let us check that (i) follows from (3.13). Namely, (3.18) is equivalent to the system

$$y(t) \to Z, \quad t \to \infty, \tag{3.19}$$

$$\dot{y}(t) \to 0, \quad t \to \infty. \tag{3.20}$$

• First let us prove (3.20). Assume the contrary, that

$$|\dot y(t_k)| \geqslant \varepsilon > 0 \tag{3.21}$$

for a sequence $t_k \to \infty$. Integrating the equation (3.10), we get that

$$m\big(\dot y(t) - \dot y(s)\big) = \int_s^t F\big(y(\tau)\big)\,d\tau - 2\int_s^t \dot y(\tau)\,d\tau + 2\int_s^t \dot w_{in}(\tau)\,d\tau, \quad s,t \geqslant 0. \tag{3.22}$$

Let us estimate each of three integrals in the RHS. The first is $\mathcal{O}(|t-s|)$ since $y(\tau)$ is a bounded function by (3.13). The second and third integrals are $\mathcal{O}(|t-s|^{1/2})$ by (3.13), (3.12) and the Cauchy–Schwartz inequality. Hence, (3.22) implies that $\dot y(t)$ is a Hölder function of degree $1/2$, i.e.

$$\big|\dot y(t) - \dot y(s)\big| \leqslant C|t-s|^{1/2}, \quad s,t \geqslant 0, \quad |t-s| \leqslant 1. \tag{3.23}$$

Therefore, $\int_0^\infty \dot y^2(t)\,dt = \infty$ by (3.21) which contradicts (3.13).

• Now we can prove (3.19). Again assume the contrary. Then

$$F\big(y(t_k)\big) \to \bar F \neq 0 \tag{3.24}$$

for a sequence $t_k \to \infty$ since $y(t)$ is a bounded function. Moreover, (3.20) implies the uniform convergence

$$F\big(y(\tau)\big) \to \bar F, \quad |\tau - t_k| \leqslant T \tag{3.25}$$

for any $T > 0$. Now (3.22) and (3.20), (3.12) imply that

$$m\big(\dot y(t_k+T) - \dot y(t_k - T)\big) = 2T\bar F + o(1), \quad t_k \to \infty, \tag{3.26}$$

which contradicts (3.20) since $T\bar F \neq 0$. □

## 4. Examples

Let us illustrate Lemma 3.5 by an example. For simplicity let us assume that

$$u_0(x) = C_\pm, \qquad v_0(x) = 0, \quad \pm x > r_0 \tag{4.1}$$

with some $C_\pm \in \mathbb{R}$ and $r_0 \geqslant 0$. Then (3.11) implies that $\dot w(t) \equiv 0$ for $t > r_0$ and (3.10) is an autonomous equation for $t > r_0$. In the phase plane $(y, \dot y)$ the orbits of the reduced equation (3.10) are determined by the following system:

$$\begin{cases} \dot y(t) = v(t), \\ m\dot v(t) = F(y(t)) - 2v(t), \end{cases} \quad t > r_0. \tag{4.2}$$

Let us compare this system with a *free* oscillator which is not coupled to a string,

$$\begin{cases} \dot y = v, \\ m\dot v = F(y). \end{cases} \tag{4.3}$$

Let us establish some simple relationships between phase portraits of these two systems.

**A.** These system have the same stationary points.

**B.** The vertical component $\dot v$ of the phase velocity vector of (4.2) is less than that of (4.3) if $v > 0$, and is greater if $v < 0$. The horizontal components of these vectors are equal.

**C.** Hence the orbits of (4.2) intersect those of (4.3) from above in the halfplane $v > 0$ and from below in the halfplane $v < 0$. Let us consider, for instance, a nondegenerate potential of Ginzburg–Landau type

$$V(y) = \frac{1}{4}\big(y^2 - 1\big)^2, \quad y \in \mathbb{R}. \tag{4.4}$$

It satisfies conditions (2.3) and (2.4). Then the system (4.3) has the following orbits:

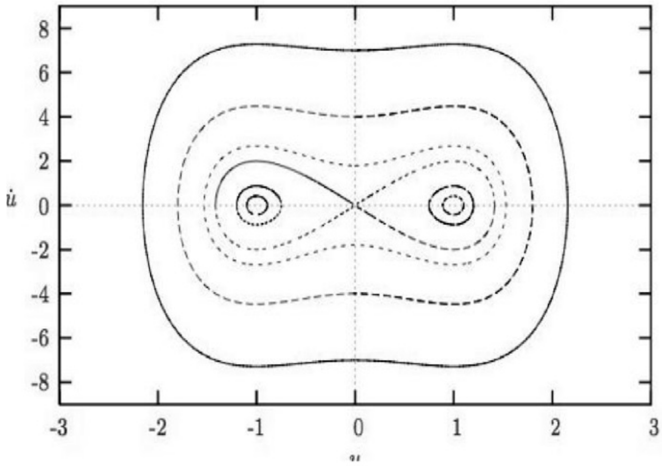


**Fig. 2.** Hamiltonian system.

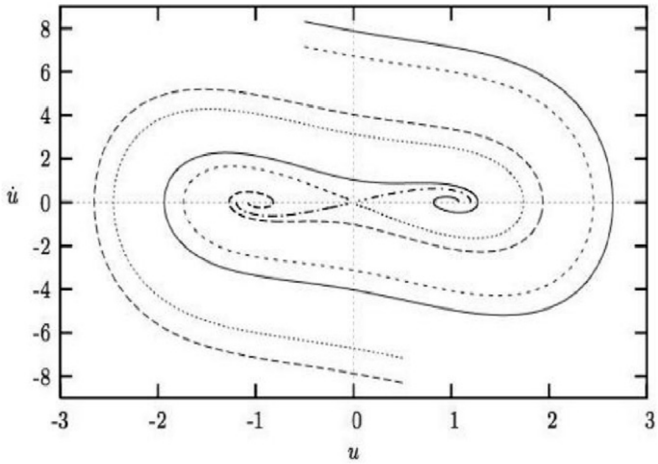


**Fig. 3.** System with a friction.

- closed curves corresponding to periodic solutions,
- two separatrices both leaving and entering the point $(0,0)$,
- three stationary points: a saddle at the point $(0,0)$ and two centers at the points $(\pm 1, 0)$, see Fig. 2. Taking into account the property **C**, we see that for the system (4.2) with potential (4.4):
  - the points $(\pm 1, 0)$ are stable foci,
  - the point $(0,0)$ is a saddle, see Fig. 3.

## 5. Convergence to global attractor

Now we can prove Theorem 3.1 for $t \to \infty$.

**Lemma 5.1.** *Let all the assumptions of Theorem 3.1 hold. Then* $Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}$ *as* $t \to \infty$.

**Proof.** It suffices to construct $z(t) \in Z$ for $t \geqslant 0$ such that

$$\big\|Y(t) - S_{z(t)}\big\|_R \to 0 \quad \text{as } t \to \infty.$$

The convergence (3.19) means that there exists a function $z(t) \in Z$, $t \geqslant 0$, such that

$$\big|y(t) - z(t)\big| \to 0, \quad t \to \infty. \tag{5.1}$$

By definitions (2.2) and (3.1),

$$\big\|Y(t) - S_{z(t)}\big\|_R = \big\|u'(\cdot,t)\big\|_R + \big|u(0,t) - z(t)\big| + \big\|\dot u(\cdot,t)\big\|_R + \big|\dot y(t)\big|.$$

Here both norms $\|\cdots\|_R \to 0$ due to (2.6), (3.7), (3.16) and (3.17). Therefore, (5.1) and (3.20) complete the proof. □

Now Theorem 3.1(i) is proved. Then Theorem 3.1(ii) follows since the set $\mathcal{S}$, isomorphic to $Z$, is discrete.

**Remark 5.2.** The bound (3.13) is provided by the friction term in the reduced equation (3.10) for the nonlinear oscillator. The friction means the energy radiation by the oscillator, and the integral in (3.13) represents the energy radiated to infinity. Thus, our proof of Theorem 3.1 relies on the energy radiation to infinity.

## 6. Divergent wave

Here we prove Theorem 3.2 for $t \to \infty$. First, let us construct the divergent wave

$$\tilde{W}(t)\Psi_+ = \big(\mathrm{w}_{\mathrm{out}}(x,t), \dot{\mathrm{w}}_{\mathrm{out}}(x,t), 0\big), \quad t \geqslant 0. \tag{6.1}$$

Here $\mathrm{w}_{\mathrm{out}}(x,t)$ is a finite energy solution to the free d'Alembert equation. Let us set

$$\mathrm{w}_{\mathrm{out}}(x,t) = C_0 + f_+(x-t) + g_-(x+t), \tag{6.2}$$

where the constant $C_0$ will be chosen below. It remains to check (3.3) and (3.4) for $t \to \infty$ that means the representation

$$\big(u(x,t), \dot{u}(x,t), \dot{y}(t)\big) = \big(s_+(x), 0, 0\big) + \big(\mathrm{w}_{\mathrm{out}}(x,t), \dot{\mathrm{w}}_{\mathrm{out}}(x,t), 0\big) + r_+(t), \quad t > 0, \tag{6.3}$$

where

$$s_+(x) \equiv z_+ := \lim_{t\to+\infty} y(t), \tag{6.4}$$

and

$$\big\|r_+(t)\big\|_{\mathcal{E}} \to 0, \quad t \to +\infty. \tag{6.5}$$

By definition of the norm (2.1), (6.5) is equivalent to

$$\big\|u'(\cdot,t) - \mathrm{w}'_{\mathrm{out}}(\cdot,t)\big\|_{L^2(\mathbb{R},\mathbb{R}^d)} + \big|u(0,t) - z_+ - \mathrm{w}_{\mathrm{out}}(0,t)\big| + \big\|\dot{u}(\cdot,t) - \dot{\mathrm{w}}_{\mathrm{out}}(\cdot,t)\big\|_{L^2(\mathbb{R},\mathbb{R}^d)} \to 0, \quad t \to \infty \tag{6.6}$$

since $\dot{y}(t) \to 0$ by (3.20).

*Step (i).* Let us start with the second term in the LHS of (6.6). Since $u(0,t) = y(t) \to z_+$, it suffices to prove that

$$\mathrm{w}_{\mathrm{out}}(0,t) = C_0 + f_+(-t) + g_-(t) \to 0, \quad t \to +\infty. \tag{6.7}$$

First, (1.5) and (3.6) imply that

$$\lim_{t\to\infty} f_-(-t) = \frac{u_0^-}{2} - \frac{1}{2}\int_0^{-\infty} v_0(y)\,dy,$$

$$\lim_{t\to+\infty} g_+(t) = \frac{u_0^+}{2} + \frac{1}{2}\int_0^{\infty} v_0(y)\,dy. \tag{6.8}$$

Second, we have by (3.8) and (6.4) that

$$\lim_{t\to\infty} f_+(-t) = z_+ - \lim_{t\to+\infty} g_+(t),$$

$$\lim_{t\to+\infty} g_-(t) = z_+ - \lim_{t\to\infty} f_-(-t). \tag{6.9}$$

Substituting (6.8), we obtain

$$\begin{cases} \lim_{t\to\infty} f_+(-t) = z_+ - \frac{u_0^+}{2} - \frac{1}{2}\int_0^\infty v_0(y)\,dy, \\ \lim_{t\to+\infty} g_-(t) = z_+ - \frac{u_0^-}{2} + \frac{1}{2}\int_0^{-\infty} v_0(y)\,dy. \end{cases} \tag{6.10}$$

Hence, (6.7) holds if we choose

$$C_0 := \frac{u_0^+}{2} + \frac{u_0^-}{2} + \frac{I_0}{2} - 2z_+, \tag{6.11}$$

where $I_0$ is defined in (1.5).

*Step (ii).* Now, let us consider the first term in the LHS of (6.6). It suffices to prove for example that

$$\big\|u'(\cdot,) - \mathrm{w}'_{\mathrm{out}}(\cdot,t)t\big\|_{L^2(\mathbb{R}^+,\mathbb{R}^d)} \to 0, \quad t \to \infty. \tag{6.12}$$

Using (6.2) and the d'Alembert representation (2.6) for $x > 0$, we get

$$u'(x,t) - \mathrm{w}'_{\mathrm{out}}(x,t) = g'_+(x+t) - g'_-(x+t), \quad x \geqslant t. \tag{6.13}$$

Finally, (3.7) and (3.16) imply that

$$\big\|g'_+(x+t) - g'_-(x+t)\big\|^2_{L^2(\mathbb{R}^+,\mathbb{R}^d)} \leqslant C\int_0^\infty \big[|g'_+(x+t)|^2 + |g'_-(x+t)|^2\big]\,dx = C\int_t^\infty \big[|g'_+(z)|^2 + |g'_-(z)|^2\big]\,dz \to 0, \quad t \to \infty. \tag{6.14}$$

*Step (iii).* The third term in the LHS of (6.6) can be handled similarly. □

**Definition 6.1.** $\mathcal{E}_\infty$ is the space of $(u,v,p) \in \mathcal{E}$ such that the limits (1.5) exist.

Let us express the asymptotic states in initial data and the function $y(t)$. The asymptotic state $\Psi_+(x) := (\Psi_0(x), \Psi_1(x), 0)$ is determined by

$$\Psi_0(x) := \mathrm{w}_{\mathrm{out}}(x,0), \qquad \Psi_1(x) := \dot{\mathrm{w}}_{\mathrm{out}}(x,0). \tag{6.15}$$

Substituting the expression (3.6), (3.8) into (6.2), we obtain

**Corollary 6.2.** *For $(u_0, v_0, p_0) \in \mathcal{E}_\infty$ the asymptotic state $\Psi_+ = (\Psi_0, \Psi_1, 0)$ is expressed by the formulas:*

$$\Psi_0(x) = C_0 + \begin{cases} y(x) + \frac{u_0(x)-u_0(-x)}{2} - \frac{1}{2}\int_{-x}^x v_0(y)\,dy, & x \geqslant 0, \\ y(-x) + \frac{u_0(x)-u_0(-x)}{2} + \frac{1}{2}\int_{-x}^x v_0(y)\,dy, & x \leqslant 0, \end{cases}$$

$$\Psi_1(x) = \begin{cases} y'(x) - \frac{u'_0(x)-u'_0(-x)}{2} + \frac{v_0(x)-v_0(-x)}{2}, & x \geqslant 0, \\ y'(-x) + \frac{u'_0(x)-u'_0(-x)}{2} + \frac{v_0(x)-v_0(-x)}{2} & x \leqslant 0, \end{cases} \tag{6.16}$$

*where $C_0$ is given by (6.11).*

**Remark 6.3.** The outgoing wave $\mathrm{w}_{\mathrm{out}}$ admits the D'Alembert representation

$$\mathrm{w}_{\mathrm{out}}(x,t) = W(t)(\Psi_0, \Psi_1) = \frac{\Psi_0(x-t) + \Psi_0(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \Psi_1(y)\,dy, \quad y, t \in \mathbb{R}, \tag{6.17}$$

because $\mathrm{w}_{\mathrm{out}}$ is a solution to the D'Alembert equation.

**Definition 6.4.** Let $Y(t) = (u(x,t), \dot{u}(x,t), \dot{y}(t)) \in C(\mathbb{R}, \mathcal{E}_\infty)$ with $Y(0) = Y_0 \in \mathcal{E}_\infty$ be such that the asymptotics (3.3) holds with $S_+ = (s_+(x), 0, 0)$, where $s_+(x) \equiv z_+ \in Z$, and $\Psi_+ \in \mathcal{E}_0$. Let us set

$$W_+ Y_0 = (\Psi_+, z_+) \in \mathcal{E}_0 \times Z. \tag{6.18}$$

The map $W_+ : \mathcal{E}_\infty \to \mathcal{E}_0 \times Z$ is called wave operator, and $(\Psi_+, z_+)$ – scattering data, corresponding to $Y_0$.

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