On completeness of scattering in nonlinear Lamb system with nonzero mass

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Abstract

We establish the asymptotic completeness in the nonlinear Lamb system with nonzero mass for hyperbolic stationary states. For the proof we construct a trajectory of a reduced equation (which is a nonlinear nonautonomous ODE) converging to a hyperbolic stationary point using the Inverse Function Theorem in a Banach space, and a priori estimates. We give a counterexample showing that the hyperbolicity condition is essential.

[1]Supported partly by Austrian Science Fund (FWF): P28152-N35, and the grant of RFBR 16-01-00100.
[2]Supported by CONACYT and CIC of UMSNH, México.

1 Introduction

In this paper we consider the asymptotic completeness of scattering in the nonlinear Lamb system for the case of nonzero oscillator mass $m > 0$. In [?]-[?] we have analyzed the case of $m = 0$.

This system describes a string coupled to a n-dimensional nonlinear oscillator with the force function $F(y)$, $y \in \mathbb{R}^n$

$$(1.1) \qquad \begin{cases} \ddot{u}(x,t) &= u''(x,t), \quad x \in \mathbb{R} \setminus \{0\}, \\ m\ddot{y} &= F(y(t)) + u'(0+,t) - u'(0-,t); \quad y(t) := u(0,t), \end{cases}$$

where $m > 0$. Here $\dot{u} := \dfrac{\partial u}{\partial t}$, $u' := \dfrac{\partial u}{\partial x}$. The solution $u(x,t)$ takes the values in $\mathbb{R}^n$ with $n \geq 1$.

The system (**??**) has been introduced originally by H. Lamb [**?**] in the linear case when $F(y) = -\omega^2 y$ and $n = 1$. The Lamb system with general nonlinear function $F(y)$ and the oscillator of mass $m \geq 0$ has been considered in [**?**] where the questions of irreversibility and nonrecurrence were discussed. The system was studied further in [**?**] where the global attraction to stationary states has been established for the first time, and in [**?**] where metastable regimes were studied for the stochastic Lamb system.

We consider the Cauchy problem for the system (**??**) with the initial conditions

$$(1.2) \qquad u|_{t=0} = u_0(x); \ \dot{u}|_{t=0} = v_0(x), \ \dot{y}|_{t=0} = p_0,$$

where $y(t) := u(0,t)$. Let us denote $Y(t) = (u(x,t), \dot{u}(x,t), \dot{y}(t))$. Then the Cauchy problem (**??**), (**??**) *formally* reads

$$(1.3) \qquad \dot{Y}(t) = \mathbf{F}(Y(t)) \ \text{ for } \ t > 0, \quad Y(0) = Y_0,$$

where $Y_0 = (u_0, v_0, p_0)$ is the initial data, and

$$\mathbf{F}(Y(t)) = ((\dot{u}(\cdot,t), u''(x,t)|_{x \neq 0}, F(y(t)) + u'(+0,t) - u'(-0,t)).$$

The following scattering asymptotics with a diverging free wave were established in [**?**] for $m = 0$ and in [**?**] for $m > 0$:

$$(1.4) \qquad Y(t) = S_+ + W(t)\Psi_+ + r_+(t), \qquad t \geq 0$$

with some limit stationary state $S_+ = (z_+, 0, 0)$. Here $W(t)$ is a dynamical group of the free wave equation, and Ψ_+ is an asymptotic state. The remainder is small in the global energy norm. The exact formulation can be found in Section 2.

In the present paper we continue the study of asymptotic completeness in nonlinear scattering for the Lamb system with a nonzero mass. The case $m = 0$ for $n = 1$ was studied in [**?**] under condition $F'(z_+) \neq 0$. The case of all $n > 1$ was completed in [**?**].

We will call (S_+, Ψ_+) the scattering data of the solution $Y(t)$. Our goal is to describe all *admissible pairs* (S_+, Ψ_+) such that there exists $Y(t)$ satisfying (**??**) for $m > 0$. For $m = 0$ we have solved this problem in Theorem 5.1 from [**?**]).

In paper [**?**] we have proved that for any $n \geq 1$ and $m = 0$ a pair (S_+, Ψ_+) is admissible if the stationary state S_+ is *hyperbolic*, i.e. $\det F'(z_+) \neq 0$, and Ψ_+ is arbitrary from an appropriate space. In this paper we prove the corresponding result for the case of $m > 0$.

The asymptotic completeness for nonlinear wave equations was considered in [**?**] for small initial states. We prove the asymptotic completeness without the smallness assumption.

The paper is organized as follows. In Section 2 we introduce basic notations, and we recall some statements and constructions from [**?**, **?**, **?**, **?**]. In Section 3 we introduce the inverse reduced ODE, and we reduce the asymptotic completeness to the existence of incoming trajectory of a reduced ODE. In Section 4 we introduce the hyperbolicity condition. In Section 5 we prove the existence of the incoming trajectory for small perturbations. First, we prove this for linear F, and then for nonlinear F using the Inverse Function Theorem. In Section 6 we extend the results of Section 5 to arbitrary perturbations without the smallness assumption. First, the solution is constructed for large t, and then it is continued back using a priori estimates. In Section 7 we give a counterexample which show that the hyperbolicity condition is essential.

2 Scattering asymptotics for the Lamb system

Denote by $\|\cdot\|_{L^2}$ the norm in the Hilbert space $L^2(\mathbb{R}_+, \mathbb{R}^n)$.

Definition 2.1. *The phase space $\mathcal{E}$ of finite energy states for the system (**??**) is the Hilbert space of the triples $(u(x), v(x), p) \in C(\mathbb{R}_+, \mathbb{R}^n) \oplus L^2(\mathbb{R}_+, \mathbb{R}^n) \oplus \mathbb{R}^d$ with $u'(x) \in L^2(\mathbb{R}_+, \mathbb{R}^n)$ and the global energy norm*

$$\|(u, v, p)\|_{\mathcal{E}} = \|u'\| + |u(0)| + \|v\| + |p|. \tag{2.1}$$

The stationary states $S(x) = (s(x), 0, 0) \in \mathcal{E}$ for (**??**) are evidently determined by

$$s(x) \equiv z \in Z := \{z \in \mathbb{R}^n : \ F(z) = 0\}. \tag{2.2}$$

We denote by $\mathcal{S}$ the set of all stationary states of system (**??**). We assume that

$$F(u) = -\nabla V(u), \quad V(u) \in C^2(\mathbb{R}^n, \mathbb{R}), \quad \text{and} \quad V(u) \to +\infty, \quad |u| \to \infty, \tag{2.3}$$

and the following limits exist:

$$u_0^+ := \lim_{x \to +\infty} u_0(x), \quad u_0^- := \lim_{x \to -\infty} u_0(x), \quad \overline{v}_0 := \int_{-\infty}^{\infty} v_0(y) dy. \tag{2.4}$$

Definition 2.2. *$\mathcal{E}_\infty$ is the space of $(u, v, p) \in \mathcal{E}$ such that the limits (**??**) exist.*

Each solution to the first equation of (**??** admits the d'Alembert representation

$$u(x, t) = f_\pm(x, t) + g_\pm(x + t), \quad \pm x > 0, \quad f_\pm, g_\pm \in C(\mathbb{R}, \mathbb{R}^n). \tag{2.5}$$

Here $f_\pm(z), g_\pm(z)$ for $\pm z > 0$ are given by the d'Alembert formulas

$$f_\pm(z) := \frac{u_0(z)}{2} - \frac{1}{2}\int_0^z v_0(y)\, dy, \quad g_\pm(z) := \frac{u_0(z)}{2} + \frac{1}{2}\int_0^z v_0(y)\, dy, \quad \pm z > 0. \tag{2.6}$$

These formulas imply that

$$f'_\pm(z), g'_\pm(z) \in L^2(\mathbb{R}^\pm, \mathbb{R}^d) \tag{2.7}$$

since $(u_0, v_0) \in \mathcal{E}$. The formulas (**??**) , (**??**) give the solution for $|x| > t > 0$. For $|x| < t$ the solution is expressed in the reflected waves $f_+(z)$ for $z < 0$ and $g_-(z)$ for $z > 0$ given by

$$f_+(-t) := y(t) - g_+(t), \qquad g_-(t) := y(t) - f_-(-t), \qquad t > 0 \tag{2.8}$$

since $y(t) := u(0,t) = f_+(-t) + g_+(t) = f_-(-t) + g_-(t)$. Hence, for $|x| < t$ the solution reads

$$u(x,t) = \left\{ \begin{array}{ll} y(t-x) + g_+(x+t) - g_+(t-x), & 0 < x < t \\ y(t+x) + f_-(x-t) - f_-(-x-t), & -t < x < 0 \end{array} \right| \qquad t > 0. \tag{2.9}$$

The function $y(t)$ can be determined from the Cauchy problem for the “reduced equation” (see [**?**])

$$m\ddot{y}(t) = F(y(t)) - 2\dot{y}(t) + 2\dot{w}_{\rm in}(t), \qquad t > 0; \;\; y(0) = u_0(0); \;\; \dot{y}(0) = p_0, \tag{2.10}$$

where

$$w_{\rm in}(t) = g_+(t) + f_-(-t), \;\; t > 0 \tag{2.11}$$

is the “incident wave”. Note that

$$\dot{w}_{\rm in} \in L^2(\mathbb{R}^+, \mathbb{R}^n) \tag{2.12}$$

by (**??**), hence the Cauchy problem (**??**) admits a unique solution for all $t > 0$, and a priori bound holds:

$$\sup_{t>0}|y(t)| + m\sup_{t>0}|\dot{y}(t)| + \int_0^\infty |\dot{y}(t)|^2 dt \le B < \infty, \tag{2.13}$$

where B is bounded for bounded $\|(u_0, v_0, p_0)\|_{\mathcal{E}}$.

These arguments imply that the Cauchy problem (**??**) admits a unique solution $Y(t) = (u(x,t), \dot{u}(x,t), \dot{y}(t)) \in C(\mathbb{R}, \mathcal{E})$ for any $Y_0 \in \mathcal{E}$, where $u(x,t)$ is defined by (**??**), (**??**), and (**??**) (see [**?**]).

Let $W(t)$ be the dynamical group of the free wave equation. Introduce the local energy norm: for $R > 0$

$$\|(u,v)\|_{\mathcal{E},R} := \|u'\|_R + |u(0)| + \|v\|_R \;\text{ for }\; (u,v) \in C(\mathbb{R}, \mathbb{R}^n) \oplus L^2(R, \mathbb{R}^n),$$

where $\|\cdot\|_R$ stands for the norm in $L^2((-R,R); \mathbb{R}^d)$.

Theorem 2.3. *Let $m \ge 0$ and the assumptions (**??**) and (**??**) hold, the set Z be a discrete subset in $\mathbb{R}^d$, and initial state $Y_0 \in \mathcal{E}_\infty$. Then the scattering asymptotics (**??**) holds, where*
i) The dispersive wave $W(t)\Psi_+$ converges to zero in local energy seminorms, i.e.

$$\|W(t)\Psi_+\|_{\mathcal{E},R} \to 0, \qquad t \to \infty, \quad \forall R > 0, \tag{2.14}$$

and $W(t)\Psi_+$ *admits the representation* $W(t)\Psi_+ = (W_{\rm out}(x,t), \dot W_{\rm out}(x,t))$ *where*

$$W_{\rm out}(x,t) = C_0 + f_+(x-t) + g_-(x+t), \qquad C_0 := \frac{u_0^+ + u_0^- + \overline{v}_0}{2} - 2z_+. \tag{2.15}$$

ii) The rermainder admits the following asymptotics

$$\|r_+(t)\|_{\mathcal{E}} \to 0, \;\; t\to\infty. \tag{2.16}$$

iii) The asymptotic state $\Psi_+ = (\Psi_0, \Psi_1) \in \mathcal{E}_\infty$*, i.e. there exist the finite limits*

$$\Psi_0^+ = \lim_{x\to+\infty} \Psi_0(x), \qquad \Psi_0^- = \lim_{x\to-\infty} \Psi_0(x), \qquad \overline{\Psi}_1 = \int_{-\infty}^{\infty} \Psi_1(y)dy \;, \tag{2.17}$$

and the following identity holds:

$$\Psi_0^+ + \Psi_0^- + \overline{\Psi}_1 = 0. \tag{2.18}$$

Proof (**??**) implies that there exists $z_+ \in Z$ such that

$$y(t) := u(0,t) \to z_+, \;\; t\to\infty. \tag{2.19}$$

The statements i)-ii) follow from (**??**) and (**??**), see [**?**, Theorem 4.5 ii) b)] for $m > 0$, (and in [**?**, Theorem 3.1.], and [**?**, Theorem 3.2], for $m = 0$.)

Let us prove iii). Substituting the expressions (**??**), (**??**) into (**??**), we obtain that the asymptotic state $\Psi_+ = (\Psi_0, \Psi_1) \in \mathcal{E}_\infty$ is expressed in the initial data $(u_0, v_0) \in \mathcal{E}_\infty$ by the formulas

$$\Psi_0(x) = C_0 + \left\{ \begin{array}{ll} y(x) + \dfrac{u_0(x) - u_0(-x)}{2} - \dfrac{1}{2}\displaystyle\int_{-x}^{x} v_0(y)dy \;, & x \ge 0 \\[2ex] y(-x) + \dfrac{u_0(x) - u_0(-x)}{2} + \dfrac{1}{2}\displaystyle\int_{-x}^{x} v_0(y)dy \;, & x \le 0 \end{array} \right. \tag{2.20}$$

$$\Psi_1(x) = \left\{ \begin{array}{l} y'(x) - \dfrac{u_0'(x) - u_0'(-x)}{2} + \dfrac{v_0(x) - v_0(-x)}{2} \;, \; x > 0 \\[2ex] y'(-x) + \dfrac{u_0'(x) - u_0'(-x)}{2} + \dfrac{v_0(x) - v_0(-x)}{2} \;, \; x < 0. \end{array} \right. \tag{2.21}$$

The existence of the limits (**??**) follows from (**??**) and (**??**) by (**??**) and (**??**). Finally, the identity (**??**) follows from the d'Alembert formula

$$W_{\rm out}(x,t) = \frac{\Psi_0(x-t) + \Psi_0(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \Psi_1(y)dy \tag{2.22}$$

since $W_{\rm out}(x,t) \to 0$ as $t\to\infty$ for all $x \in \mathbb{R}$ by (**??**). ■

3 Asymptotic completeness via the reduced equation

Let $Y(t) \in C(\mathbb{R}, \mathcal{E})$ be a solution to (**??**) with $Y_0 \in \mathcal{E}_\infty$. Let us define the wave operator

$$W_+ Y_0 := (S_+, \Psi_+) \in \mathcal{S} \times \mathcal{E}_\infty^+, \tag{3.1}$$

where Ψ_+ , S_+ are the corresponding asymptotics from (**??**). For $\Psi_+ = (\Psi_0, \Psi_1) \in \mathcal{E}_\infty^+$, let us set

$$w(t) = W_{out}(0,t) := \frac{\Psi_0(t) + \Psi_0(-t)}{2} + \frac{1}{2}\int_{-t}^{t} \Psi_1(y)dy, \quad t > 0. \tag{3.2}$$

Let us note that

$$\dot{w} \in L^2(\mathbb{R}_+, \mathbb{R}^n) \tag{3.3}$$

since $\Psi_+ \in \mathcal{E}_\infty^+$. In [**?**] we have proved the following lemma for $m = 0$. For $m > 0$ the proof is similar.

Lemma 3.1. *[[**?**], Lemma 3.1] Let $Y(t) \in C(\mathbb{R}, \mathcal{E})$ be a solution of (**??**) with $Y(0) = Y_0 \in \mathcal{E}_\infty$ and (**??**) holds with $S_+ =: (z_+, 0, 0)$. Then the function $y(t) := u(0,t)$ satisfies the conditions*

$$\frac{1}{2}m\ddot{y}(t) - \dot{y}(t) = \frac{1}{2}F(y(t)) - \dot{w}(t), \quad t > 0\ ; \ \ \dot{y} \in L^2(\mathbb{R}_+, \mathbb{R}^n)\ ; \ \ y(t) \to z_+, \ t \to +\infty. \tag{3.4}$$

We call the differential equation (**??**) the *inverse reduced equation*.

Definition 3.2. *The Lamb system (**??**) is asymptotically complete at a stationary state S_+ if for any $\Psi_+ \in \mathcal{E}_\infty^+$ there exists initial data $Y_0 \in \mathcal{E}_\infty$ such that (**??**) holds.*

Lemma 3.3. *Let $S_+ \in \mathcal{S}$, (**??**) holds and $w(t)$ is given by (**??**). Then the Lamb problem (**??**) is asymptotically complete at the stationary state S_+ iff for any $w(t)$ with $\dot{w} \in L^2(\mathbb{R}_+, \mathbb{R}^n)$ there exists an incoming trajectory $y(t) = u(0,t)$ satisfying conditions (**??**).*

Remark 3.4. *This Lemma has been proved in [**?**] for $m = 0$. For $m > 0$ the proof is similar.*

It plays a crucial role in the proof of the asymptotics completeness.

4 Hyperbolic stationary states

The equation (**??**) is equivalent to the following system:

$$\begin{cases} \dot{y} = p \\ \frac{1}{2}m\dot{p} = \frac{1}{2}F(y) + p - \dot{w}(t), \end{cases}$$

which in the vector form reads

$$\dot{Q} = \mathcal{F}(Q,t), \ \ \mathcal{F}(Q,t) = \begin{pmatrix} 0 \\ \frac{1}{2}F(y) + \frac{2}{m}p - \frac{2}{m}\dot{w}(t) \end{pmatrix}, \quad Q = \begin{pmatrix} y \\ p \end{pmatrix}. \tag{4.1}$$

Let us consider $z_+ \in Z$. Without loss of generality we will assume in the following that $z_+ = 0$. Then $F(0) = 0$ by (**??**), and

$$F(y) = F'(0)y + G(y), \ \ |G(y)| \leq Cy^2, \ \ y \to 0.$$

The linear homogeneous system corresponding to (**??**) reads:

$$\begin{cases} \dot{y} = p \\ \dot{p} = \frac{1}{m}F'(0)y + \frac{2}{m}p, \end{cases}$$

or in the vector form: $\dot{Q} = AQ$, where

$$A = \begin{pmatrix} 0 & ... & 0 & 1 & ... & 0 \\ & ... & & & ... & \\ 0 & ... & 0 & 0 & ... & 1 \\ \frac{1}{m}V_{11} & ... & \frac{1}{m}V_{1n} & \frac{2}{m} & ... & 0 \\ & ... & & & ... & \\ \frac{1}{m}V_{n1} & ... & \frac{1}{m}V_{nn} & 0 & ... & \frac{2}{m} \end{pmatrix}, \ \ V_{lk} := \partial_l \partial_k V(0), \ \ l, k \leq n.$$

Respectively, system (**??**) reads:

$$\dot{Q} = AQ + N(Q) + B(t), \tag{4.2}$$

where

$$N(Q) = \begin{pmatrix} 0 \\ ... \\ 0 \\ G_1(y) \\ ... \\ G_n(y) \end{pmatrix}, \ \ B(t) = \begin{pmatrix} 0 \\ ... \\ 0 \\ -\dot{w}_1(t) \\ ... \\ -\dot{w}_n(t) \end{pmatrix}. \tag{4.3}$$

Let us find the eigenvalues of matrix A. First, consider the case $n = 1$. Then

$$A = \begin{pmatrix} 0 & 1 \\ \frac{1}{m}F'(0) & \frac{2}{m} \end{pmatrix}.$$

We have:

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ \frac{1}{m}F'(0) & \frac{2}{m} - \lambda \\ . & \end{pmatrix}.$$

The characteristic equation reads:

$$\lambda^2 - \frac{2}{m}\lambda - \frac{1}{m}F'(0) = 0. \tag{4.4}$$

Hence,

$$\lambda_{1,2} = \frac{1}{m} \pm \sqrt{\frac{1}{m^2} + \frac{F'(0)}{m}}. \tag{4.5}$$

Obviously,

$$\text{(4.6)} \qquad \operatorname{Re}\lambda_{1,2} \neq 0, \ \text{ iff } \ \mathrm{F}'(0) = \mathrm{V}''(0) \neq 0.$$

Now consider the case $n > 1$. Then the matrix $A - \lambda I$ takes the form

$$A - \lambda I = \begin{pmatrix} -\lambda & ... & 0 & 1 & ... & 0 \\ & ... & & & ... & \\ 0 & ... & -\lambda & 0 & ... & 1 \\ \frac{1}{m}V_{11} & ... & \frac{1}{m}V_{1n} & \frac{2}{m} - \lambda & ... & 0 \\ & ... & & & ... & \\ \frac{1}{m}V_{n1} & ... & \frac{1}{m}V_{nn} & 0 & ... & \frac{2}{m} - \lambda \end{pmatrix},$$

where

$$V_{lk} = \partial_l \partial_k V(0), \ \ l,k \leq n.$$

Definition 4.1. *The stationary state* $S_+ = (0,0,0)$ *is hyperbolic if* $\operatorname{Re}\lambda_j \neq 0, \ \ \forall j = 1,...,n.$

Lemma 4.2. *The stationary state* $S_+ = (0,0,0)$ *is hyperbolic iff* $\det V''(0) \neq 0.$

Proof. After a nondegenerate change of variable $y = Ty_1$ we transform the matrix $F'(0) = V''(0)$ to a diagonal form:

$$T^{-1}V''(0)T = K = \begin{pmatrix} K_1 & ... & 0 \\ & ... & \\ 0 & ... & K_n \end{pmatrix}.$$

Obviously,

$$\text{(4.7)} \qquad \det K = \det F'(0),$$

In this case the matrix $A - \lambda I$ is transformed to the matrix

$$\begin{pmatrix} -\lambda & ... & 0 & 1 & ... & 0 \\ & ... & & & ... & \\ 0 & ... & -\lambda & 0 & ... & 1 \\ \frac{1}{m}K_1 & ... & 0 & \frac{2}{m} & ... & 0 \\ & ... & & & ... & \\ 0 & ... & \frac{1}{m}K_n & 0 & ... & \frac{2}{m} \end{pmatrix}.$$

Hence,

$$\det(A - \lambda I) = \prod_1^n (\lambda^2 - \frac{2}{m} - \frac{1}{m}K_i).$$

Then $\operatorname{Re}\lambda_j \neq 0$ for all $l = 1,...2n$ iff $K_i \neq 0$, $i = 1,2...n$, by (**??**). This condition is equivalent to condition the $\det K \neq 0$. This implies Lemma **??** by (**??**). ■

5 Incoming trajectories

In this section we prove the existence of a solution to (**??**) for small $\|B\|_{L^2}$ in the case of hyperbolic stationary state $S_+ = (z_+, 0, 0)$. We adapt to our case the methods of [**?**] and [**?**] for the construction of stable and unstable invariant manifolds in the hyperbolic case. First we prove the existence for a linear $F(y)$ and then for a nonlinear $F(y)$ with small $\|B\|_{L^2}$. Further, we will extend these results to arbitrary perturbations $B \in L^2(\mathbb{R}, \mathbb{R}^n)$ in the next section.

5.1 Linear equation

Let A be a linear operator $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and

$$\operatorname{Re}\lambda \neq 0, \quad \lambda \in \operatorname{Spec}(A).$$

Then

$$\operatorname{Spec} A = \sigma_- \cup \sigma_+,$$

where $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma_-$ and $\operatorname{Re}\lambda > 0$ for all $\lambda \in \sigma_+$. Let $\varepsilon > 0$ be such that

$$|\operatorname{Re}\lambda| > \varepsilon, \quad \lambda \in \operatorname{Spec}(A). \tag{5.1}$$

Denote by $P_\pm$ the projectors of $\mathbb{R}^{2n}$ to the subspaces generated by the eigenvectors corresponding to $\sigma_\pm$ respectively. Then the operator A is decomposed as

$$A = A_+ + A_-, \quad A_\pm = P_\pm A.$$

Let us denote

$$L^2 := L^2(\mathbb{R}_+, \mathbb{R}^{2n}), C_b^0 := \{y \in C_b(\mathbb{R}_+, \mathbb{R}^{2n}) : \ Q(t) \to 0, \quad t \to +\infty\}.$$

Definition 5.1. *Define the Banach space $\mathcal{Q} := L^2 \cap C_b^0$, with the norm*

$$\|Q\|_{\mathcal{Q}} := \|Q\|_{L^2} + \|Q\|_{C_b}, \quad Q \in \mathcal{Q}. \tag{5.2}$$

Consider the equation

$$\dot{Q}(t) = AQ(t) + B(t), \quad t \geq 0, \tag{5.3}$$

where $B(t)$ is defined in (**??**).

Lemma 5.2. *For any $B \in L^2$ the solution $Q \in \mathcal{Q}$ to equation (**??**) exists and $Q = \mathcal{R}B$, where $\mathcal{R} : L^2 \to \mathcal{Q}$ is a continuous linear operator.*

Proof. Let us introduce a fundamental solution of system (**??**)

$$E(t) := \theta(-t)e^{A_+ t}P_+ + \theta(t)e^{A_- t}P_-.$$

By (**??**) we have

$$|E(t)| \leq Ce^{-\varepsilon|t|}, \quad t \in \mathbb{R}. \tag{5.4}$$

Obviously,

$$Q = \mathcal{R}B := E * B \tag{5.5}$$

is a solution to (**??**). It remains to check that $Q \in \mathcal{Q}$.

i) First, let us prove that $Q \in C_b^0$. By (**??**) we have:

$$|Q(t)| = |(E * B)(t)| \leq C \int_{-\infty}^{\infty} e^{-\varepsilon|t-s|}|B(s)|ds \leq C\|B\|_{L^2} \tag{5.6}$$

by the Cauchy-Schwartz inequality. Let us prove that $Q(t) \to 0$, as $t \to \infty$. By (**??**) it suffices to check that

$$\int_{-\infty}^{t/2} e^{-\varepsilon|t-s|}|B(s)|ds \to 0, \quad \int_{t/2}^{\infty} e^{-\varepsilon|t-s|}|F(s)|ds \to 0, \quad t \to \infty.$$

The second limit follows from the Cauchy-Schwartz inequality since

$$\|B\|_{L^2(t/2,\infty)} \to 0, \quad t \to \infty.$$

It remains to prove the first limit. The limit holds since

$$\int_{-\infty}^{t/2} e^{-\varepsilon|t-s|}|B(s)|ds \le Ce^{-\varepsilon t/2}\|B\|_{L^2} \to 0, \quad t \to \infty.$$

ii) It remains to check that $Q \in L^2$. Denote $M(t) := |B(t)|$. Using (**??**) and (**??**) we obtain

$$\|Q\|_{L^2} \le \|e^{-\varepsilon|t|} * M(t)\|_{L^2} \le \int e^{-\varepsilon|t|}dt \cdot \|M\|_{L^2}.$$

■

5.2 Nonlinear term

Now, let us consider the nonlinear term $N(Q)$ in equation (**??**). Conditions (**??**) imply that

$$N \in C^2(\mathbb{R}^{2n}, \mathbb{R}^{2n}), \quad N(0) = 0 \text{ and } N'(0) = 0. \tag{5.7}$$

Define the functional map $\mathcal{N}$: $\mathcal{Q} \to \mathcal{Q}$ for $Q \in \mathcal{Q}$, by

$$\mathcal{N}(Q)(t) := N(Q(t)), \quad t \in \mathbb{R}.$$

Lemma 5.3. *i) The map $\mathcal{N} : \mathcal{Q} \to \mathcal{Q}$ is continuous.*

ii) There exists the Frechet derivative $\mathcal{N}'(Q_0) \in \mathcal{L}(\mathcal{Q}, \mathcal{Q})$ for $Q_0 \in \mathcal{Q}$.

$$\mathcal{N}'(Q_0)Q(t) = \nabla N(Q_0)Q(t) \tag{5.8}$$

iii) Moreover, $\mathcal{N}'(0) = 0$.

iv) $\mathcal{N}' \in C(\mathcal{Q}, \mathscr{L}(\mathcal{Q}, \mathcal{Q}))$.

Proof. i) Conditions (**??**) imply that for any $\delta > 0$

$$|N(Q)| \le C_\delta |Q|^2, \quad |Q| < \delta. \tag{5.9}$$

Hence $\mathcal{N}(Q) \in \mathcal{Q}$ for $Q \in \mathcal{Q}$. The Lagrange formula implies that the map $\mathcal{N}$ is continuous from $\mathcal{Q}$ to $\mathcal{Q}$ by (**??**).

ii) By (**??**) we have

$$N(Q) - N(Q_0) = N'(Q_0)(Q - Q_0) + r(Q, Q_0), \quad |r(Q, Q_0)| \le C|Q - Q_0|^2, \quad |Q| + |Q_0| \le \delta.$$

for any $\delta > 0$. Hence,

$$|r(t)| := |r(Q(t), Q_0(t))| \le C|Q(t) - Q_0(t)|^2.$$

Therefore,

$$\|r\|_{C_b} \le C\|Q - Q_0\|^2_{C_b}, \quad \|r\|_{L^2} \le C\|Q - Q_0\|_{C_b}\|Q - Q_0\|_{L^2}.$$

Hence,

$$\mathcal{N}(Q) - \mathcal{N}(Q_0) = \mathcal{N}'(Q_0)(Q - Q_0)) + r(Q, Q_0), \ \|r\|_{\mathcal{Q}} \le C\|Q - Q_0\|^2_{\mathcal{Q}}, \ \|Q\|_{\mathcal{Q}} + \|Q_0\|_{\mathcal{Q}} \le \delta, \tag{5.10}$$

which implies (**??**).

iii) Let us check that $\mathcal{N}'(0) = 0$. By (**??**) it suffices to prove that

$$\frac{\|\mathcal{N}(Q)\|_{\mathcal{Q}}}{\|Q\|_{\mathcal{Q}}} \to 0, \quad \|Q\|_{\mathcal{Q}} \to 0.$$

This follows from (**??**) with $Q_0 = 0$.

iv) We should prove that the map $Q \to \mathcal{N}'(Q)$ is continuous: $\mathcal{Q} \to \mathcal{L}(\mathcal{Q}, \mathcal{Q})$, i.e.

$$\|\mathcal{N}'(Q_1) - \mathcal{N}'(Q_2)\|_{\mathcal{L}(\mathcal{Q},\mathcal{Q})} \to 0, \quad \|Q_1 - Q_2\|_{\mathcal{Q}} \to 0. \tag{5.11}$$

Indeed, (**??**) means that

$$\sup_{\|Q\|_{\mathcal{Q}} \le 1} \|[\mathcal{N}'(Q_1) - \mathcal{N}'(Q_2)]Q\|_{\mathcal{Q}} \to 0, \quad \|Q_1 - Q_2\|_{\mathcal{Q}} \to 0. \tag{5.12}$$

Now (**??**) implies that (**??**) is equivalent to

$$\sup_{t \in \mathbb{R}} |\nabla N(Q_1(t)) - \nabla N(Q_2(t))| \to 0, \quad \|Q_1 - Q_2\|_{\mathcal{Q}} \to 0.$$

Finally, this follows from (**??**). ■.

5.3 Nonlinear equation and Inverse function theorem

By Lemma **??**, equation (**??**) with $Q \in \mathcal{Q}$ is equivalent to $Q = \mathcal{R}(\mathcal{N}Q + BQ)$, or to

$$\Phi(Q) = \mathcal{R}B \in \mathcal{Q}, \tag{5.13}$$

where

$$\Phi(Q) := Q - (R\mathcal{N})(Q).$$

The map $\mathcal{R}\mathcal{N} : \mathcal{Q} \to \mathcal{Q}$ is continuous and admits the Fréchet differential $(\mathcal{R}\mathcal{N})' \in C(\mathcal{Q}, \mathscr{L}(\mathcal{Q}, \mathcal{Q}))$ by Lemma **??**, and

$$(\mathcal{R}\mathcal{N})' = \mathcal{R}\mathcal{N}', \qquad (\mathcal{R}\mathcal{N})'(0) = 0.$$

Therefore, the map Φ is continuous $\mathcal{Q} \to \mathcal{Q}$, $\Phi' \in C(\mathcal{Q}, \mathcal{L}(\mathcal{Q}, \mathcal{Q}))$, and $\Phi'(0) = I$ by Lemma **??** iii), where I is the identity operator.

Theorem 5.4. *Let $B \in L^2$. There exist $\varepsilon > 0$, $C > 0$ such that equation (**??**) admits a unique solution $Q \in \mathcal{Q}$ with $\|Q\|_{\mathcal{Q}} < C$ for $\|B\|_{L^2} < \varepsilon$. This solution depends continuously on $B \in L^2$.*

Proof. The map $\Phi : \mathcal{Q} \to \mathcal{Q}$ is continuously differentiable, $\Phi(0) = 0$ and $\Phi'(0) = I$. Hence, by the Inverse Function Theorem (Theorem 10.4, [**?**]) there exist $\varepsilon,\ C > 0$ such that for $\|\mathcal{R}B\| < \varepsilon$ there exists a unique $Q \in \mathcal{Q}$ with $\|Q\|_{\mathcal{Q}} < C$ satisfying (**??**) and depending continuously on $\mathcal{R}B \in \mathcal{Q}$. It remains to note that $\mathcal{R}$ is a continuous operator $L^2 \to \mathcal{Q}$ by Lemma **??**. ■

6 Asymptotic completeness

In this section we prove asymptotic completeness for any hyperbolic stationary state. First, we construct the incoming trajectory for large t using Theorem **??**, and next we continue the trajectory backwards using a priori estimate.

Theorem 6.1. *Let conditions (**??**) hold, and a stationary state $S_+ = (z_+, 0, 0) \in \mathcal{S}$ be hyperbolic. Then system (**??**) is asymptotically complete at S_+.*

Proof. We assume that $z_+ = 0$ as above. According to Lemma **??**, it suffices to prove that there exists $y(t)$, $t > 0$ satisfying (**??**). Let us recall that the equation (**??**) is equivalent to equation (**??**).

i) First, we construct a solution $Q \in \mathcal{Q}$ to (**??**) for large $t > 0$. Let $T > 0$ be such that $\|B\|_{L^2(T,\infty)} < \varepsilon$, for $t \geq T$, where ε is chosen as in Theorem **??** and B is given by (**??**). Let us define

$$B_1(t) = B(T + t), \ \ t \geq 0.$$

Then by Theorem **??** there exists $Q_1(t) = \begin{pmatrix} y_1(t) \\ p_1(t) \end{pmatrix} \in \mathcal{Q}$ satisfying the inverse reduced equation (**??**) for $t \geq 0$ with B_1 instead of B. In particular, y_1 satisfies (**??**).

ii) It remains to construct a solution $y_2(t)$ to equation (**??**) for $t \in [0, T]$ with $w(T + t)$ instead of $w(t)$ such that

$$\dot{y}_2 \in L^2(0, T), \ \ y_2(T) = y_1(0), \ \ \dot{y}_2(T) = \dot{y}_1(0). \tag{6.1}$$

It suffices to prove a priori estimate for y_2 and $\dot{y}_2$ for $t \in [0, T]$. First, we prove the a priori estimate for y_2. Multiplying equation (**??**) for y_2 by $2\dot{y}_2(t)$ and using (**??**), we obtain

$$\frac{1}{2} m \frac{d}{dt} |\dot{y}_2(t)|^2 - |\dot{y}_2(t)|^2 = -\nabla V(y_2(t)) - \dot{w}(t)\dot{y}_2(t), \ \ 0 \leq t \leq T.$$

Integrating and using the initial condition, we obtain

$$\frac{1}{2} m |\dot{y}_2(T)|^2 - \frac{1}{2} m |\dot{y}_2(t)|^2 - \int_t^T |\dot{y}_2(t)|^2 = V(y_2(t)) - V(y_2(T)) - 2 \int_t^T \dot{w}(t)\dot{y}_2(t) dt.$$

Hence,

$$V(y_2(t) = \frac{1}{2} m |\dot{y}_2(T)|^2 - \frac{1}{2} m |\dot{y}_2(t)|^2 - \int_t^T \dot{y}_2(t)^2 dt + V(y_2(T)) + 2 \int_t^T \dot{w}(t)\dot{y}_2(t) dt$$

Using the Young inequality, we estimate the last term on the right hand side as

$$2 \left| \int_t^T \dot{w}(\tau)\dot{y}_2(\tau) d\tau \right| \leq \int_t^T |\dot{w}(\tau)|^2 d\tau + \int_t^T |\dot{y}_2(\tau)|^2 d\tau.$$

Hence,

$$(6.2)\quad V(y_2(t) \le \frac{1}{2}m|\dot y_2(T)|^2 - \frac{1}{2}m|\dot y_2(t)|^2 - \int_t^T |y_2(t)|^2 dt + V(y_2(T)) + \int_t^T |\dot w(\tau)|^2 d\tau + \int_t^T |\dot y_2(\tau)|^2 d\tau,$$

and so

$$(6.3)\quad V(y_2(t) \le \frac{1}{2}m|\dot y_2(T)|^2 - \frac{1}{2}m|\dot y_2(t)|^2 + V(y_2(T)) + \int_t^T |\dot w(\tau)|^2 d\tau \le B < \infty, \quad t \in [0,T],$$

since $\dot w \in L^2(\mathbb{R}_+)$ by (**??**). Therefore, $y_2(t)$ is bounded for $t \in [0,T]$ by (**??**).

It remains to prove a priori estimate for $\dot y_2$. From (**??**) we obtain

$$\frac{1}{2}m|\dot y_2(t)|^2 \le -V(y_2(t) + \frac{1}{2}m|\dot y_2(T)|^2 + V(y_2(T)) + \int_t^T |\dot w(\tau)|^2 d\tau, \quad t \in [0,T].$$

Hence

$$(6.4)\qquad \dot y_2(t) \text{ is bounded in } [0,T],$$

since $y_2(t)$ is bounded there, V is continuous by (**??**), and $\dot w \in L^2(\mathbb{R}_+, \mathbb{R}^n)$. Now (**??**) follows.

iii) Finally, defining

$$(6.5)\qquad y(t) := \begin{cases} y_2(t), & t \in [0,T] \\ y_1(t-T), & t > T \end{cases}$$

we obtain that $y(t)$ satisfies (**??**) by (**??**), since $y_1(t)$ satisfies (**??**) by (**??**) and (**??**). ■

7 Counterexample

In this section we give an example which shows that the incoming solution may not exist for nonhyperbolic stationary state. This means that the system is not asymptotically complete in this state.

Example 7.1. *Let us consider equation (**??**) with F satisfying (**??**), and such that*

$$(7.1)\qquad F(0) = 0, \quad F(y) \ge 0 \text{ for } |y| < 1.$$

Then $F'(0) = 0$, so $z_+ = 0$ is the nonhyperbolic stationary point. Let us choose

$$(7.2)\qquad \dot w(t) = \frac{1}{1+t} \in L^2(\mathbb{R}_+).$$

In this case a trajectory satisfying condition

$$(7.3)\qquad y(t) \to 0, \quad t \to \infty, \quad \dot y \in L^2(\mathbb{R}^+)$$

does not exist.

Proof Let y satisfy (**??**) with $z_+ = 0$

$$\frac{m}{2}\ddot{y} - \dot{y} = F(y) + \frac{1}{1+t}, \ \ t > 0.$$

Then by (**??**) and (**??**) there exists $T > 0$ such that $|y(t)| < 1$ for $t > T$, hence

$$\frac{m}{2}\ddot{y} \geq \dot{y} + \frac{1}{1+t}, \ \ t > T.$$

Integrating, we obtain

$$\frac{m}{2}(\dot{y}(t) - \dot{y}(0)) \geq y(t) - y(0) + \log(1+t), \ \ t \geq 0.$$

Then (**??**) implies that

$$\frac{m}{2}\dot{y}(t) \geq \log(1+t) + C,$$

hence

$$\dot{y} \notin L^2(\mathbb{R}_+),$$

which contradicts (**??**). ■

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