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On uniqueness and stability of Sobolev's solution in scattering by wedges

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Abstract. We prove that the solution of the scattering problem of pulses, constructed by Sobolev (Proc Seismol Inst Acad Sci URSS 41:1–15, 1934), coincides with the solution obtained by our method of complex characteristics. The coincidence holds for the DD- and NN-boundary conditions. The method of complex characteristics has been developed by Komech and Merzon in 2002–2007. Its main advantage is that it provides (a) the existence and uniqueness of the solutions in suitable functional classes and (b) the limiting amplitude principle. The uniqueness in the Sobolev approach was not considered. Our result means that Sobolev's solution belongs to our functional classes and agrees with the limiting amplitude principle.

Mathematics Subject Classification. 35Q60 · 78A45.

Keywords. Wave equation scattering · Wedge · Plane wave · Sobolev · Exact solution · Limiting amplitude · Stability.

1. Introduction

In 1934, Sobolev published his seminal paper [7] developing a scattering theory of plane waves by wedges. In particular, an exact solution is constructed for an incident wave with the pulse profile. The scattering is described by a mixed problem for the wave equation in the complement Q of a convex angle W with suitable boundary conditions and the incident plane wave

$$u_{\text{in}}(y,t) := F(t - n_0 \cdot y), \quad y \in Q,\ t < 0. \tag{1.1}$$

Here F is a given profile function, $F(s) = 0$ for $s < 0$ and n_0 is the unit vector in the direction of the incident wave. Sobolev considered the Heaviside profile function $F(s) = h(s)$ and obtained a particular solution using the Smirnov–Sobolev representation [15] for general solutions of the wave equation [16, formulas (15)–(17)]. In 1951, Keller and Blank [5] independently considered the diffraction of Heaviside incident wave by a wedge developing Buseman's method [1] which is similar to Sobolev's approach. However, the functional spaces of the solutions are not specified, and uniqueness of solutions was not analyzed. The uniqueness fails if we do not specify singularity of the solutions at the vertex. On the other hand, the existence breaks down if we require an excessive regularity of the solution.

This is why we have developed a rigorous theory [3,4,6,7,12,13] providing the uniqueness and existence of a solution in suitable functional spaces for the smooth Heaviside-type function profile $F \in C^\infty$, $\text{supp}F \subset [0,\infty)$, $F(s) = 1$ for $s > s_0 > 0$ of the incident wave.

The theory relies on reduction in the time-dependent problem to a stationary one by means of the Fourier–Laplace transform in time. The most general results on solution to such problems in Sobolev spaces are obtained in [2]. Our approach relies on the method of complex characteristics, which gives more explicit representation of solutions, and allows to investigate their properties [8,11].

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In [9], we have extended the results to the case of general tempered distribution F. The corresponding general formula is obtained in [9], see (2.7)–(2.5) below. This solution belongs to a space of distributions $\mathcal{M}_\varepsilon$ with $\varepsilon = 1-q$ for the $DD-$ and NN-problems and $\varepsilon = 1-2q$ for DN-problem, where $q = \pi/(2\Phi)$, $\Phi = 2\pi - \phi$ and $\phi \in (0,\pi)$ is the magnitude of the wedge. Roughly speaking, $\mathcal{M}_\varepsilon$ is a space of functions with the asymptotics $|\nabla u(y,t)| = \mathcal{O}(|y|^{-\varepsilon})$ at the vertex, i.e., as $|y| \to 0$. Moreover, in [9], we have established the stability of the diffracted wave with respect to local perturbations of the profile function F.

Our main results in the present paper are the following.

I. In Theorem 6.1, we establish coincidence of our solution constructed in [9], with the Sobolev solution [16] in the case of $F(s) \equiv h(s)$.

II. We apply our results [9] to prove the uniqueness of the Sobolev solution in an appropriate functional class (Corollary 7.2).

III. We also establish the stability of the solution with respect to local perturbations of the profile function (Corollary 7.4).

Finally, we note that Sobolev's motivation for the diffraction of discontinuous incident waves stemmed from a concrete applied problem. In the next paper [17], Sobolev introduced his famous theory of the "weak derivatives" of discontinuous functions. Next development resulted in the Schwartz theory of distributions [14].

2. Scattering problem

We consider the scattering of plane waves (1.1) by two-dimensional wedge

$$W := \{y = (y_1, y_2) \in \mathbb{R}^2 : y_1 = \rho\cos\theta,\ y_2 = \sin\theta,\ \rho \geq 0,\ 0 \leq \theta \leq \phi\}, \tag{2.1}$$

where $\phi < \pi/2$ is the wedge angle. Then $Q := \mathbb{R}^2 \setminus W$ is an open angle of magnitude $\Phi = 2\pi - \phi$. Throughout the paper, we will consider the case of the DD-boundary conditions. The case of the NN-problem can be considered similarly by methods [4] (the DNA-problem is not considered in [16]).

We will restrict ourselves to the case when the direction vector $n_0 = (\cos\alpha, \sin\alpha)$ of the incident wave (1.1) satisfies the condition considered by Sobolev [16, Fig. 2] which in our notations is

$$\phi < \alpha < \pi/2. \tag{2.2}$$

The scattering is described by the mixed problem

$$\begin{cases} \Box u(y,t) = 0, & y \in Q; u(y,t)\Big|_{\partial Q} = 0,\ t \in \mathbb{R} \\ u(y,t) = u_{\rm in}(y,t), & y \in Q,\ t < 0, \end{cases} \tag{2.3}$$

where $\Box = \partial_t^2 - \triangle$. For $t > 0$, we define the incident wave by (see Fig. 1)

$$u_{\rm in}(y,t) = \begin{cases} F(t - \rho\cos(\theta - \alpha)), & \theta \notin (\phi, \theta_1) \\ 0, & \theta \in (\phi, \theta_1) \end{cases} \Bigg| (\rho,\theta) \in Q,\ t > 0, \tag{2.4}$$

where $\theta_1 := \alpha$.

The reflected wave u_r for $t > 0$ is defined by

$$u_r(\rho,\theta,t) := \begin{cases} 0, & \theta_1 < \theta < \theta_2 \\ -F(t - \rho\cos(\theta - \theta_2)), & \phi < \theta < \theta_2 \end{cases} \Bigg| \rho, t > 0, \tag{2.5}$$

where $\theta_2 = 2\pi - \alpha$.

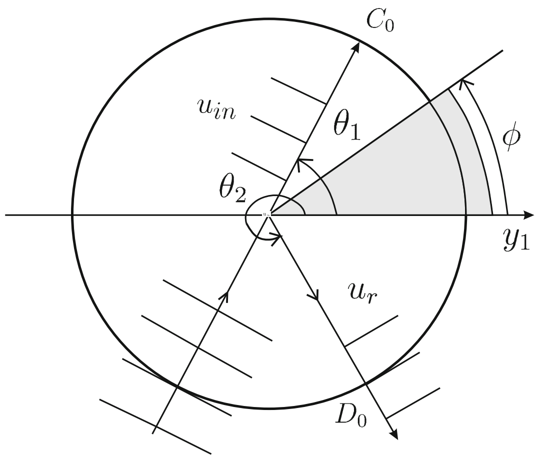

FIG. 1. The incident and reflected waves

In [9, Section 3], we have established the existence and uniqueness of a solution u to the problem (2.3) in a suitable space of distributions $\mathcal{M}_\varepsilon$ with $\varepsilon = 1 - q$.

Let us introduce the space of solutions to problem (2.3). Let us take an arbitrary $\varepsilon > 0$.

Definition 2.1. (see Def. 2.4 of [9]) *(i) E_ε is the Banach space of functions $u(y) \in C(\overline{Q}) \cap C^1(\dot{\overline{Q}})$ with finite norm*

$$\|u\|_\varepsilon = \sup_{y\in\overline{Q}} |u(y)| + \sup_{y\in\dot{\overline{Q}}} \{y\}^\varepsilon |\nabla u(y)| < \infty, \tag{2.6}$$

where $\{y\} := \frac{|y|}{1+|y|}$ and $\dot{\overline{Q}} := \overline{Q} \setminus 0$.
(ii) $\mathcal{M}_\varepsilon$ is the space of tempered distributions $u(y,t) \in S'(\overline{Q} \times \overline{\mathbb{R}^+})$, such that its Fourier–Laplace transform $\hat{u}(y,\omega)$ is a holomorphic function of $\omega \in \mathbb{C}^+$ with values in E_ε.

The solution of (2.3) admits the splitting

$$u(y,t) = u_{\text{in}}(y,t) + u_r(y,t) + u_d(y,t) \tag{2.7}$$

which is a definition of the diffracted wave u_d.

The diffracted wave is given by formula (1.10) from [9]:

$$u_d(\rho,\theta,t) = \int_{-l(t/\rho)}^{l(t/\rho)} Z(s+i\theta) F(t - \rho\cosh\beta) \mathrm{d}s, \quad \theta \in \Theta = [\phi, 2\pi] \setminus \{\theta_1, \theta_2\}, \tag{2.8}$$

$$Z(s) = -H(-i\pi/2 + s) - H(-5i\pi/2 + s), \quad s \in \mathbb{C} \tag{2.9}$$

$$H(s) = \coth(q(s + i\pi/2 - i\alpha)) - \coth(q(s - 3i\pi/2 + i\alpha)) \tag{2.10}$$

and

$$q = \frac{\pi}{2\Phi}, \tag{2.11}$$

$$l(\lambda) = \begin{cases} \log(\lambda + \sqrt{\lambda^2 - 1}), & \lambda > 1 \\ 0, & \lambda \in (0,1). \end{cases} \tag{2.12}$$

Let us denote by S_t the "diffraction sector" of the plane:

$$S_t = \{(\rho,\theta) : 0 < \rho < t, \theta \in (\phi, 2\pi)\}. \tag{2.13}$$

Note that formulas (2.8)–(2.10) imply that $u_d(\rho,\theta,t)$ for each $t>0$ is a continuous function in every sector Θ_k, where

$$\begin{array}{l}\Theta_1 := \{(\rho,\theta) : \rho>0,\ \theta\in(\phi,\theta_1)\},\\ \Theta_2 := \{(\rho,\theta) : \rho>0,\ \theta\in(\theta_1,\theta_2)\},\\ \Theta_3 := \{(\rho,\theta) : \rho>0,\ \theta\in(\theta_2,2\pi)\}.\end{array} \tag{2.14}$$

Moreover,

$$u_d(\rho,\theta,t)=0,\ \theta\in\Theta,\ \rho\ge t, \tag{2.15}$$

since $l(t/\rho)=0$ in this case by our definition (2.12).

We consider the case $F\equiv h$. Then the integral (2.8) can be easily calculated (see [10]):

$$u_d(\rho,\theta,t)=\frac{i}{2\pi}\Big[-\log U_0-\log U_1+\log U_2+\log U_3\Big],\quad 0<\rho<t,\ \theta\in\Theta, \tag{2.16}$$

where $\mathrm{Im}\log(\cdot)\in[-\pi,\pi)$, and

$$U_k=\frac{r^{-q}e^{ic_k(\theta)}-r^{q}e^{-ic_k(\theta)}}{-\big(r^{-q}e^{-ic_k(\theta)}-r^{q}e^{ic_k(\theta)}\big)},\quad k=\overline{0,3};\quad r=\frac{t}{\rho}-\sqrt{\left(\frac{t}{\rho}\right)^2-1},\quad 0<\rho<t; \tag{2.17}$$

$$c_0(\theta)=q(\theta-\theta_1),\quad c_1(\theta)=q(\theta-2\phi+\alpha),\quad c_2(\theta)=q(\theta-\theta_2),\quad c_3(\theta)=q(\theta-2\pi-\alpha). \tag{2.18}$$

For the NN-boundary conditions, the formulas similar to (2.8)–(2.12) and (2.15)–(2.17) are also obtained in [9].

Let us note that the diffracted wave (2.8) is discontinuous on the critical rays $\theta=\theta_k$, $k=1,2$. Nevertheless, the sum (2.7), which is the total solution, is continuous in $\overline{S_t}$ with the exception of two points on its boundary.

In what follows, we consider the incident wave (1.1) with the profile function $F=h$ as in Sobolev's paper [16].

Lemma 2.2. *For $F=h$, the total solution $u(\rho,\theta,t)$ given by (2.4)–(2.8) is continuous in $\overline{S}_t\backslash\{(t,\theta_1),(t,\theta_2)\}$ for any $t>0$.*

Proof. In the case $F=h$, Lemma 4.1 of [16] implies that the diffracted wave is continuous outside the rays $\theta=\theta_k$, $k=1,2$ and its jumps are given by

$$u_d(\rho,\theta_k+0,t)-u_d(\rho,\theta_k-0,t)=-1^{(k+1)},\quad k=1,2,\ \rho<t$$

in the sense of distributions of $\rho>0$. Moreover, this relation holds also pointwise as it is shown in the proof of [9, Lemma 4.1] for the Hölder functions $F(s)$ of $s\ge 0$.

Further, the function $u_{\rm in}$ is also continuous outside the ray $\theta=\theta_1$ by (2.4). Hence,

$$\begin{aligned}&u(\rho,\theta_1+0,t)-u(\rho,\theta_1-0,t)\\&\quad=(u_d(\rho,\theta_1+0,t)+u_{\rm in}(\rho,\theta_1+0,t))-(u_d(\rho,\theta_1-0,t)+u_{\rm in}(\rho,\theta_1-0,t))\\&\quad=u_d(\rho,\theta_1+0,t)+1-u_d(\rho,\theta_1-0,t)=0,\quad \rho<t,\end{aligned}$$

so u is continuous on the ray $\{\theta=\theta_1\}$. Similarly, the continuity of u on the ray $\theta=\theta_2$ can be proved by using (2.5). Thus, the total solution $u(\cdot,\cdot,t)$ is continuous in $\overline{S}_t\setminus\{(t,\theta_1),t,\theta_2\}$ for all $t>0$. □

Finally, let us note that for $F=h$, we have

$$u(\rho,\theta,t)=\left.\begin{cases}1+u_d(\rho,\theta,t), & \theta\in(\theta_1,\theta_2)\\ u_d(\rho,\theta,t), & \theta\in(\phi,\theta_1)\cap(\theta_2,2\pi),\end{cases}\right|\rho,t>0 \tag{2.19}$$

by (2.4)–(2.7), where u_d is given by (2.16)–(2.17).

Remark 2.3. Formulas (2.8)–(2.18) were obtained in [9] for $\alpha < \phi$, though in the present paper we consider $\alpha > \phi$ which corresponds to Sobolev's geometry (see Fig. 2 and 5 from [16]). Nevertheless formulas (2.8)–(2.18) remain correct in the case $\alpha > \phi$ too, and function (2.7) is a solution to problem (2.3) which belongs to $\mathcal{M}_\varepsilon$ with $\varepsilon = 1 - q$. The proofs in this case are almost identical to the ones of [9].

3. Sobolev's formula

The main goal of the present paper is identification of our solution (2.7) with Sobolev's formula [16, (34)]. Sobolev considered the scattering in variables (x_1, x_2) which differ from (y_1, y_2) by the rotation over an angle of $\pi/2 - \phi/2$. The corresponding Sobolev's polar angle is given by

$$\theta_s := \theta - \frac{3\pi}{2} - \frac{\phi}{2}. \tag{3.1}$$

Let us denote $\Theta_s := [-3\pi/2 + \phi/2, \pi/2 - \phi/2]$. Sobolev's solution for $t \in \mathbb{R}$ is defined in the region

$$S := \Big\{(\rho, \theta_s) : \ \rho > 0, \ \theta_s \in \Theta_s\Big\} \tag{3.2}$$

which complements the wedge $|\theta_s - \pi/2| \le \phi/2$.

The incident wave (1.1) with $F = h$ coincides with Sobolev's incident wave [16, (7)]

$$w_0(x_1, x_2, t) = h(t - x_1 \sin\beta + x_2 \cos\beta), \quad t < 0. \tag{3.3}$$

Here

$$\beta = \alpha + \Phi/2, \tag{3.4}$$

and Φ is given by (5.4), since $\alpha - \phi/2 = \beta - \pi$ (see Fig. 2) (we take here Sobolev's parameter $b = 1$, and h denotes the Heaviside function).

The Sobolev solution to (2.3) is given by formulas (26) and (34) of [16]:

$$w_0(\rho, \theta_s, t) = \operatorname{Re} \frac{1}{\pi i} \log \frac{(e^{i\gamma_2} - \xi)(e^{-i\gamma_2} - \xi)}{(e^{i\gamma_1} - \xi)(e^{-i\gamma_1} - \xi)}, \quad \rho < t, \ \theta_s \in \Theta_s, \tag{3.5}$$

where $\operatorname{Im} \log(\cdot) \in [-\pi, \pi)$, and γ_1, γ_2 are some parameters depending on β from (3.3). However, Sobolev's expressions (33) of [16] for γ_1 and γ_2 do not correspond to his definition of β by (3.3). Rather, [16, formulas

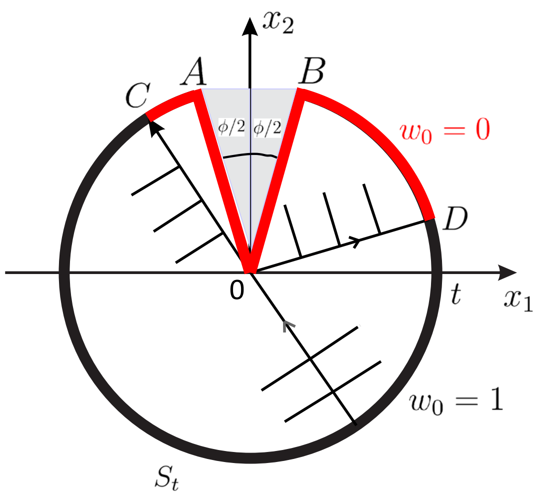

Fig. 2. Boundary values of w_0

(33)] correspond to another Sobolev's definition of β by Fig. 2 of [16]. These two values of β differ by π. The Sobolev's definition of γ_1 and γ_2 reads

$$\gamma_1 = \frac{\pi}{2} - 2q\beta + 2q\pi, \quad \gamma_2 = \frac{\pi}{2} + 2q\beta + 2q\pi. \tag{3.6}$$

We show that the definition, corresponding to (3.3), reads

$$\gamma_1 = \frac{\pi}{2} - 2q\beta + 4q\pi, \quad \gamma_2 = -\frac{\pi}{2} + 2q\beta. \tag{3.7}$$

Obviously, (3.6) follows if we replace β by $\beta + \pi$ in (3.7). These discrepancies force us to trace Sobolev's calculations to justify the expressions (3.7). This is necessary for proving the coincidence of our solution (2.6) with Sobolev's formula (34) of [16].

4. Sobolev's calculations

Here we trace briefly Sobolev's calculations [16].

(i) According to the Smirnov–Sobolev theory [15], any function

$$w_0(x_1, x_2, t) = \operatorname{Re} W_0(\zeta(x_1, x_2, t)) \tag{4.1}$$

is a solution to the wave equation $\Box w_0(x_1, x_2, t) = 0$ if $W_0(\zeta)$ is an analytic function of $|\zeta| < 1$ and $\zeta(x, y, t)$ is given by formula (20) of [16]:

$$\zeta = \zeta(\rho, \theta_s, t) = (t/\rho - \sqrt{t^2/\rho^2 - 1})e^{i\theta_s} = r(\rho, t)e^{i\theta_s}, \tag{4.2}$$

where $r(\rho, t)$ agrees with (2.17).

(ii) It remains to find an appropriate $W_0(\zeta)$ satisfying the needed boundary conditions.

Namely, formulas (1.1) and (2.5) give the values of u_{in} and u_r on the diffraction sector S_t boundary for any fixed $t > 0$ [see (2.13)].

Finally, Sobolev suggested the boundary values for the diffracted wave

$$u_d(\rho, \theta_s, t) = 0, \quad \rho \geq t$$

arguing by the Huygens principle. Thus, Sobolev takes the following values for the solution $w_0(x, y, t)$ on the diffraction sector S_t boundary as shown in Fig. 2:

$$w_0(\rho, \theta_s, t) = \begin{cases} 1, & (\rho, \theta_s) \in \overset{\frown}{\mathrm{CD}} \\ 0 & (\rho, \theta_s) \in \overset{\frown}{\mathrm{CA}} \cup OA \cup AO \cup OB \cup \overset{\frown}{\mathrm{BD}} \,. \end{cases} \tag{4.3}$$

Two "critical" points C and D lie on the rays which are the lines of discontinuity of the incident and reflected waves according to (1.1) and (2.5), see Fig. 2.

Remark 4.1. These points C and D are the points of discontinuity for the solution $w_0(\rho, \theta, t)$ according to the boundary values (4.3).

In the next step, these boundary conditions are "transmitted" to the variable ζ given by (4.2). Namely, the map (4.2) is a "ray-preserving" one-to-one correspondence of the region S_t onto the sector of the unit circle $S_1 := \Big\{|\zeta| < 1, \arg\ \zeta \in \Theta_s\Big\}$ (see Fig. 3).

The Sobolev's solution $w_0(\rho, \theta_s, t)$ in the diffraction sector S_t is given by (4.1):

$$w_0(\rho, \theta_s, t) = \operatorname{Re} W_0(\zeta), \quad \zeta \in S_1, \tag{4.4}$$

where ζ is defined by (4.2).

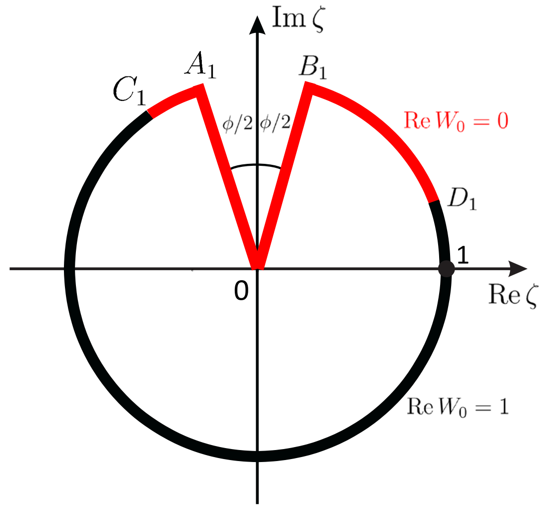

Fig. 3. Boundary values of $W_0(\zeta)$

Hence, the transmitted boundary conditions (4.3) read as

$$\operatorname{Re} W_0(\zeta) = \begin{cases} 0, \ \zeta \in \overset{\frown}{A_1 C_1} \cup OA_1 \cup OB_1 \cup \overset{\frown}{B_1 D_1} \\ 1, \ \zeta \in \overset{\frown}{C_1 E_1 D_1}, \end{cases} \tag{4.5}$$

where $A_1 = \zeta(A)$, $B_1 = \zeta(B)$, $C_1 = \zeta(C)$, $D_1 = \zeta(D)$. It is easy to see that

$$C_1 = e^{i\delta_1}, \quad D_1 = e^{i\delta_2}, \quad \delta_1 = -\frac{5\pi}{2} + \beta, \quad \delta_2 = \frac{3}{2}\pi - 2\frac{\phi}{2} - \beta. \tag{4.6}$$

Let us stress that we should choose the arguments $\delta_1, \delta_2 \in \Theta_s$.

(iii) Next, Sobolev constructs an analytic function $W_0(\zeta)$, of $\zeta \in S_1$ with boundary values (4.5).

The function $W_0(\zeta)$ is obtained by means of a conformal map of S_1 onto the lower half-disk $\{|\xi| < 1, \ \operatorname{Im} \xi < 0\}$,

$$\xi = \xi(\zeta) = e^{-i\frac{\pi}{2}}(e^{i\frac{\pi}{2}}\zeta)^{2q}, \quad |\zeta| < 1 \tag{4.7}$$

(see [16, formula (30)]). Here q is given by (2.11), and the branch of the power function $(\cdot)^{2q}$ is s.t. $1^{2q} = 1$.

Thus, the problem is reduced to construction of a function

$$W_1(\xi) = W_0(\zeta(\xi)) \tag{4.8}$$

in the lower half-disk (see Fig. 4).

with the boundary conditions corresponding to (4.5) (formulas (32) of [16]):

$$\operatorname{Re} W_1(\xi) = \begin{cases} 0, \quad \xi \in (\overset{\frown}{C_2, A_2}) \cup [A_2, B_2] \cup (\overset{\frown}{B_2, D_2}) \\ 1, \quad \xi \in (\overset{\frown}{C_2, D_2}), \end{cases}$$

where $C_2 = \xi(C_1)$, $D_2 = \xi(D_1)$. Let us calculate the arguments of C_2 and D_2.

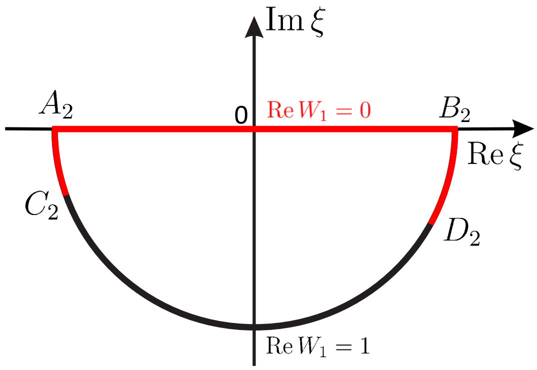

Fig. 4. Boundary values of $W_1(\xi)$

Using (4.6) and (4.7), we obtain

$$\begin{aligned} C_2 &= e^{-i\pi/2}(e^{i\pi/2}e^{i(-5\pi/2+\beta)})^{2q} = e^{-i\pi/2}(e^{-2\pi i+i\beta})^{2q} \\ &= e^{-i\pi/2}e^{-4\pi iq}e^{2iq\beta} = e^{-i(\frac{\pi}{2}-2q\beta+4q\pi)} = e^{-i\gamma_1}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} D_2 &= e^{-i\pi/2}(e^{i\pi/2}e^{3i\pi/2-i\phi-i\beta})^{2q} = e^{-i\pi/2}(e^{2i\pi-i(2\pi-\Phi)-i\beta})^{2q} = e^{-i\pi/2}e^{(i\Phi-i\beta)2q} \\ &= e^{-i\pi/2}e^{i\pi}e^{-2iq\beta} = e^{-i(-\pi/2+2q\beta)} = e^{-i\gamma_2} \end{aligned} \tag{4.10}$$

since $-\pi < \beta - 2\pi < -\pi/2$ and $0 < \Phi - \beta < \pi$. Here γ_1 and γ_2 are given by (3.7). It is important that $0 < \gamma_2 < \gamma_1 < \pi$.

Let us recall that our formulas (3.7) for γ_1 and γ_2 differ from (33) of [16] since we use the first Sobolev's definition of β corresponding to the formula (3.3) for the incident wave. It is easy to check that we get (33) if we replace β by $\beta + \pi$ in (3.7).

Finally, Sobolev obtains the following expression for W_1:

$$W_1(\xi) = \frac{1}{\pi i}\log\frac{(e^{i\gamma_2}-\xi)(e^{-i\gamma_2}-\xi)}{(e^{i\gamma_1}-\xi)(e^{-i\gamma_1}-\xi)}, \quad |\xi| < 1 \tag{4.11}$$

using the Schwarz symmetry principle. Now, (4.8) and (4.4) imply formula (3.5) with γ_1 and γ_2 given by (3.7).

5. Evaluation of Sobolev's formula

Here we evaluate formula (3.5) and transform it to our polar coordinate θ, see (3.1). We will denote this function as $w_0(\rho,\theta,t)$ though formally we should write it as $\tilde{w}_0(\rho,\theta,t) := w_0(\rho,\theta_s,t)$.

Lemma 5.1. *Sobolev's solution (3.5) in our coordinate can be written as follows:*

$$w_0(\rho,\theta,t) = \frac{1}{\pi}\left[\sum_{k=0}^{3}(-1)^{[\frac{k}{2}]}\arctan\frac{B\sin 2c_k}{1-B\cos 2c_k}\right], \quad \rho < t, \tag{5.1}$$

where c_k are given by (2.18), $B := r^{2q}$ and

$$\arctan(\cdot) \in (-\pi/2, \pi/2). \tag{5.2}$$

Proof. Let us express function (4.2) in our polar coordinate θ and with the angle α instead of β. From (3.1), we obtain that

$$\zeta(\rho,\theta,t) = r(\rho,t)e^{i\theta-\frac{5}{2}i\pi+i\frac{\Phi}{2}}, \tag{5.3}$$

since

$$\Phi = 2\pi - \phi. \tag{5.4}$$

Now, we can express ξ, γ_1 and γ_2 in our coordinates. First, (4.7) implies that

$$\xi(\rho,\theta,t) = e^{-i\pi/2}(e^{i\pi/2} r e^{i\theta-(5/2)i\pi+i\Phi/2})^{2q} = r^{2q}e^{-i\pi/2}e^{-4i\pi q}e^{2iq\theta}e^{i\pi/2} = r^{2q}e^{2iq(\theta-2\pi)}, \tag{5.5}$$

since $q = \pi/(2\Phi)$, and $\theta - (5/2)\pi + \Phi/2 \in (-\pi,\pi)$ for $\theta \in (\phi, 2\pi)$. Further, (3.7) gives

$$\gamma_1 = 2q(2\pi-\alpha), \quad \gamma_2 = 2q\alpha \tag{5.6}$$

by (3.4). Substituting these expressions of ξ and γ_1, γ_2 into (4.11), we obtain

$$W_1(\xi(\rho,\theta,t)) = \frac{1}{\pi i}\log A(\rho,\theta,t), \tag{5.7}$$

where

$$\begin{aligned} A(\rho,\theta,t) &= \frac{\left(e^{2iq\alpha} - r^{2q}e^{2iq(\theta-2\pi)}\right)\left(e^{-2iq\alpha} - r^{2q}e^{2iq(\theta-2\pi)}\right)}{\left(e^{2iq(2\pi-\alpha)} - r^{2q}e^{2iq(\theta-2\pi)}\right)\left(e^{-2iq(2\pi-\alpha)} - r^{2q}e^{2iq(\theta-2\pi)}\right)} \\ &= \frac{\left(1 - Be^{2iq(\theta-2\pi+\alpha)}\right)\left(1 - Be^{2iq(\theta-2\pi-\alpha)}\right)}{\left(1 - Be^{2iq(\theta-\alpha)}\right)\left(1 - Be^{2iq(\theta-4\pi+\alpha)}\right)}, \quad B := r^{2q}. \end{aligned} \tag{5.8}$$

Now (5.7), (4.4) and (4.11) imply

$$\begin{aligned} w_0(\rho,\theta,t) &= \mathrm{Re}\left(\frac{1}{\pi i}\log\frac{\left(1 - Be^{2iq(\theta-2\pi+\alpha)}\right)\left(1 - Be^{2iq(\theta-2\pi-\alpha)}\right)}{\left(1 - Be^{2iq(\theta-\alpha)}\right)\left(1 - Be^{2iq(\theta-4\pi+\alpha)}\right)}\right) \\ &= \frac{1}{\pi}\arg\left(\frac{\left(1 - Be^{2iq(\theta-2\pi+\alpha)}\right)\left(1 - Be^{2iq(\theta-2\pi-\alpha)}\right)}{\left(1 - Be^{2iq(\theta-\alpha)}\right)\left(1 - Be^{2iq(\theta-4\pi+\alpha)}\right)}\right), \quad \rho < t, \end{aligned} \tag{5.9}$$

where $\arg(\cdot) \in (-\pi,\pi)$. Hence,

$$\begin{aligned} w_0(\rho,\theta,t) = \frac{1}{\pi}\Bigg[&-\arctan\frac{B\sin 2q(\theta-2\pi+\alpha)}{1-B\cos 2q(\theta-2\pi+\alpha)} - \arctan\frac{B\sin 2q(\theta-2\pi-\alpha)}{1-B\cos 2q(\theta-2\pi-\alpha)}\Bigg] \\ &+\arctan\frac{B\sin 2q(\theta-\alpha)}{1-B\cos 2q(\theta-\alpha)} + \arctan\frac{B\sin 2q(\theta-4\pi+\alpha)}{1-B\cos 2q(\theta-4\pi+\alpha)} + \kappa, \quad \rho < t, \end{aligned} \tag{5.10}$$

where arctan satisfies (5.2), and $\kappa \in \mathbb{Z}$ since w_0 is a continuous function in S_t by (4.1), and $0 < B < 1$ for $\rho < t$. Substituting $\theta = 2\pi - 0$ into (5.10), we obtain $\kappa = 0$.

Now, we rewrite the representation (5.10) in terms of the parameters c_k from (2.18). First, we note that

$$\begin{aligned} \sin 2q(\theta-\alpha) = \sin 2c_0, &\quad \sin 2q(\theta-4\pi+\alpha) = \sin 2c_1, \\ \sin 2q(\theta-2\pi+\alpha) = \sin 2c_2, &\quad \sin 2q(\theta-2\pi-\alpha) = \sin 2c_3. \end{aligned} \tag{5.11}$$

Here the first and the last identities follow from (2.18) directly, and the third identity follows from (2.18) and (2.11). Further, using (2.18), (5.4) and (2.11), we obtain that

$$c_1 = q(\theta - 2\phi + \alpha) = q(\theta - 2(2\pi - \Phi) + \alpha)) = q(\theta - 4\pi + \alpha) + \pi,$$

that gives the second identity of (5.11). Similarly, identities (5.11) hold with $\cos(\cdot)$ instead of $\sin(\cdot)$. Now, function (5.10) can be written as (5.1). Lemma 5.1 is proved. □

6. Proof of the coincidence

The main result of our paper is the following.

Theorem 6.1. *Solution (2.7) to problem (2.3) in the case of $F = h$ coincides with Sobolev's solution (3.5).*

Proof. It suffices to check that our solution $u(y,t)$ coincides with $w_0(\rho,\theta,t)$ given by (5.1). It suffices to prove this coincidence for $\rho < t$ since for $\rho > t$, both solutions are reduced to the sum of the incident and reflected waves which obviously coincide.

For $\rho < t$, formulas (2.17) can be rewritten as

$$U_k = e^{2ic_k}\frac{1 - B\cos 2c_k + iB\sin 2c_k}{-1 + B\cos 2c_k + iB\sin 2c_k}, \quad 0 < \rho < t, \tag{6.1}$$

where $0 < B := r^{2q} < 1$. Obviously,

$$|U_k| = 1, \quad k = \overline{0,3}, \tag{6.2}$$

hence $\log U_k$ is a purely imaginary number, and

$$\log U_k(\rho,\theta,t) = 2i\Big(c_k(\theta) + \arctan\frac{B(\rho,\theta)\sin 2c_k}{1 - B\cos 2c_k} + \pi m_k(\rho,\theta,t)\Big),$$

where $m_k(\rho,\theta,t) \in \mathbb{Z}$ and $\arctan(\cdot)$ satisfies (5.2). Therefore, (2.16) implies that

$$u_d(\rho,\theta,t) = \frac{1}{\pi}\left[\sum_{k=0}^{3}(-1)^{[\frac{k}{2}]}\arctan\frac{B\sin 2c_k}{1 - B\cos 2c_k}\right] + m(\rho,\theta,t), \quad (\rho,\theta) \in S_t. \tag{6.3}$$

Here $m(\rho,\theta,t) \in \mathbb{Z}$, and it is constant in every sector $R_k := S_t \cap \Theta_k$, in which u_d is continuous by Lemma 2.2.

Hence,

$$m(\rho,\theta,t) = m_k, \quad \theta \in \Theta_k,\ k = 1,2,3. \tag{6.4}$$

It remains to calculate m_k for $k = 1,2,3$. Let us note that

$$u_d(\rho,\theta,t) = 0, \quad \rho \le t, \quad \theta = \phi + 0, \quad \theta = 2\pi - 0. \tag{6.5}$$

In fact, $u(\rho,\phi+0,t) = 0$ and $u(\rho,2\pi-0,t) = 0$ by boundary conditions (2.3). Further, $u_{\rm in}(\rho,\phi+0,t) = 0$ and $u_{\rm in}(\rho,2\pi-0,t) = 1$ by (2.4) with $F = h$; $u_r(\rho,\phi+0,t) = 0$, $u_r(\rho,\phi-0,t) = -1$ by (2.5). Thus (6.5) follows from (2.7).

First, we calculate $m_1 = 0$. Let us consider $\theta = \phi + 0$. Then

$$\begin{array}{l} c_0 = q(\phi - \theta_1), \quad c_1 = q(\phi - 2\phi + \alpha)) = q(-\phi + \alpha) = -c_0, \\ c_2 = q(\phi - \theta_2) = q(-\Phi + \alpha), \quad c_3 = q(\phi - 2\pi - \alpha) = q(-\Phi - \alpha) \end{array} \tag{6.6}$$

by (2.18), (5.4) and (2.11). This implies that

$$\arctan\frac{B\sin 2c_1(\phi)}{1 - B\cos 2c_1(\phi)} = -\arctan\frac{B\sin 2c_0}{1 - B\cos 2c_0}.$$

Further,

$$\arctan\frac{B\sin 2c_2(\phi)}{1 - B\cos 2c_2(\phi)} = \arctan\frac{B\sin 2q(-\Phi+\alpha)}{1 - B\cos(-\Phi+\alpha)},$$

$$\arctan\frac{B\sin 2c_3(\phi)}{1 - B\cos 2c_3(\phi)} = \arctan\frac{B\sin 2q(-\Phi-\alpha)}{1 - B\cos 2q(-\Phi-\alpha)} = -\arctan\frac{B\sin 2c_2(\phi)}{1 + B\cos 2c_2(\phi)}$$

by (2.18), (2.11) and (5.4). Therefore, for $\theta = \phi + 0$, we have

$$\sum_{k=0}^{3} (-1)^{[\frac{k}{2}]} \arctan \frac{B \sin 2c_k(\phi)}{1 - B \cos 2c_k(\phi)} = 0.$$

Finally, (6.5), (6.3) and (6.4) imply that $m_1 = 0$.

Similarly we calculate $m_3 = 0$ considering $\theta \in \Theta_3$. Namely, let us take $\theta = 2\pi - 0$. In this case (2.18), (5.4) and (2.11) imply that

$$\begin{aligned} c_0 &= q(2\pi - \alpha), \quad c_1 = q(2\pi - (2\phi - \alpha)) = q(-2\pi + 2\Phi + \alpha), \\ c_2 &= q(2\pi - 2\pi + \alpha) = q\alpha, \quad c_3 = q(2\pi - 2\pi - \alpha) = -q\alpha. \end{aligned} \tag{6.7}$$

Hence,

$$\arctan \frac{B \sin 2c_1}{1 - B \cos 2c_1} = \arctan \frac{B \sin 2q(-2\pi + 2\Phi + \alpha)}{1 - B \cos 2q(-2\pi + 2\Phi + \alpha)} = -\arctan \frac{B \sin 2c_0}{1 - B \cos 2c_0}, \tag{6.8}$$

and

$$\arctan \frac{B \sin 2c_3}{1 - B \cos 2c_3} = -\arctan \frac{B \sin 2c_2}{1 - B \cos 2c_2}. \tag{6.9}$$

Therefore, (6.8) implies that $m_3 = 0$ by (6.3) and (6.4).

Finally, let us calculate $m_2 = 0$ considering $\theta \in \Theta_2$. In this case,

$$u_d(\rho, \theta, t) = 0, \quad \rho = t, \ \theta \in \Theta_2. \tag{6.10}$$

In particular,

$$\lim_{\theta \to \theta_1 + 0} u_d(t - 0, \theta, t) = 0. \tag{6.11}$$

On the other hand, we can calculate this limit using our formula (6.3), where $B = 1$. We have:

$$c_0 = q(\alpha + 0 - \alpha) = +0, \quad c_1 = q(2\alpha - 4\pi) + \pi \in (-\pi, 0), \quad c_2 = q(2\alpha - 2\pi), \quad c_3 = -2q\pi$$

by (2.18) and (5.4), (2.11). Hence, calculating the terms of (6.3) with $\theta = \theta_1 + 0$ and $B = 1$, we obtain

$$\begin{aligned} \arctan \frac{\sin 2c_0}{1 - \cos 2c_0} &= \arctan \frac{\sin 2(0+)}{1 - \cos 2(+0)} = \frac{\pi}{2}, \\ \arctan \frac{\sin 2c_1}{1 - \cos 2c_1} &= \arctan(\cot c_1) = -\frac{\pi}{2} - 2q\alpha + 4\pi q \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big), \\ \arctan \frac{\sin 2c_2}{1 - \cos 2c_2} &= \arctan(\cot c_2) = -\frac{\pi}{2} - c_2 = -\frac{\pi}{2} - 2\alpha q + 2\pi q \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big), \\ \arctan \frac{\sin 2c_3}{1 - \cos 2c_3} &= \arctan(\cot c_3) = -\frac{\pi}{2} - c_3 = -\frac{\pi}{2} + 2q\pi \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big) \end{aligned}$$

by (5.2), (5.4) and (2.11). Hence, (6.3) implies that

$$\begin{aligned} \lim_{\theta \to \theta_1 + 0} u_d(t - 0, \theta, t) &= \frac{1}{\pi} \sum_{k=0}^{3} (-1)^{[\frac{k}{2}]} \arctan \frac{\sin 2c_k(\theta_1 + 0)}{1 - B \cos 2c_k(\theta_1 + 0)} + m_2 \\ &= 1 + m_2, \quad \theta \in \Theta_2. \end{aligned} \tag{6.12}$$

Comparing with (6.11), we obtain $m_2 = -1$.

As the result, (6.4) with $m_1 = m_3 = 0$ and $m_2 = -1$ gives

$$u(\rho,\theta,t) = \frac{1}{\pi}\left[\sum_{k=0}^{3}(-1)^{[\frac{k}{2}]}\arctan\frac{B\sin 2c_k}{1-B\cos 2c_k}\right] = w_0(\rho,\theta,t), \quad (\rho,\theta)\in S_t$$

by (2.19), (6.3) and (5.1). The same identity obviously holds for $\rho > t$ since in this region, the diffracted wave vanishes, while the incident and reflected waves coincide with the same ones of Sobolev. Theorem 6.1 is proved. □

7. Uniqueness and stability of Sobolev's solution

7.1. Uniqueness

In [9, Sections 3.1 and 3.2], we have proved that there exists a unique solution $u \in \mathcal{M}_\varepsilon$ with $\varepsilon = 1-q$ (Definition 2.1), and this solution is expressed by (2.7), (2.4), (2.5) and (2.8). The Sobolev solution w_0 coincides with our solution u by Theorem 6.1. Therefore we obtain the following corollary.

Corollary 7.1. *The Sobolev solution (3.5) belongs to $\mathcal{M}_{1-q}$, and it is unique in this class of solutions.*

7.2. Stability

Let us consider the incident profiles $F \in L^1_{loc}(\mathbb{R})$, for which $F(x) = 0$ for $x < 0$, and

$$F(s) \to C, \quad s \to \infty. \tag{7.1}$$

In [9, Thm 4.2], we have proved that the diffracted wave (2.16) converges in the long time limit:

$$u_d(\rho,\theta,t) \underset{t\to\infty}{\longrightarrow} u_d(\theta,\infty) = \frac{iC}{4\Phi}\int\limits_{\mathbb{R}} Z(\beta+i\theta)\mathrm{d}\beta, \quad \rho>0,\ \theta\in\Theta. \tag{7.2}$$

Lemma 7.2. *(i) Let w_0 be Sobolev's solution (3.5). Then*

$$w_0(\rho,\theta,t) \to w_\infty(\theta), \quad t\to\infty, \tag{7.3}$$

where

$$w_\infty(\theta) = \begin{cases} \dfrac{1}{4\Phi}\displaystyle\int\limits_{\mathbb{R}} Z(s+i\theta)ds + 1, \ \rho>0, & \theta\in[\theta_1,\theta_2] \\[2ex] \dfrac{1}{4\Phi}\displaystyle\int\limits_{\mathbb{R}} Z(s+i\theta)ds, \ \rho>0, & \theta\in[\phi,\theta_1]\cup[\theta_2,2\pi] \end{cases} \tag{7.4}$$

with Z given by (2.8).

(ii) Let $\tilde h(s) \in L^1_{loc}(\overline{\mathbb{R}^+})$ s.t.

$$\tilde h(s) = h(s), \quad s\ge T \tag{7.5}$$

for some $T\in\mathbb{R}$, and $\tilde u \in \mathcal{M}_{1-q}$ be a solution to diffraction problem (2.3) with $F = \tilde h$. Then

$$\tilde u(\rho,\theta,t) \to w_\infty(\theta), \quad t\to\infty. \tag{7.6}$$

Proof. (i) Our main result, Theorem 6.1, implies that Sobolev's solution w_0 admits representation (2.19). Moreover, in our case $F \equiv h$, convergence (7.1) holds with $C = 1$. Hence, (7.2) with $C = 1$ and (2.19) imply (7.3) and (7.4).

(ii) There exists the splitting $\tilde{u} = \tilde{u}_{\rm in} + \tilde{u}_r + \tilde{u}_d$, where

$$\lim_{t\to\infty} \tilde{u}_{\rm in}(\rho,\theta,t) = \lim_{t\to\infty} u_{\rm in}(\rho,\theta,t), \ \lim_{t\to\infty} \tilde{u}_r(\rho,\theta,t) = \lim_{t\to\infty} u_r(\rho,\theta,t), \ \lim_{t\to\infty} \tilde{u}_d(\rho,\theta,t) = \lim_{t\to\infty} u_d(\rho,\theta,t).$$

Here the first and the second identities follow by (7.5). The last identity follows from (7.2) since (7.1) holds with $C = 1$ for both profile functions $F = h$ and $F = \tilde{h}$. Thus, (7.6) follows from (7.3). □

This lemma shows that the longtime asymptotics of Sobolev's solution does not depend on a local perturbation of the incident wave. In this sense, the Sobolev solution is asymptotically stable.

8. Conclusion

There are different approaches to non-stationary scattering of plane waves by two-dimensional wedges. Some particular solutions for the pulse incident wave were obtained by Sobolev in 1930', and Keller and Blank in 1950'. However, the uniqueness of the solutions in an appropriate functional class was not established up to now. Moreover, it is well known that the solution is not unique if its singularity is not specified.

Recently we have proposed a universal approach that gives explicit formulas for the solution with general incident waves and guarantees the existence and uniqueness of solutions in suitable functional classes.

In the present paper, we have checked that for the pulse incident wave, our solution coincides with Sobolev's formula, and specify its uniqueness and stability properties.

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