Weighted Energy Decay for 3D Klein-Gordon Equation

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Abstract

We obtain a dispersive long-time decay in weighted energy norms for solutions of the 3D Klein-Gordon equation with generic potential. The decay extends the results obtained by Jensen and Kato for the 3D Schrödinger equation. For the proof we modify the spectral approach of Jensen and Kato to make it applicable to relativistic equations.

Keywords: dispersion, Klein-Gordon equation, relativistic equations, resolvent, spectral representation, weighted spaces, continuous spectrum, Born series, convolution, long-time asymptotics, asymptotic completeness.

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1 Introduction

In this paper, we establish a dispersive long time decay for the solutions to 3D Klein-Gordon equation

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - m^2\psi(x,t) + V(x)\psi(x,t), \quad x \in \mathbb{R}^3, \quad m > 0 \tag{1.1}$$

in weighted energy norms. In vectorial form, equation (1.1) reads

$$i\dot{\Psi}(t) = \mathcal{H}\Psi(t) \tag{1.2}$$

where

$$\Psi(t) = \left(\begin{array}{c} \psi(t) \\ \dot{\psi}(t) \end{array}\right), \qquad \mathcal{H} = \left(\begin{array}{cc} 0 & i \\ i(\Delta - m^2 + V) & 0 \end{array}\right) \tag{1.3}$$

For $s, \sigma \in \mathbb{R}$, let us denote by $H^s_\sigma = H^s_\sigma(\mathbb{R}^3)$ the weighted Sobolev spaces introduced by Agmon, [1], with the finite norms

$$\|\psi\|_{H^s_\sigma} = \|\langle x\rangle^\sigma \langle\nabla\rangle^s \psi\|_{L^2} < \infty, \qquad \langle x\rangle = (1 + |x|^2)^{1/2}$$

We assume that $V(x)$ is a real function, and

$$|V(x)| + |\nabla V(x)| \le C\langle x\rangle^{-\beta}, \qquad x \in \mathbb{R}^3 \tag{1.4}$$

for some $\beta > 3$. Then the multiplication by $V(x)$ is bounded operator $H^1_s \to H^1_{s+\beta}$ for any $s \in \mathbb{R}$.

We restrict ourselves to the "regular case" in the terminology of [13] (or "nonsingular case" in [23]) which holds for *generic potentials*. Equivalently, the truncated resolvent of the Schrödinger operator $H = -\Delta + V(x)$ is bounded at the end point $\lambda = 0$ of the continuous spectrum by [23, Theorem 7.2]. In other words, the point $\lambda = 0$ is neither eigenvalue nor resonance for the operator H.

Definition 1.1. *$\mathcal{F}_\sigma$ is the complex Hilbert space $H^1_\sigma \oplus H^0_\sigma$ of vector-functions $\Psi = (\psi, \pi)$ with the norm*

$$\|\,\Psi\|_{\mathcal{F}_\sigma} = \|\psi\|_{H^1_\sigma} + \|\pi\|_{H^0_\sigma} < \infty \tag{1.5}$$

Our main result is the following long time decay of the solutions to (1.2): in the "regular case",

$$\|\mathcal{P}_c\Psi(t)\|_{\mathcal{F}_{-\sigma}} = \mathcal{O}(|t|^{-3/2}), \quad t \to \pm\infty \tag{1.6}$$

for initial data $\Psi_0 = \Psi(0) \in \mathcal{F}_\sigma$ with $\sigma > 5/2$ where $\mathcal{P}_c$ is a Riesz projector onto the continuous spectrum of the operator $\mathcal{H}$. The decay is desirable for the study of asymptotic stability and scattering for the solutions to nonlinear hyperbolic equations. The study has been started in 90' for nonlinear Schrödinger equation, [5, 24, 25, 29, 30], and continued last decade [6, 7, 16]. The study has been extended to the Klein-Gordon equation in [10, 31]. Further extension need more information on the decay for the corresponding linearized equations that stipulated our investigation.

Let us comment on previous results in this direction. Local energy decay has been established first in the scattering theory for linear Schrödinger equation developed since 50' by Birman, Kato, Simon, and others.

For free 3D Klein-Gordon equation, the decay $\sim t^{-3/2}$ in L^∞ norm has been proved first by Morawetz and Strauss [22, Appendix B]. For wave and Klein-Gordon equations with magnetic potential, the decay $\sim t^{-3/2}$ has been established primarily by Vainberg [32] in local energy norms for initial data with compact support. The results were extended to general hyperbolic partial differential equations by Vainberg in [33]. The decay in the L^p norms for wave and Klein-Gordon equations has been obtained in [3, 4, 8, 15, 21, 35, 36].

However, applications to asymptotic stability of solutions to the nonlinear equations also require an exact characterization of the decay for the corresponding linearized equations in weighted norms (see e.g.[5, 6, 7, 31]).

The decay of type (1.6) in weighted norms has been established first by Jensen and Kato [13] for the Schrödinger equation in the dimension $n = 3$. The result has been extended to all other dimensions by Jensen and Nenciu [11, 12, 14],and to more general PDEs of the Schrödinger type by Murata [23]. The survey of the results can be found in [28].

For free wave equations corresponding to $m = 0$, some estimates in weighted L^p-norms have been established in [2]. The Strichartz weighted estimates for the perturbed Klein-Gordon equations were established in [19].

For the free 3D Klein-Gordon equation, the decay (1.6) in the weighted energy norms has been proved first in [10, Lemma 18.2]. However, for the perturbed relativistic equations the decay was not proved until now. The problem was that the Jensen-Kato approach is not applicable directly to the relativistic equations. The difference reflects distinct character of wave propagation in the relativistic and nonrelativistic equations (see below).

Let us comment on the disctinction and our techniques. The Jensen-Kato approach [13] relies on the spectral Fourier-Laplace representation

$$P_c\Psi(t) = \frac{1}{2\pi i}\int_0^\infty e^{-i\omega t}\Big[R(\omega+i0) - R(\omega-i0)\Big]\Psi_0 d\omega, \qquad t\in\mathbb{R} \tag{1.7}$$

where $R(\omega)$ is the resolvent of the Schrödinger operator $H = -\Delta + V$, and P_c is the corresponding projector onto the continuous spectrum of H. Integration by parts implies the time decay of type (1.6)since the resolvent $R(\omega)$ is sufficiently smooth and its derivatives $\partial_\omega^k R(\omega)$ have a good decay at $|\omega|\to\infty$ for large k in the weighted norms. On the other hand, in the case of the Klein-Gordon, the derivatives do not decay though the smoothness of the resolvent also follows from the results [13].

Let us illustrate this difference in the case of the corresponding free 3D equations:
i) the resolvent of the free Schrödinger equation is the integral operator with the kernel

$$R_{\rm S}(\omega, x-y) = \frac{e^{i\sqrt{\omega}|x-y|}}{4\pi|x-y|}$$

ii) the resolvent of the free Klein-Gordon equation is the integral operator with the matrix kernel

$$R_{\rm KG}(\omega, x-y) = \left(\begin{array}{cc} 0 & 0 \\ -i\delta(x-y) & 0\end{array}\right) + \frac{e^{i\sqrt{\omega^2-m^2}|x-y|}}{4\pi|x-y|}\left(\begin{array}{cc}\omega & i\\ -i\omega^2 & \omega\end{array}\right) \tag{1.8}$$

and the region of integration in the corresponding formula (1.7) is changed to $|\omega| > m$. Leading singularities of the both resolvents are almost identical: $\sqrt{\omega}$ at $\omega = 0$ for $R_{\rm S}$, and $\sqrt{\omega\mp m}$ at

$\omega = \pm m$ for $R_{\rm KG}$. Hence, the contribution of law frequencies into the integral (1.7) decays like $t^{-3/2}$ both for the Schrödinger and Klein-Gordon case.

Now let us discuss the contribution of high frequencies into the integral (1.7). For the Schrödinger case, the contribution decays like $\sim t^{-N}$ with any $N > 0$. This follows by partial integration since the derivatives $\partial_\omega^k R_{\rm S}(\omega, x-y)$ decay like $|\omega|^{-k/2}$ as $\omega \to \infty$.

On the other hand, the kernel $R_{\rm KG}(\omega, x-y)$ does not decay for large $|\omega|$, and differentiation in ω does not improve the decay (cf. the bounds (2.25) and (3.8)). Hence, for the Klein-Gordon equation the integration by parts does not provide the long time decay.

This difference is not only technical. It reflects the fact that the multiplication by t^N, with large N, improves the smoothness of the solutions to the Schrödinger equation in contrast to the Klein-Gordon equation. This corresponds to distinct character of the wave propagation in the relativistic and nonrelativistic equations:
i) for a solution $\psi(x,t)$ to the Schrödinger equation, main singularity is concentrated at $t = 0$ and disappears at infinity for $t \neq 0$ due to infinite speed of propagation.
ii) for a solution $\psi(x,t)$ to the Klein-Gordon equation, the singularities move with bounded speed, thus they are present forever in the space.

Thus, the proof of the decay for the high energy component of the solution requires novel robust ideas. This problem is resolved at present paper with a modification of the Jensen and Kato technique. Our modification relies on a version of the Huygens principle, the Born series and the convolution. Namely, the resolvent $\mathcal{R}(\omega)$ of the operator $\mathcal{H}$ admits the finite Born expansion

$$\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega) + \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega) \tag{1.9}$$

where $\mathcal{R}_0(\omega)$ stands for the free resolvent with the integral kernel (1.8) corresponding to $V = 0$, and $\mathcal{V} = \left(\begin{array}{cc} 0 & 0 \\ V & 0 \end{array}\right)$. Taking the inverse Fourier-Laplace transform, we obtain the corresponding expansion for the dynamical group $\mathcal{U}(t)$ of the Klein-Gordon equation (1.2),

$$\mathcal{U}(t) = \mathcal{U}_0(t) + i\int_0^t \mathcal{U}_0(t-s)\mathcal{V}\mathcal{U}_0(s)ds - iF^{-1}_{\omega\to t}\Big[\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega)\Big] \tag{1.10}$$

where $\mathcal{U}_0(t)$ stands for the free dynamical group corresponding to $V = 0$. The expansion corresponds to iterative procedure in solving the perturbed Klein-Gordon equation (1.2). Further we consider separately each term in the right hand side of (1.10):

I. As we noted above, for the first term $\mathcal{U}_0(t)$ we cannot deduce the time decay (1.6) from the spectral representation of type (1.7). On the other hand, the decay has been established in [10, Lemma 18.2] using an analog of the strong Huygens principle extending Vainberg's trick [33] from the wave to the Klein-Gordon equation.

II. For the second term we also cannot deduce the time decay from the spectral representation. However, the decay follows by standard estimates for the convolution using the decay of the first term and the condition (1.4) on the potential.

III. Finally, the time decay for the last term follows from the spectral representation by the Jensen-Kato technique since $\|\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\| \sim |\omega|^{-2}$ as $|\omega| \to \infty$ that follows from the (expected) lucky structure of the matrix $\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}$ (see (3.13)).

Our paper is organized as follows. In Section 2 we obtain the time decay for the solution to the free Klein-Gordon equation and state the spectral properties of the free resolvent which follow from the corresponding known properties of the free Schrödinger resolvent. In Section 3 we obtain spectral properties of the perturbed resolvent and prove the decay (1.6). In Section 4 we apply the obtained decay to the asymptotic completeness.

In Appendix A we prove a revised version of Agmon-Jensen-Kato high energy decay for the free Schrödinger resolvent which we use in Section 2. Finally, in Appendix B we give a streamlined proof of the Jensen-Kato lemma on the decay of the Fourier integrals which we need in Section 3.

The asymptotic decay (1.6) is proved in [17] for 1D Klein-Gordon equation. For the 3D wave equation corresponding to $m = 0$, the weighted energy decay of type (1.6) was established in [18].

2 Free Klein-Gordon equation

2.1 Time decay

First, we prove the time decay (1.6) for the free Klein-Gordon equation:

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - m^2\psi(x,t), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R} \tag{2.1}$$

In vectorial form equation (2.1) reads

$$i\dot{\Psi}(t) = \mathcal{H}_0\Psi(t) \tag{2.2}$$

where

$$\Psi(t) = \begin{pmatrix} \psi(t) \\ \dot{\psi}(t) \end{pmatrix}, \qquad \mathcal{H}_0 = \begin{pmatrix} 0 & i \\ i(\Delta - m^2) & 0 \end{pmatrix} \tag{2.3}$$

Denote by $\mathcal{U}_0(t)$ the dynamical group of the equation (2.2). It is strongly continuous group in the Hilbert space $\mathcal{F}_0$. The group is unitary after a suitable modification of the norm that follows from the energy conservation.

Proposition 2.1. *(cf.[10], Lemma 18.2) Let $\sigma > 3/2$. Then for $\Psi_0 \in \mathcal{F}_\sigma$*

$$\|\mathcal{U}_0(t)\Psi_0\|_{\mathcal{F}_{-\sigma}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}}, \quad t \in \mathbb{R} \tag{2.4}$$

Proof. Step i) It suffices to consider $t > 0$. In this case the matrix kernel of the dynamical group $\mathcal{U}_0(t)$ can be written as $\mathcal{U}_0(x-y,t)$ where

$$\mathcal{U}_0(z,t) = \begin{pmatrix} \dot{U}(z,t) & U(z,t) \\ \ddot{U}(z,t) & \dot{U}(z,t) \end{pmatrix}, \quad z \in \mathbb{R}^3 \tag{2.5}$$

and

$$U(z,t) = \frac{\delta(t-|z|)}{4\pi t} - \frac{m}{4\pi}\frac{\theta(t-|z|)J_1(m\sqrt{t^2-|z|^2})}{\sqrt{t^2-|z|^2}}, \qquad t > 0 \tag{2.6}$$

where J_1 is the Bessel function of order 1, and θ is the Heavyside function. Let us fix an arbitrary $\varepsilon \in (0,1)$. Well known asymptotics of the Bessel function imply that

$$|\partial_z^\alpha \mathcal{U}_0(z,t)| \le C(\varepsilon)(1+t)^{-3/2}, \quad |z| \le \varepsilon t, \quad t \ge 1 \tag{2.7}$$

for $|\alpha|\le 1$.
Step ii) Now we consider an arbitrary $t\ge 1$. Let us split the initial function Ψ_0 in two terms, $\Psi_0=\Psi'_{0,t}+\Psi''_{0,t}$ such that

$$\|\Psi'_{0,t}\|_{\mathcal{F}_\sigma}+\|\Psi''_{0,t}\|_{\mathcal{F}_\sigma}\le C\|\Psi_0\|_{\mathcal{F}_\sigma},\quad t\ge 1 \tag{2.8}$$

and

$$\Psi'_{0,t}(x)=0\ \text{ for } |x|>\frac{\varepsilon t}{2},\qquad \text{and}\qquad \Psi''_{0,t}(x)=0\ \text{ for } |x|<\frac{\varepsilon t}{4} \tag{2.9}$$

The estimate (2.4) for $\mathcal{U}_0(t)\Psi''_{0,t}$ follows by energy conservation for the Klein-Gordon equation, (2.9) and (2.8):

$$\begin{aligned}\|\mathcal{U}_0(t)\Psi''_{0,t}\|_{\mathcal{F}_{-\sigma}} &\le \|\mathcal{U}_0(t)\Psi''_{0,t}\|_{\mathcal{F}_0}\le C\|\Psi''_{0,t}\|_{\mathcal{F}_0}\\ &\le \frac{C_1(\varepsilon)\|\Psi''_{0,t}\|_{\mathcal{F}_\sigma}}{(1+t)^\sigma}\le\frac{C_2(\varepsilon)\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+t)^{3/2}},\quad t\ge 1\end{aligned} \tag{2.10}$$

since $\sigma>3/2$.
Step iii) Next we consider $\mathcal{U}_0(t)\Psi'_{0,t}$. Now we split the operator $\mathcal{U}_0(t)$ in two terms:

$$\mathcal{U}_0(t)=(1-\zeta)\mathcal{U}_0(t)+\zeta\mathcal{U}_0(t),\qquad t\ge 1$$

where ζ is the operator of multiplication by the function $\zeta(|x|/t)$ such that $\zeta=\zeta(s)\in C_0^\infty(\mathbb{R})$, $\zeta(s)=1$ for $|s|<\varepsilon/4$, $\zeta(s)=0$ for $|s|>\varepsilon/2$. Obviously, for any α we have

$$|\partial_x^\alpha\zeta(|x|/t)|\le C<\infty,\qquad t\ge 1$$

Furthermore, $1-\zeta(|x|/t)=0$ for $|x|<\varepsilon t/4$, hence

$$\|(1-\zeta)\mathcal{U}_0(t)\Psi'_{0,t}\|_{\mathcal{F}_{-\sigma}}\le\frac{C_3(\varepsilon)\|(1-\zeta)\mathcal{U}_0(t)\Psi'_{0,t}\|_{\mathcal{F}_0}}{(1+t)^\sigma}\le\frac{C_4(\varepsilon)\|\mathcal{U}_0(t)\Psi'_{0,t}\|_{\mathcal{F}_0}}{(1+t)^\sigma} \tag{2.11}$$

Applying here the energy conservation for the group $\mathcal{U}_0(t)$, we obtain by (2.8) that

$$\|(1-\zeta)\mathcal{U}_0(t)\Psi'_{0,t}\|_{\mathcal{F}_{-\sigma}}\le\frac{C_5(\varepsilon)\|\Psi'_{0,t}\|_{\mathcal{F}_0}}{(1+t)^\sigma}\le\frac{C_6(\varepsilon)\|\Psi'_{0,t}\|_{\mathcal{F}_\sigma}}{(1+t)^\sigma}\le\frac{C_7(\varepsilon)\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+t)^{3/2}},\qquad t\ge 1 \tag{2.12}$$

since $\sigma>3/2$.
Step iv) It remains to estimate $\zeta\mathcal{U}_0(t)\Psi'_{0,t}$. Let $\chi_{\varepsilon t/2}$ be the characteristic function of the ball $|x|\le\varepsilon t/2$. We will use the same notation for the operator of multiplication by this characteristic function. By (2.9), we have

$$\zeta\mathcal{U}_0(t)\Psi'_{0,t}=\zeta\mathcal{U}_0(t)\chi_{\varepsilon t/2}\Psi'_{0,t} \tag{2.13}$$

The matrix kernel of the operator $\zeta\mathcal{U}_0(t)\chi_{\varepsilon t/2}$ is equal to

$$\mathcal{U}'_0(x-y,t)=\zeta(|x|/t)\mathcal{U}_0(x-y,t)\chi_{\varepsilon t/2}(y)$$

Since $\zeta(|x|/t)=0$ for $|x|>\varepsilon t/2$ and $\chi_{\varepsilon t/2}(y)=0$ for $|y|>\varepsilon t/2$, the estimate (2.7) implies that

$$|\partial_x^\alpha\mathcal{U}'_0(x-y,t)|\le C(1+t)^{-3/2},\quad |\alpha|\le 1,\quad t\ge 1 \tag{2.14}$$

The norm of the operator $\zeta\mathcal{U}_0(t)\chi_{\varepsilon t/2} : \mathcal{F}_\sigma \to \mathcal{F}_{-\sigma}$ is equivalent to the norm of the operator

$$\langle x\rangle^{-\sigma}\zeta\mathcal{U}_0(t)\chi_{\varepsilon t/2}(y)\langle y\rangle^{-\sigma} : \mathcal{F}_0 \to \mathcal{F}_0$$

The norm of the latter operator does not exceed the sum in α, $|\alpha| \le 1$, of the norms of operators

$$\partial_x^\alpha[\langle x\rangle^{-\sigma}\zeta\mathcal{U}_0(t)\chi_{\varepsilon t/2}(y)\langle y\rangle^{-\sigma}] : L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \tag{2.15}$$

The estimates (2.14) imply that operators (2.15) are Hilbert-Schmidt operators since $\sigma > 3/2$, and their Hilbert-Schmidt norms do not exceed $C(1+t)^{-3/2}$. Hence, (2.13) and (2.8) imply that

$$\|\zeta\mathcal{U}_0(t)\Psi'_{0,t}\|_{\mathcal{F}_{-\sigma}} \le C(1+t)^{-3/2}\|\Psi'_{0,t}\|_{\mathcal{F}_\sigma} \le C(1+t)^{-3/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \ge 1 \tag{2.16}$$

Finally, the estimates (2.16), (2.11) and (2.10) imply (2.4). □

2.2 Spectral properties

We state spectral properties of the free Klein-Gordon dynamical group $\mathcal{U}_0(t)$ applying known results of [1, 13] which concern the corresponding spectral properties of the free Schrödinger dynamical group. For $t > 0$ and $\Psi_0 = \Psi(0) \in \mathcal{F}_0$, the solution $\Psi(t)$ to the free Klein-Gordon equation (2.2) admits the spectral Fourier-Laplace representation

$$\theta(t)\Psi(t) = \frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i(\omega+i\varepsilon)t}\mathcal{R}_0(\omega+i\varepsilon)\Psi_0\,d\omega, \quad t \in \mathbb{R} \tag{2.17}$$

with any $\varepsilon > 0$ where $\theta(t)$ is the Heavyside function, $\mathcal{R}_0(\omega) = (\mathcal{H}_0 - \omega)^{-1}$ for $\omega \in \mathbb{C}^+ := \{\omega \in \mathbb{C} : \text{Im }\omega > 0\}$ is the resolvent of the operator $\mathcal{H}_0$. The representation follows from the stationary equation $\omega\tilde{\Psi}^+(\omega) = \mathcal{H}_0\tilde{\Psi}^+(\omega) + i\Psi_0$ for the Fourier-Laplace transform $\tilde{\Psi}^+(\omega) := \int_{\mathbb{R}}\theta(t)e^{i\omega t}\Psi(t)dt$, $\omega \in \mathbb{C}^+$. The solution $\Psi(t)$ is continuous bounded function of $t \in \mathbb{R}$ with the values in $\mathcal{F}_0$ by the energy conservation for the free Klein-Gordon equation (2.2). Hence, $\tilde{\Psi}^+(\omega) = -i\mathcal{R}(\omega)\Psi_0$ is analytic function of $\omega \in \mathbb{C}^+$ with the values in $\mathcal{F}_0$, and bounded for $\omega \in \mathbb{R} + i\varepsilon$. Therefore, the integral (2.17) converges in the sense of distributions of $t \in \mathbb{R}$ with the values in $\mathcal{F}_0$. Similarly to (2.17),

$$\theta(-t)\Psi(t) = -\frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i(\omega-i\varepsilon)t}\mathcal{R}_0(\omega-i\varepsilon)\Psi_0\,d\omega, \quad t \in \mathbb{R} \tag{2.18}$$

The resolvent $\mathcal{R}_0(\omega)$ can be expressed in terms of the resolvent $R_0(\zeta) = (-\Delta - \zeta)^{-1}$ of the free Schrödinger operator

$$\mathcal{R}_0(\omega) = \begin{pmatrix} \omega R_0(\omega^2 - m^2) & iR_0(\omega^2 - m^2) \\ -i(1 + \omega^2 R_0(\omega^2 - m^2)) & \omega R_0(\omega^2 - m^2) \end{pmatrix} \tag{2.19}$$

The free Schrödinger resolvent $R_0(\zeta)$ is an integral operator with the integral kernel

$$R_0(\zeta, x-y) = \exp(i\zeta^{1/2}|x-y|)/4\pi|x-y|, \quad \zeta \in \mathbb{C}^+, \quad \text{Im}\,\zeta^{1/2} > 0 \tag{2.20}$$

Definition 2.2. *Denote by $\mathcal{L}(B_1, B_2)$ the Banach space of bounded linear operators from a Banach space B_1 to a Banach space B_2.*

The explicit formula (2.20) implies the properties of $R_0(\zeta)$ which are obtained in [13, Lemmas 2.1 and 2.2]:

i) $R_0(\zeta)$ is analytic function of $\zeta \in \mathbb{C} \setminus [0, \infty)$ with the values in $\mathcal{L}(H_0^{-1}, H_0^1)$;
ii) For $\zeta > 0$, the convergence holds $R_0(\zeta \pm i\varepsilon) \to R_0(\zeta \pm i0)$ as $\varepsilon \to 0+$ in $\mathcal{L}(H_\sigma^{-1}, H_{-\sigma}^1)$ with $\sigma > 1/2$;
iii) The asymptotics hold for $\zeta \in \mathbb{C} \setminus [0, \infty)$,

$$\|R_0(\zeta)\|_{\mathcal{L}(H_\sigma^{-1}, H_{-\sigma}^1)} = \mathcal{O}(1), \qquad \zeta \to 0, \qquad \sigma > 1 \tag{2.21}$$

$$\|R_0^{(k)}(\zeta)\|_{\mathcal{L}(H_\sigma^{-1}, H_{-\sigma}^1)} = \mathcal{O}(\zeta^{1/2-k}), \quad \zeta \to 0, \qquad \sigma > 1/2 + k, \quad k = 1,\ 2, ... \tag{2.22}$$

Let us denote $\Gamma := (-\infty, -m) \cup (m, \infty)$ Then the properties i) – iv) and (2.19) imply the following lemma.

Lemma 2.3. *i) The resolvent $\mathcal{R}_0(\omega)$ is analytic function of $\omega \in \mathbb{C} \setminus \overline{\Gamma}$ with the values in $\mathcal{L}(\mathcal{F}_0, \mathcal{F}_0)$;*
ii) For $\omega \in \Gamma$, the convergence holds $\mathcal{R}_0(\omega \pm i\varepsilon) \to \mathcal{R}_0(\omega \pm i0)$ as $\varepsilon \to 0+$ in $\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$ with $\sigma > 1/2$;
iii) The asymptotics of type (2.21), (2.22) hold for $\omega \in \mathbb{C} \setminus \overline{\Gamma}$,

$$\|\mathcal{R}_0(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})} = \mathcal{O}(1), \quad \omega \pm m \to 0, \quad \sigma > 1 \tag{2.23}$$

$$\|\mathcal{R}_0^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})} = \mathcal{O}(|\omega \pm m|^{1/2-k}), \quad \omega \pm m \to 0, \quad \sigma > 1/2 + k, \quad k = 1,\ 2, ... \tag{2.24}$$

Finally, we state the asymptotics of $\mathcal{R}_0(\omega)$ for large ω which follow from the corresponding asymptotics of R_0, given in Proposition A.1.

Lemma 2.4. *The bounds hold*

$$\|\mathcal{R}_0^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})} = \mathcal{O}(1), \quad |\omega| \to \infty, \quad \omega \in \mathbb{C} \setminus \Gamma \tag{2.25}$$

with $\sigma > 1/2 + k$ for $k = 0, 1, 2, ...$

Proof. The bounds follow from representation (2.19) for $\mathcal{R}_0(\omega)$ and asymptotics (A.1) for $R_0(\zeta)$ with $\zeta = \omega^2 - m^2$. □

Corollary 2.5. *For $t \in \mathbb{R}$ and $\Psi_0 \in \mathcal{F}_\sigma$ with $\sigma > 1$, the group $\mathcal{U}_0(t)$ admits the integral representation*

$$\mathcal{U}_0(t)\Psi_0 = \frac{1}{2\pi i} \int\limits_\Gamma e^{-i\omega t} \Big[\mathcal{R}_0(\omega + i0) - \mathcal{R}_0(\omega - i0)\Big] \Psi_0 \, d\omega \tag{2.26}$$

where the integral converges in the sense of distributions of $t \in \mathbb{R}$ with the values in $\mathcal{F}_{-\sigma}$.

Proof. Summing up the representations (2.17) and (2.18), and sending $\varepsilon \to 0+$, we obtain (2.26) by the Cauchy theorem and Lemmas 2.3 and 2.4. □

Remark 2.6. *The estimates (2.25) do not allow obtain the decay (2.4) by partial integration in (2.26). This is why we deduce the decay in Section 2.1 from explicit formulas (2.5) and (2.6).*

3 Perturbed Klein-Gordon equation

To prove the long time decay for the perturbed Klein-Gordon equation, we first establish the spectral properties of the generator.

3.1 Spectral properties

According [13, p. 589] and [23, formula (3.1)], let us introduce a generalized eigenspace $\mathbf{M}$ for the perturbed Schrödinger operator $H = -\Delta + V$:

$$\mathbf{M} = \{\psi \in H^1_{-1/2-0} : \ (1 + A_0 V)\psi = 0\}$$

where A_0 is the operator with the integral kernel $1/4\pi|x-y|$. Below we assume that

$$\mathbf{M} = 0 \tag{3.1}$$

In [13, p. 591] the point $\lambda = 0$ is called then "regular point" for the Schrödinger operator H (it corresponds to the "nonsingular case" in [23, Section 7]). The condition holds for *generic potentials* V satisfying (1.4) (see[13, p. 589]).

Denote by $R(\zeta) = (H - \zeta)^{-1}$, $\zeta \in \mathbb{C} \setminus \mathbb{R}$, the resolvent of the Schrödinger operator H.

Remark 3.1. i) By [23, Theorem 7.2], the condition (3.1) is equivalent to the boundedness of the resolvent $R(\zeta)$ at $\zeta = 0$ in the norm of $\mathcal{L}(H^{-1}_\sigma, H^1_{-\sigma})$ with a suitable $\sigma > 0$.
ii) By Lemma 3.2 in [13], the condition (3.1) is equivalent to absence of nonzero solutions $\psi \in H^1_{-\sigma}$, with $\sigma \le 3/2$, to the equation $H\psi = 0$.
iii) $N(H) \subset \mathbf{M}$ where $N(H)$ is the zero eigenspace of the operator H. The imbedding is obtained in [13, Theorem 3.6]. The functions from $\mathbf{M}\setminus N(H)$ are called *zero resonance functions*. Hence, the condition (3.1) means that $\lambda = 0$ is neither eigenvalue nor resonance for the operator H.

Let us collect the properties of $R(\zeta)$ obtained in [1, 13, 23] under conditions (1.4) and (3.1):

R1. $R(\zeta)$ is meromorphic function of $\zeta \in \mathbb{C} \setminus [0, \infty)$ with the values in $\mathcal{L}(H^{-1}_0, H^1_0)$; the poles of $R(\zeta)$ are located at a finite set of eigenvalues $\zeta_j < 0$, $j = 1, ..., N$, of the operator H with the corresponding eigenfunctions $\psi^1_j(x), ..., \psi^{\kappa_j}_j(x) \in H^2_s$ with any $s \in \mathbb{R}$, where κ_j is the multiplicity of ζ_j.
R2. For $\zeta > 0$, the convergence holds $R(\zeta \pm i\varepsilon) \to R(\zeta \pm i0)$ as $\varepsilon \to 0+$ in $\mathcal{L}(H^{-1}_\sigma, H^1_{-\sigma})$ with $\sigma > 1/2$.
R3. The asymptotics hold for $\zeta \in \mathbb{C} \setminus [0, \infty)$,

$$\|R(\zeta)\|_{\mathcal{L}(H^{-1}_\sigma, H^1_{-\sigma})} = \mathcal{O}(1), \qquad \zeta \to 0, \qquad \sigma > 1 \tag{3.2}$$

$$\|R^{(k)}(\zeta)\|_{\mathcal{L}(H^{-1}_\sigma, H^1_{-\sigma})} = \mathcal{O}(|\zeta|^{1/2-k}), \quad \zeta \to 0, \qquad \sigma > 1/2 + k, \quad k = 1,\ 2 \tag{3.3}$$

Remark 3.2. The asymptotics (3.3) is deduced in [13, Remark 6.7] from (2.21), (2.22), (3.2) and the identities

$$R' = (1 - RV)R'_0(1 - VR), \quad R'' = \Big[(1 - RV)R''_0 - 2R'VR'_0\Big](1 - VR)$$

Further, the resolvent $\mathcal{R}(\omega)=(\mathcal{H}-\omega)^{-1}$ can be expressed similarly to (2.19):

$$\mathcal{R}(\omega)=\left(\begin{array}{cc}\omega R(\omega^2-m^2) & iR(\omega^2-m^2)\\ -i(1+\omega^2R(\omega^2-m^2)) & \omega R(\omega^2-m^2)\end{array}\right) \tag{3.4}$$

Hence, the properties **R1** – **R3** imply the corresponding properties of $\mathcal{R}(\omega)$:

Lemma 3.3. *Let the potential V satisfy conditions* (1.4) *and* (3.1)*. Then*
i) $\mathcal{R}(\omega)$ is meromorphic function of $\omega\in\mathbb{C}\setminus\overline{\Gamma}$ with the values in $\mathcal{L}(\mathcal{F}_0,\mathcal{F}_0)$;
ii) The poles of $\mathcal{R}(\omega)$ are located at a finite set

$$\Sigma=\{\omega_j^\pm=\pm\sqrt{m^2+\zeta_j},\ j=1,...,N\}$$

of eigenvalues of the operator $\mathcal{H}$ with the corresponding eigenfunctions $\left(\begin{array}{c}\psi_j^\kappa(x)\\ \omega_j^\pm\psi_j^\kappa(x)\end{array}\right)$, $\kappa=1,...,\kappa_j$;
iii) For $\omega\in\Gamma$, the convergence holds $\mathcal{R}(\omega\pm i\varepsilon)\to\mathcal{R}(\omega\pm i0)$ as $\varepsilon\to 0+$ in $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ with $\sigma>1/2$;
iv) The asymptotics of type (2.23), (2.24) *hold for $\omega\in\mathbb{C}\setminus\overline{\Gamma}$,*

$$\|\mathcal{R}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(1),\qquad \omega\pm m\to 0,\qquad \sigma>1 \tag{3.5}$$

$$\|\mathcal{R}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(|\omega\pm m|^{1/2-k}),\quad \omega\pm m\to 0,\qquad \sigma>1/2+k,\quad k=1,\ 2 \tag{3.6}$$

Now we obtain the asymptotics of $R(\zeta)$ and $\mathcal{R}(\omega)$ for large ζ and ω.

Lemma 3.4. *Let the potential V satisfy* (1.4)*. Then for $s=0,1$ and $l=-1,0,1$ with $s+l\in\{0;1\}$, we have*

$$\|R^{(k)}(\zeta)\|_{\mathcal{L}(H^s_\sigma,H^{s+l}_{-\sigma})}=\mathcal{O}(|\zeta|^{-\frac{1-l+k}{2}}),\quad \zeta\to\infty,\quad \zeta\in\mathbb{C}\setminus[0,\infty) \tag{3.7}$$

with $\sigma>1/2+k$ for $k=0,1,2$.

Proof. The lemma follows from Proposition A.1 in appendix A by the arguments from the proof of Theorem 9.2 in [13], where the bounds are proved for $s=0$ and $l=0,1$. □

Hence (3.4) implies

Corollary 3.5. *Let the potential V satisfy* (1.4)*. Then the following bounds hold*

$$\|\mathcal{R}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})}=\mathcal{O}(1),\quad |\omega|\to\infty,\quad \omega\in\mathbb{C}\setminus\Gamma \tag{3.8}$$

with $\sigma>1/2+k$ for $k=0,1,2$.

Finally, let us denote by $\mathcal{V}$ the matrix

$$\mathcal{V}=\left(\begin{array}{cc}0 & 0\\ iV & 0\end{array}\right) \tag{3.9}$$

Then the vectorial equation (1.2) reads

$$i\dot{\Psi}(t)=(\mathcal{H}_0+\mathcal{V})\Psi(t) \tag{3.10}$$

where $\mathcal{H}_0$ is defined in (2.3). The resolvents $\mathcal{R}(\omega)$, $\mathcal{R}_0(\omega)$ are related by the Born perturbation series

$$\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega) + \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega), \quad \omega \in \mathbb{C}\setminus[\Gamma\cup\Sigma] \tag{3.11}$$

which follows by iteration of $\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega)$. An important role in (3.11) plays the product $\mathcal{W}(\omega) := \mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}$. We obtain the asymptotics of $\mathcal{W}(\omega)$ for large ω.

Lemma 3.6. *Let the potential V satisfy (1.4) with $\beta > 1/2+k+\delta$ where $\delta > 0$ and $k = 0, 1, 2$. Then the following asymptotics hold*

$$\|\mathcal{W}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_{-\delta},\mathcal{F}_\delta)} = \mathcal{O}(|\omega|^{-2}), \quad |\omega|\to\infty, \quad \omega\in\mathbb{C}\setminus\Gamma \tag{3.12}$$

Proof. The asymptotics follow from the algebraic structure of the matrix

$$\mathcal{W}^{(k)}(\omega) = \mathcal{V}\mathcal{R}_0^{(k)}(\omega)\mathcal{V} = \begin{pmatrix} 0 & 0 \\ -iV\partial_\omega^k R_0(\omega^2-m^2)V & 0 \end{pmatrix} \tag{3.13}$$

since (A.1) with $s = 1$ and $l = -1$ implies that

$$\|VR_0^{(k)}(\zeta)Vf\|_{H^0_\delta} \le C\|R_0^{(k)}(\zeta)Vf\|_{H^0_{\delta-\beta}} = \mathcal{O}(|\zeta|^{-1-\frac{k}{2}})\|Vf\|_{H^1_{\beta-\delta}} = \mathcal{O}(|\zeta|^{-1-\frac{k}{2}})\|f\|_{H^1_{-\sigma}}$$

since $1/2 + k < \beta - \delta$. □

3.2 Time decay

In this section we combine the spectral properties of the perturbed resolvent and time decay for the unperturbed dynamics using the (finite) Born perturbation series. Our main result is the following.

Theorem 3.7. *Let conditions (1.4) and (3.1) hold. Then*

$$\|e^{-it\mathcal{H}} - \sum_{\omega_J\in\Sigma} e^{-i\omega_J t}P_J\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(|t|^{-3/2}), \quad t\to\pm\infty \tag{3.14}$$

with $\sigma > 5/2$, where P_J are the Riesz projectors onto the corresponding eigenspaces.

Proof. Step i) Let us substitute the series (3.11) into the spectral representation of type (2.17) for the solution to (1.1) with $\Psi(0) = \Psi_0 \in \mathcal{F}_\sigma$ where $\sigma > 3/2$. Then Lemma 3.3 and asymptotics (3.5) and (3.8) with $k = 0$ imply similarly to (2.26), that

$$\begin{aligned}
\Psi(t) \;-\; \sum_{\omega_J\in\Sigma} e^{-i\omega_J t}P_J\Psi_0 &= \frac{1}{2\pi i}\int_\Gamma e^{-i\omega t}\Big[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\Big]\Psi_0\,d\omega \qquad (3.15)\\
&= \frac{1}{2\pi i}\int_\Gamma e^{-i\omega t}\Big[\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\Big]\Psi_0\,d\omega\\
&+ \frac{1}{2\pi i}\int_\Gamma e^{-i\omega t}\Big[\mathcal{R}_0(\omega+i0)\mathcal{V}\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\mathcal{V}\mathcal{R}_0(\omega-i0)\Big]\Psi_0\,d\omega\\
&+ \frac{1}{2\pi i}\int_\Gamma e^{-i\omega t}\Big[[\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}](\omega+i0)-[\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}](\omega-i0)\Big]\Psi_0\,d\omega\\
&= \Psi_1(t)+\Psi_2(t)+\Psi_3(t), \qquad t\in\mathbb{R}
\end{aligned}$$

where P_J stands for the corresponding Riesz projector

$$P_J\Psi_0 := -\frac{1}{2\pi i}\int_{|\omega-\omega_J|=\delta}\mathcal{R}(\omega)\Psi_0 d\omega$$

with a small $\delta>0$. Further we analyze each term Ψ_k separately.
Step ii) The first term $\Psi_1(t)=\mathcal{U}_0(t)\Psi_0$ by (2.26). Hence, Proposition 2.1 implies that

$$\|\Psi_1(t)\|_{\mathcal{F}_{-\sigma}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}}, \quad t\in\mathbb{R}, \quad \sigma>3/2. \tag{3.16}$$

Step iii) The second term $\Psi_2(t)$ can be rewritten as a convolution.

Lemma 3.8. *The convolution representation holds*

$$\Psi_2(t) = i\int_0^t \mathcal{U}_0(t-\tau)\mathcal{V}\Psi_1(\tau)\ d\tau, \qquad t\in\mathbb{R} \tag{3.17}$$

where the integral converges in $\mathcal{F}_{-\sigma}$ with $\sigma>3/2$.

Proof. The term $\Psi_2(t)$ can be rewritten as

$$\Psi_2(t) = \frac{1}{2\pi i}\int_{\mathbb{R}}\Big[e^{-i\omega t}\mathcal{R}_0(\omega+i0)\mathcal{V}\mathcal{R}_0(\omega+i0) - e^{-i\omega t}\mathcal{R}_0(\omega-i0)\mathcal{V}\mathcal{R}_0(\omega-i0)\Big]\Psi_0\ d\omega \tag{3.18}$$

Let us denote

$$\mathcal{U}_0^{\pm}(t) := \theta(\pm t)\mathcal{U}_0(t), \quad \Psi_1^{\pm}(t) := \theta(\pm t)\Psi_1(t), \qquad t\in\mathbb{R}$$

We know that $\mathcal{R}_0(\omega+i0)\Psi_0 = i\tilde{\Psi}_1^+(\omega)$, hence the first term in the right hand side of (3.18) reads

$$\begin{aligned}
\Psi_{21}(t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t}\mathcal{R}_0(\omega+i0)\mathcal{V}\tilde{\Psi}_1^+(\omega)\ d\omega \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t}\mathcal{R}_0(\omega+i0)\mathcal{V}\Big[\int_{\mathbb{R}} e^{i\omega\tau}\Psi_1^+(\tau)d\tau\Big]d\omega \\
&= \frac{1}{2\pi}(i\partial_t+i)^2\int_{\mathbb{R}}\frac{e^{-i\omega t}}{(\omega+i)^2}\mathcal{R}_0(\omega+i0)\mathcal{V}\Big[\int_{\mathbb{R}} e^{i\omega\tau}\Psi_1^+(\tau)d\tau\Big]d\omega
\end{aligned} \tag{3.19}$$

The last double integral converges in $\mathcal{F}_{-\sigma}$ with $\sigma>3/2$ by (3.16), Lemma 2.3 ii), and (2.25) with $k=0$. Hence, we can change the order of integration by the Fubini theorem. Then we obtain that

$$\Psi_{21}(t) = i\int_{\mathbb{R}}\mathcal{U}_0^+(t-\tau)\mathcal{V}\Psi_1^+(\tau)d\tau = \begin{cases} i\int_0^t \mathcal{U}_0(t-\tau)\mathcal{V}\Psi_1(\tau)d\tau &, \ t>0 \\ 0 &, \ t<0\end{cases} \tag{3.20}$$

since

$$\mathcal{U}_0^+(t-\tau)=\frac{1}{2\pi i}(i\partial_t+i)^2\int\limits_{\mathbb{R}}\frac{e^{-i\omega(t-\tau)}}{(\omega+i)^2}\mathcal{R}_0(\omega+i0)\,d\omega$$

by (2.17). Similarly, integrating the second term in the right hand side of (3.18), we obtain

$$\Psi_{22}(t)=i\int_{\mathbb{R}}\mathcal{U}_0^-(t-\tau)\mathcal{V}\Psi_1^-(\tau)d\tau=\left\{\begin{array}{ll}0 & ,\ t>0\\ i\displaystyle\int_0^t\mathcal{U}_0(t-\tau)\mathcal{V}\Psi_1(\tau)d\tau & ,\ t<0\end{array}\right. \tag{3.21}$$

Now (3.17) follows since $\Psi_2(t)$ is the sum of two expressions (3.20) and (3.21). $\square$

Further, let us consider $\sigma\in(3/2,\beta/2]$. Applying Proposition 2.1 to the integrand in (3.17), we obtain that

$$\|\mathcal{U}_0(t-\tau)\mathcal{V}\Psi_1(\tau)\|_{\mathcal{F}_{-\sigma}}\le\frac{C\|\mathcal{V}\Psi_1(\tau)\|_{\mathcal{F}_\sigma}}{(1+|t-\tau|)^{3/2}}\le\frac{C_1\|\Psi_1(\tau)\|_{\mathcal{F}_{-\sigma}}}{(1+|t-\tau|)^{3/2}}\le\frac{C_2\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t-\tau|)^{3/2}(1+|\tau|)^{3/2}}$$

Therefore, integrating here in τ, we obtain by (3.17) that

$$\|\Psi_2(t)\|_{\mathcal{F}_{-\sigma}}\le\frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}},\quad t\in\mathbb{R},\quad \sigma>3/2 \tag{3.22}$$

Step iv) Finally, let us rewrite the last term in (3.15) as

$$\Psi_3(t)=\frac{1}{2\pi i}\int\limits_{\Gamma}e^{-i\omega t}\mathcal{N}(\omega)\Psi_0\,d\omega \tag{3.23}$$

where $\mathcal{N}(\omega):=\mathcal{M}(\omega+i0)-\mathcal{M}(\omega-i0)$ for $\omega\in\Gamma$, and

$$\mathcal{M}(\omega):=\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega)=\mathcal{R}_0(\omega)\mathcal{W}(\omega)\mathcal{R}(\omega),\qquad \omega\in\mathbb{C}\setminus[\Gamma\cup\Sigma] \tag{3.24}$$

First, we obtain the asymptotics of $\mathcal{N}(\omega)$ at the points $\pm m$.

Lemma 3.9. *i) The following asymptotics hold*

$$\|\mathcal{N}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})}=\mathcal{O}(|\omega\mp m|^{1/2}),\qquad \omega\to\pm m,\quad \omega\in\Gamma \tag{3.25}$$

for $\sigma>3/2$.
ii) The asymptotics (3.25) *can be differentiated twice:*

$$\left.\begin{array}{rcl}\|\mathcal{N}'(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})}&=&\mathcal{O}(|\omega\mp m|^{-1/2})\\ \|\mathcal{N}''(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})}&=&\mathcal{O}(|\omega\mp m|^{-3/2})\end{array}\right|\quad \omega\to\pm m,\quad \omega\in\Gamma \tag{3.26}$$

for $\sigma>5/2$.

Proof. The lemma follows from the corresponding asymptotics (2.23), (2.24) and (3.5), (3.6) of the resolvents $\mathcal{R}_0$ and $\mathcal{R}$ and their derivatives, and assumption (1.4) on the potential $V(x)$. $\square$

Second, we obtain the asymptotics of $\mathcal{N}(\omega)$ and its derivatives for large ω.

Lemma 3.10. *For $k = 0, 1, 2$ the asymptotics hold*

$$\|\mathcal{N}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})} = \mathcal{O}(|\omega|^{-2}), \quad |\omega| \to \infty, \quad \omega \in \Gamma \tag{3.27}$$

with $\sigma > 1/2 + k$.

Proof. The asymptotics (3.27) follow from the asymptotics (2.25), (3.8) and (3.12) for $\mathcal{R}_0^{(k)}(\omega)$, $\mathcal{R}^{(k)}(\omega)$ and $\mathcal{W}^{(k)}(\omega)$. For example, let us consider the case $k = 2$. Differentiating (3.24), we obtain

$$\mathcal{M}'' = \mathcal{R}_0''\mathcal{W}\mathcal{R} + \mathcal{R}_0\mathcal{W}''\mathcal{R} + \mathcal{R}_0\mathcal{W}\mathcal{R}'' + 2\mathcal{R}_0'\mathcal{W}'\mathcal{R} + 2\mathcal{R}_0'\mathcal{W}\mathcal{R}' + 2\mathcal{R}_0\mathcal{W}'\mathcal{R}' \tag{3.28}$$

For a fixed $\sigma > 5/2$, let us choose $\delta \in (5/2, \min\{\sigma, \beta - 1/2\})$. Then for the first term in (3.28) we obtain by (3.8) and (3.12)

$$\begin{aligned} \|\mathcal{R}_0''(\omega)\mathcal{W}(\omega)\mathcal{R}(\omega)f\|_{\mathcal{F}_{-\sigma}} &\le \|\mathcal{R}_0''(\omega)\mathcal{W}(\omega)\mathcal{R}(\omega)f\|_{\mathcal{F}_{-\delta}} \le C\|\mathcal{W}(\omega)\mathcal{R}(\omega)f\|_{\mathcal{F}_\delta} \\ &= \mathcal{O}(|\omega|^{-2})\|\mathcal{R}(\omega)f\|_{\mathcal{F}_{-\delta}} = \mathcal{O}(|\omega|^{-2})\|f\|_{\mathcal{F}_\delta} = \mathcal{O}(|\omega|^{-2})\|f\|_{\mathcal{F}_\sigma}, \quad |\omega| \to \infty, \quad \omega \in \mathbb{C} \setminus \Gamma \end{aligned} \tag{3.29}$$

Other terms can be estimated similarly choosing an appropriate value of δ. Namely, $\delta \in (1/2, \min\{\sigma, \beta - 5/2\})$ for the second term, $\delta \in (5/2, \min\{\sigma, \beta - 1/2\})$ for the third, $\delta \in (3/2, \min\{\sigma, \beta - 3/2\})$ for the forth and sixth terms, and $\sigma' \in (3/2, \min\{\sigma, \beta - 1/2\})$ for the fifth term. □

Now we prove the desired decay of $\Psi_3(t)$ from (3.23) using methods [13]. Let us consider the integral over (m, ∞). The integral over $(-\infty, -m)$ can be dealt in the same way.

Let us split $\Psi_3(t)$ into the low and high energy components. We choose $\phi_1(\omega), \phi_2(\omega) \in C_0^\infty(\mathbb{R})$ where supp $\phi_1 \subset [m/2, b]$ with sufficiently large $b > 0$, and supp $\phi_2 \subset [b-1, \infty)$, such that $\phi_1(\omega) + \phi_2(\omega) = 1$ for $\omega \in [m, \infty)$. Then (3.23) implies that $\Psi_3(t) = \Psi_{31}(t) + \Psi_{32}(t)$, where

$$\Psi_{31}(t) = \frac{1}{2\pi i}\int_m^b e^{-i\omega t}\phi_1(\omega)\mathcal{N}(\omega)\Psi_0 \, d\omega, \qquad \Psi_{32}(t) = \frac{1}{2\pi i}\int_{b-1}^\infty e^{-i\omega t}\phi_2(\omega)\mathcal{N}(\omega)\Psi_0 \, d\omega$$

By Lemma 3.9, we can apply to the Fourier integral $\Psi_{31}(t)$ the corresponding version of Lemma B.1 below with $a = m$, operator function $F = \phi_1(\omega)\mathcal{N}(\omega)$, and the Banach space $\mathbf{B} = \mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$ with $\sigma > 5/2$. Then we obtain that

$$\|\Psi_{31}(t)\|_{\mathcal{F}_{-\sigma}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}}, \quad t \in \mathbb{R} \tag{3.30}$$

Further, supp $\phi_2\mathcal{N} \subset [b-1, \infty)$, and $(\phi_2\mathcal{N})'' \in L^1(b-1, \infty; \mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma}))$ with $\sigma > 5/2$ by Lemma 3.10. Hence, two times partial integration implies that

$$\|\Psi_{32}(t)\|_{\mathcal{F}_{-\sigma}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^2}, \quad t \in \mathbb{R}$$

This completes the proof of Theorem 3.7. □

Corollary 3.11. *The asymptotics* (3.14) *imply* (1.6) *with the projector*

$$\mathcal{P}_c := 1 - \mathcal{P}_d, \quad \mathcal{P}_d = \sum_{\omega_J \in \Sigma} P_J \tag{3.31}$$

4 Application to the asymptotic completeness

We apply the obtained results to prove the asymptotic completeness by standard Cook's argument.

Theorem 4.1. *Let conditions* (1.4) *and* (3.1) *hold. Then*
i) For solution to (1.2) *with any initial function* $\Psi(0) \in \mathcal{F}_0$*, the following long time asymptotics hold,*

$$\Psi(t) = \sum_{\omega_J \in \Sigma} e^{-i\omega_J t}\Psi_J + \mathcal{U}_0(t)\Phi_\pm + r_\pm(t) \tag{4.1}$$

where Ψ_J *are the corresponding eigenfunctions,* $\Phi_\pm \in \mathcal{F}_0$ *are the scattering states, and*

$$\|r_\pm(t)\|_{\mathcal{F}_0} \to 0, \qquad t \to \pm\infty \tag{4.2}$$

ii) Furthermore,

$$\|r_\pm(t)\|_{\mathcal{F}_0} = \mathcal{O}(|t|^{-1/2}) \tag{4.3}$$

if $\Psi(0) \in \mathcal{F}_\sigma$ *with* $\sigma \in (5/2, \beta]$.

Proof. Denote $\mathcal{X}_d := \mathcal{P}_d\mathcal{F}_0$, $\mathcal{X}_c := \mathcal{P}_c\mathcal{F}_0$. For $\Psi(0) \in \mathcal{X}_d$ the asymptotics (4.1) obviously hold with $\Phi_\pm = 0$ and $r_\pm(t) = 0$. Hence, it remains to prove for $\Psi(0) \in \mathcal{X}_c$ the asymptotics

$$\Psi(t) = \mathcal{U}_0(t)\Phi_\pm + r_\pm(t) \tag{4.4}$$

with the remainder satisfying (4.2). Moreover, it suffices to prove the asymptotics (4.4), (4.3) for $\Psi_0 \in \mathcal{X}_c \cap \mathcal{F}_\sigma$ with $\sigma > 5/2$ since the space $\mathcal{F}_\sigma$ is dense in $\mathcal{F}_0$, while the group $\mathcal{U}_0(t)$ is unitary in $\mathcal{F}_0$ after a suitable modification of the norm. In this case Theorem 3.7 implies the decay

$$\|\Psi(t)\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-3/2}\|\Psi(0)\|_{\mathcal{F}_\sigma}, \quad t \to \pm\infty \tag{4.5}$$

We also can assume $\beta \ge \sigma$.

The function $\Psi(t)$ satisfies the equation (3.10),

$$i\dot{\Psi}(t) = (\mathcal{H}_0 + \mathcal{V})\Psi(t)$$

Hence, the corresponding Duhamel equation reads

$$\Psi(t) = \mathcal{U}_0(t)\Psi(0) + \int_0^t \mathcal{U}_0(t-\tau)\mathcal{V}\Psi(\tau)d\tau, \qquad t \in \mathbb{R} \tag{4.6}$$

Let us rewrite (4.6) as

$$\Psi(t) = \mathcal{U}_0(t)\Big[\Psi(0) + \int_0^{\pm\infty} \mathcal{U}_0(-\tau)\mathcal{V}\Psi(\tau)d\tau\Big] - \int_t^{\pm\infty} \mathcal{U}_0(t-\tau)\mathcal{V}\Psi(\tau)d\tau = \mathcal{U}_0(t)\Phi_\pm + r_\pm(t) \tag{4.7}$$

It remains to prove that $\Phi_\pm \in \mathcal{F}_0$ and (4.3) holds. Let us consider the sign "+" for the concreteness. The "unitarity" of $\mathcal{U}_0(t)$ in $\mathcal{F}_0$, the condition (1.4) and the decay (4.5) imply that

$$\begin{aligned}\int\limits_0^\infty \|\mathcal{U}_0(-\tau)\mathcal{V}\Psi(\tau)\|_{\mathcal{F}_0} d\tau &\le C\int\limits_0^\infty \|\mathcal{V}\Psi(\tau)\|_{\mathcal{F}_0} d\tau \le C_2\int\limits_0^\infty \|\Psi(\tau)\|_{\mathcal{F}_{-\sigma}} d\tau \\ &\le C_2\int\limits_0^\infty (1+\tau)^{-3/2}\|\Psi(0)\|_{\mathcal{F}_\sigma} d\tau < \infty\end{aligned} \tag{4.8}$$

since $|V(x)| \le C'\langle x\rangle^{-\beta} \le C''\langle x\rangle^{-\sigma}$. Hence, $\Phi_+ \in \mathcal{F}_0$. The estimate (4.3) follows similarly. $\square$

Remark 4.2. *i) The asymptotic completeness is proved by another methods in [20, 27, 34] for more general Klein-Gordon equations with an external Maxwell field.*
ii) A version of Theorem 4.1 *using standard* L^p *spaces and Strichartz estimates, follows also from [36]. Notice that the hypotheses in [36] can be relaxed to (*1.4*) by the methods of [9].*

A Appendix: Decay of the free Schrödinger resolvent

We revise the Agmon-Jensen-Kato decay of the resolvent [1, (A.2')], [13, (8.1)] for special case of free Schrödinger equation in arbitrary dimension $n \ge 1$.

Proposition A.1. *For* $k = 0, 1, 2, ...$ *and* $\sigma > 1/2 + k$ *the asymptotics hold*

$$\|R_0^{(k)}(\zeta)\|_{\mathcal{L}(H^s_\sigma, H^{s+l}_{-\sigma})} = \mathcal{O}(|\zeta|^{-\frac{1-l+k}{2}}), \quad |\zeta| \to \infty, \quad \zeta \in \mathbb{C}\setminus[0,\infty), \quad s \in \mathbb{R} \tag{A.1}$$

where $l = -1, 0, 1, 2$ *for* $k = 0$*, and* $l = -1, 0, 1$ *for* $k = 1, 2, ..$

We give a complete proof of the asymptotics (A.1) refining the arguments in the proof of Theorem A.1 from [1, Appendix A]. Namely, we deduce Proposition A.1 from the following two lemmas. The first lemma is well known (see [1, Lemma A.2], and [26, Lemma 4, p. 442]). Denote $\partial_j = \frac{\partial}{\partial x_j}$.

Lemma A.2. *For* $\sigma > 1/2$*, the following inequality holds for* $\psi \in C_0^\infty(\mathbb{R}^n)$

$$\|\partial_j\psi\|_{H^0_{-\sigma}} \le C(\sigma)\|(\Delta+\zeta)\psi\|_{H^0_\sigma}, \quad \zeta \in \mathbb{C} \tag{A.2}$$

Second lemma is a refinement, for special case of free Schrödinger equation, of Lemma A.3 from [1, Appendix A] which is proved for bounded $|\zeta|$.

Lemma A.3. *For any* $\delta \in \mathbb{R}$ *and* $\psi \in C_0^\infty(\mathbb{R}^n)$ *the estimate holds*

$$\|\psi\|_{H^l_\delta} \le C(s)|\zeta|^{-\frac{1-l}{2}}\Big(\|(\Delta+\zeta)\psi\|_{H^0_\delta} + \sum_{j=1}^n \|\partial_j\psi(x)\|_{H^0_\delta}\Big), \quad \zeta \in \mathbb{C}, \quad |\zeta| \ge 1, \quad l = 0, 1 \tag{A.3}$$

Proof. We will prove (A.3) for $\delta = 0$, and the extension to all $\delta \in \mathbb{R}$ follows by the arguments from [1, pp 207-208]. For the proof we use the bound (cf. [1, formula (A.15')])

$$(1+|\xi^l|)^2 \le C|\zeta|^{-(1-l)}\Big(\Big||\xi|^2-\zeta\Big|^2+|\xi|^2\Big), \quad \xi\in\mathbb{R}^n, \quad |\zeta|\ge 1, \quad l=0,1 \qquad \text{(A.4)}$$

For $l = 1$ the bound is obvious. For $l = 0$ it reduces to a quadratic inequality for $y = |\xi|^2-|\zeta| \ge -|\zeta|$ since then

$$\Big||\xi|^2-\zeta\Big|^2+|\xi|^2 \ge \Big||\xi|^2-|\zeta|\Big|^2+|\xi|^2 = y^2+y+|\zeta| \ge \min_{y\ge -|\zeta|}(y^2+y)+|\zeta| \ge \frac{|\zeta|}{2}, \quad |\zeta|\ge 1$$

Finally, let us multiply both sides of (A.4) by $|\hat\psi(\xi)|^2$ and integrate over $\mathbb{R}^n$. Then using Parseval's formula, we find for $|\zeta| \ge 1$ that

$$\sum_{|\alpha|\le l}\|D^\alpha\psi\|^2 \le C\int\limits_{\mathbb{R}^n}(1+|\xi|^l)^2|\hat\psi(\xi)|^2d\xi \le C_1|\zeta|^{-(1-l)}\Big(\|(\Delta+\zeta)\psi\|^2+\sum_{j=1}^n\|\partial_j\psi(x)\|^2\Big) \qquad \text{(A.5)}$$

□

Proof of Proposition A.1 It suffices to verify the case $s = 0$ since $R_0(\zeta)$ commutes with the operators $\langle\nabla\rangle^s$ with arbitrary $s \in \mathbb{R}$.
Step i) First, we prove (A.1) with $k = 0$ and $l = 0,1$ similarly to the proof of Theorem A.1 in [1, p. 208]. Applying Lemma A.3 with $\delta = -\sigma$, we obtain

$$\|\psi\|_{H^l_{-\sigma}} \le C(\sigma)|\zeta|^{-\frac{1-l}{2}}\Big(\|(\Delta+\zeta)\psi\|_{H^0_{-\sigma}}+\sum_{j=1}^n\|\partial_j\psi\|_{H^0_{-\sigma}}\Big), \quad |\zeta|\ge 1, \quad l=0,1. \qquad \text{(A.6)}$$

for all $\psi \in H^2_\sigma(\mathbb{R}^n)$. On the other hand, Lemma A.2 implies that

$$\sum_{j=1}^n\|\partial_j\psi\|_{H^0_{-\sigma}} \le C_1(\sigma)\|(\Delta+\zeta)\psi\|_{H^0_\sigma}, \quad j=1,...,n. \qquad \text{(A.7)}$$

Combining (A.6) and (A.7), we obtain

$$\|\psi\|_{H^l_{-\sigma}} \le C(\sigma)|\zeta|^{-\frac{1-l}{2}}\Big(\|(\Delta+\zeta)\psi\|_{H^0_{-\sigma}}+C_1(\sigma)\|(\Delta+\zeta)\psi\|_{H^0_\sigma}\Big) \le C_2(\sigma)|\zeta|^{-\frac{1-l}{2}}\|(\Delta+\zeta)\psi\|_{H^0_\sigma}$$

and then (A.1) with $k = 0$ and $l = 0,1$ is proved.
Step ii) Second, we prove (A.1) in the case $k = 0$ and $l = -1$. We use the identity $R_0(\zeta) = -(1+\Delta R_0(\zeta))/\zeta$. The bound with $l = 1$ implies that $\|R_0(\zeta)\|_{\mathcal{L}(H^0_\sigma,H^1_{-\sigma})} = \mathcal{O}(1)$, hence $\|\Delta R_0(\zeta)\|_{\mathcal{L}(H^0_\sigma,H^{-1}_{-\sigma})} = \mathcal{O}(1)$. Therefore

$$\|R_0(\zeta)\|_{\mathcal{L}(H^0_\sigma,H^{-1}_{-\sigma})} = \|(1+\Delta R_0(\zeta))/\zeta\|_{\mathcal{L}(H^0_\sigma,H^{-1}_{-\sigma})} = \mathcal{O}(|\zeta|^{-1})$$

Step iii) Third, we prove (A.1) in the case $k = 0$ and $l = 2$. Using the identity $(1-\Delta)R_0(\zeta) = 1+(1+\zeta)R_0(\zeta)$, we obtain

$$\begin{aligned}\|R_0(\zeta)\|_{\mathcal{L}(H^0_\sigma,H^2_{-\sigma})} &= \|(1-\Delta)R_0(\zeta)\|_{\mathcal{L}(H^0_\sigma,H^0_{-\sigma})} = \|1+(1+\zeta)R_0(\zeta)\|_{\mathcal{L}(H^0_\sigma,H^0_{-\sigma})}\\ &= 1+\mathcal{O}(|\zeta|)\|R_0(\zeta)\|_{\mathcal{L}(H^0_\sigma,H^0_{-\sigma})} = \mathcal{O}(|\zeta|^{1/2})\end{aligned} \qquad \text{(A.8)}$$

Step iv) Finally, we consider the case $k \geq 1$. The asymptotics (A.1) wits $k = 1$ follows from the asymptotics (A.1) with $k = 0$ and the Lavine-type identity [13, (8.2)]

$$\zeta R_0'(\zeta) = -R_0(\zeta) + \frac{1}{2}[x \cdot \nabla, R_0(\zeta)], \quad \zeta \in \mathbb{C} \setminus [0, \infty) \tag{A.9}$$

(where $[\cdot, \cdot]$ stands for the commutator) since

$$x \in \mathcal{L}(H^s_\sigma, H^s_{\sigma-1}), \quad \nabla \in \mathcal{L}(H^s_\sigma, H^{s-1}_\sigma)$$

For $k \geq 2$ the asymptotics (A.1) follow by induction from the recurrent relation [13, (8.5)]

$$2\zeta R_0^{(k)}(\zeta) = -(2k-3)R_0^{(k-1)}(\zeta) - \frac{1}{2}[x, [x, R_0^{(k-2)}(\zeta)]]$$

where the double commutator is defined as

$$[x, [x, R]] = |x|^2 R - 2\sum_j x_j R x_j + R|x|^2$$

□

B Appendix: the Jensen-Kato lemma

We prove a lemma concerning the decay of the Fourier integrals which we have used in (3.30). The lemma is a special case of [13, Lemma 10.2], and our proof is a streamlined version of the proof from [13]. Let $\mathbf{B}$ denote a Banach space with the norm $\|\cdot\|$, and $b > a$.

Lemma B.1. *Let $F \in C(a, b; \mathbf{B})$ satisfy*

$$F(a) = F(b) = 0, \quad F'' \in L^1(\delta, b; \mathbf{B}), \quad \forall \delta > 0, \quad \|F''(\omega)\| = \mathcal{O}(|\omega - a|^{-3/2}), \quad \omega \to a \tag{B.1}$$

Then

$$\int_a^b e^{-it\omega} F(\omega) d\omega = \mathcal{O}(t^{-3/2}), \quad t \to \infty \tag{B.2}$$

Proof. Extending F by $F(\omega) = 0$ for $\omega < a$ and for $\omega > b$, we obtain a continuous function F on $(-\infty, \infty)$ with $F' \in L^1(-\infty, \infty; \mathbf{B})$. Using Zygmund's trick [37, formula (4.2) p. 45], we obtain

$$\int_{-\infty}^{\infty} F'(\omega) e^{-it\omega} d\omega = -\frac{1}{2} \int_{-\infty}^{\infty} (F'(\omega + \frac{\pi}{t}) - F'(\omega)) e^{-it\omega} d\omega$$

Furthermore, the conditions (B.1) imply that

$$\int_{-\infty}^{\infty} \|F'(\omega + \frac{\pi}{t}) - F'(\omega)\| d\omega = \int_{-\infty}^{a+\pi/t} \ldots + \int_{a+\pi/t}^{\infty} \ldots$$

$$\leq 2 \int_a^{a+2\pi/t} \|F'(\omega)\| d\omega + \int_{a+\pi/t}^{\infty} d\omega \int_{\omega}^{\omega+\pi/t} \|F''(\nu)\| d\nu = \mathcal{O}(t^{-1/2}) + \frac{\pi}{t} \int_{a+\pi/t}^{\infty} \|F''(\nu)\| d\nu = \mathcal{O}(t^{-1/2})$$

Hence, (B.2) follows. □

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