Weighted Energy Decay for for 1D Klein-Gordon Equation

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Abstract

We obtain a dispersive long-time decay in weighted energy norms for solutions of the 1D Klein-Gordon equation with generic potential. The decay extends the results obtained by Jensen, Kato and Murata for the equations of Schrödinger's type by the spectral approach. For the proof we modify the approach to make it applicable to relativistic equations.

Keywords: Klein-Gordon equation, relativistic equations, resolvent, spectral representation, weighted spaces, Born series, convolution, asymptotic completeness.

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1 Introduction

In this paper, we establish a dispersive long time decay for the solutions to 1D Klein-Gordon equation

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) - V(x)\psi(x,t), \quad x \in \mathbb{R}, \quad m > 0 \tag{1.1}$$

in weighted energy norms. In vectorial form, equation (1.1) reads

$$i\dot{\Psi}(t) = \mathcal{H}\Psi(t), \tag{1.2}$$

where

$$\Psi(t) = \begin{pmatrix} \psi(t) \\ \dot{\psi}(t) \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} 0 & i \\ i(\frac{d^2}{dx^2} - m^2 - V) & 0 \end{pmatrix} \tag{1.3}$$

For $s, \sigma \in \mathbb{R}$, let us denote by $H^s_\sigma = H^s_\sigma(\mathbb{R})$ the weighted Sobolev spaces introduced by Agmon, [1], with the finite norms

$$\|\psi\|_{H^s_\sigma} = \|\langle x\rangle^\sigma \langle \frac{d}{dx}\rangle^s \psi\|_{L^2} < \infty, \qquad \langle x\rangle = (1 + |x|^2)^{1/2}$$

We assume that $V(x)$ is a real function, and

$$|V(x)| + |V'(x)| \le C\langle x\rangle^{-\beta}, \qquad x \in \mathbb{R} \tag{1.4}$$

for some $\beta > 5$. Then the multiplication by $V(x)$ is bounded operator $H^1_s \to H^1_{s+\beta}$ for any $s \in \mathbb{R}$.

We restrict ourselves to the "nonsingular case" in the terminology of [20], where the truncated resolvent of the Schrödinger operator $H = -\frac{d^2}{dx^2} + V(x)$ is bounded at the end point $\lambda = 0$ of the continuous spectrum. In other words, the point $\lambda = 0$ is neither eigenvalue nor resonance for the operator H; this holds for "generic" potentials satisfying $\int V(x)dx \neq 0$.

Definition 1.1. *$\mathcal{F}_\sigma$ is the complex Hilbert space $H^1_\sigma \oplus H^0_\sigma$ of vector-functions $\Psi = (\psi, \pi)$ with the norm*

$$\| \Psi\|_{\mathcal{F}_\sigma} = \|\psi\|_{H^1_\sigma} + \|\pi\|_{H^0_\sigma} < \infty \tag{1.5}$$

Our main result is the following long time decay of the solutions to (1.2): in the "nonsingular case", the asymptotics hold

$$\|\mathcal{P}_c\Psi(t)\|_{\mathcal{F}_{-\sigma}} = \mathcal{O}(|t|^{-3/2}), \quad t \to \pm\infty \tag{1.6}$$

for initial data $\Psi_0 = \Psi(0) \in \mathcal{F}_\sigma$ with $\sigma > 5/2$ where $\mathcal{P}_c$ is a Riesz projector onto the continuous spectrum of the operator $\mathcal{H}$. The decay is desirable for the study of asymptotic stability and scattering for the solutions to nonlinear hyperbolic equations. The study has been started in 90' for nonlinear Schrödinger equation, [4, 21, 22, 23], and continued last decade [5, 6, 14]. The study has been extended to the Klein-Gordon equation in [8, 24]. Further extension need more information on the decay for the corresponding linearized equations that stipulated our investigation.

Let us comment on previous results in this direction. Local energy decay has been established first in the scattering theory for linear Schrödinger equation developed since 50' by Birman, Kato, Simon, and others.

For free 3D Klein-Gordon equation, the decay $\sim t^{-3/2}$ in L^∞ norm has been proved first by Morawetz and Strauss [18, Appendix B]. For wave and Klein-Gordon equations with magnetic potential, the decay $\sim t^{-3/2}$ has been established primarily by Vainberg [27] in local energy norms for initial data with compact support. The results were extended to general hyperbolic partial differential equations by Vainberg in [28]. The decay in the L^p norms for wave and Klein-Gordon equations was obtained in [3, 7, 13, 19, 31, 32].

However, applications to asymptotic stability of solutions to the nonlinear equations also require an exact characterization of the decay for the corresponding linearized equations in weighted norms (see e.g. [4, 5, 6, 24]).

The decay of type (1.6) in weighted norms has been established first by Jensen and Kato [11] for the Schrödinger equation in the dimension $n = 3$. The result has been extended to all other dimensions by Jensen and Nenciu [9, 10, 12], and to more general PDEs of the Schrödinger type by Murata [20]. The survey of the results can be found in [26].

For free wave equations corresponding to $m = 0$, some estimates in weighted L^p-norms have been established in [2]. The Strichartz weighted estimates for the perturbed Klein-Gordon equations were established in [16].

For the free 3D Klein-Gordon equation, the decay (1.6) in the weighted energy norms has been proved first in [8, Lemma 18.2]. However, for the perturbed relativistic equations, the decay was an open problem. The problem was that the Jensen-Kato approach is not applicable directly to the relativistic equations. The difference reflects distinct character of wave propagation in the relativistic and nonrelativistic equations (see the discussion in [15, Introduction]).

In [15] the decay of type (1.6) in the weighted energy norms has been proved for the Klein-Gordon equation in the dimension $n = 3$. The approach develops the Jensen-Kato techniques to make it applicable to the reativistic equations. Namely, the decay of the low energy component of the solution follows by the Jensen-Kato techniques while the decay for the high energy component requires novel robust ideas. This problem has been resolved with a modified approach based on the Born series and convolution.

Here we extend our approach [15] to the dimension $n = 1$. The extension is not straightforward since the decay (1.6) violates for the free 1D Klein-Gordon equation corresponding to $V(x) = 0$ when the solutions decay slow, like $\sim t^{-1/2}$. Hence, the decay (1.6) cannot be deduced by pertrubation arguments from the corresponding estimate for the free equation. This difficulty is well known, and it hindered the study of the asymptotic stability of many important 1D problems. The slow decay is caused by the "zero resonance function" $\psi(x) = \text{const}$ corresponding to the end point $\lambda = 0$ of the continuous spectrum of the 1D Schrödinger operator $-d^2/dx^2$.

Main idea of our approach to $n = 1$ is a spectral analysis of the "bad" term, with the slow decay $\sim t^{-1/2}$. Namely, we show that the bad term does not contribute to the high energy component. Therefore, the decay $\sim t^{-3/2}$ for the high energy component follows by our approach [15]. On the other hand, for the low energy component, the decay $\sim t^{-3/2}$ holds for the "generic" potentials by methods [11, 20].

Our paper is organized as follows. In Section 2 we obtain the time decay for the solution to the free Klein-Gordon equation and state the spectral properties of the free resolvent which follow from the corresponding known properties of the free Schrödinger resolvent. In Section 3 we obtain spectral properties of the perturbed resolvent and prove the decay (1.6). In Section 4 we apply the obtained decay to the asymptotic completeness.

2 Free Klein-Gordon equation

First, we consider the free Klein-Gordon equation:

$$\ddot{\psi}(x,t)=\psi''(x,t)-m^2\psi(x,t),\quad x\in\mathbb{R},\quad t\in\mathbb{R} \tag{2.1}$$

In vectorial form equation (2.1) reads

$$i\dot{\Psi}(t)=\mathcal{H}_0\Psi(t), \tag{2.2}$$

where

$$\Psi(t)=\begin{pmatrix}\psi(t)\\ \dot{\psi}(t)\end{pmatrix},\quad \mathcal{H}_0=\begin{pmatrix}0 & i\\ i(\frac{d^2}{dx^2}-m^2) & 0\end{pmatrix} \tag{2.3}$$

2.1 Spectral properties

We state spectral properties of the free Klein-Gordon dynamical group $\mathcal{G}(t)$ applying known results of [1, 20] which concern the corresponding spectral properties of the free Schrödinger dynamical group. For $t>0$ and $\Psi_0=\Psi(0)\in\mathcal{F}_0$, the soluton $\Psi(t)$ to the free Klein-Gordon equation (2.2) admits the spectral Fourier-Laplace representation

$$\theta(t)\Psi(t)=\frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i(\omega+i\varepsilon)t}\mathcal{R}_0(\omega+i\varepsilon)\Psi_0\,d\omega,\quad t\in\mathbb{R} \tag{2.4}$$

with any $\varepsilon>0$ where $\theta(t)$ is the Heaviside function, $\mathcal{R}_0(\omega)=(\mathcal{H}_0-\omega)^{-1}$ for $\omega\in\mathbb{C}^+:=\{\mathrm{Im}\,\omega>0\}$ is the resolvent of the operator $\mathcal{H}_0$. The representation follows from the stationary equation $\omega\tilde{\Psi}^+(\omega)=\mathcal{H}_0\tilde{\Psi}^+(\omega)+i\Psi_0$ for the Fourier-Laplace transform $\tilde{\Psi}^+(\omega):=\int\limits_{\mathbb{R}}\theta(t)e^{i\omega t}\Psi(t)dt$, $\omega\in\mathbb{C}^+$. The solution $\Psi(t)$ is continuous bounded function of $t\in\mathbb{R}$ with the values in $\mathcal{F}_0$ by the energy conservation for the free Klein-Gordon equation (2.2). Hence, $\tilde{\Psi}^+(\omega)=-i\mathcal{R}_0(\omega)\Psi_0$ is analytic function of $\omega\in\mathbb{C}^+$ with the values in $\mathcal{F}_0$, and bounded for $\omega\in\mathbb{R}+i\varepsilon$. Therefore, the integral (2.4) converges in the sense of distributions of $t\in\mathbb{R}$ with the values in $\mathcal{F}_0$. Similarly to (2.4),

$$\theta(-t)\Psi(t)=-\frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i(\omega-i\varepsilon)t}\mathcal{R}_0(\omega-i\varepsilon)\Psi_0\,d\omega,\quad t\in\mathbb{R} \tag{2.5}$$

The resolvent $\mathcal{R}_0(\omega)$ can be expressed in terms of the resolvent $R_0(\zeta)=(-\frac{d^2}{dx^2}-\zeta)^{-1}$ of the free Schrödinger operator

$$\mathcal{R}_0(\omega)=\begin{pmatrix}\omega R_0(\omega^2-m^2) & iR_0(\omega^2-m^2)\\ -i(1+\omega^2R_0(\omega^2-m^2)) & \omega R_0(\omega^2-m^2)\end{pmatrix} \tag{2.6}$$

The free Schrödinger resolvent $R_0(\zeta)$ is an integral operator with the integral kernel

$$R_0(\zeta,x-y)=-\frac{\exp(i\sqrt{\zeta}|x-y|)}{2i\sqrt{\zeta}},\quad \zeta\in\mathbb{C}^+,\quad \mathrm{Im}\,\zeta^{1/2}>0 \tag{2.7}$$

Definition 2.1. *Denote by $\mathcal{L}(B_1,B_2)$ the Banach space of bounded linear operators from a Banach space B_1 to a Banach space B_2.*

The explicit formula (2.7) implies the properties of $R_0(\zeta)$ which are obtained in [1, 20]:
i) $R_0(\zeta)$ is strongly analytic function of $\zeta\in\mathbb{C}\setminus[0,\infty)$ with the values in $\mathcal{L}(H^{-1}_0,H^1_0)$;
ii) For $\zeta>0$, the convergence holds $R_0(\zeta\pm i\varepsilon)\to R_0(\zeta\pm i0)$ as $\varepsilon\to 0+$ in $\mathcal{L}(H^{-1}_\sigma,H^1_{-\sigma})$ with $\sigma>1/2$, uniformly in $\zeta\ge r$ for any $r>0$.
iii) For any $M\ge 0$ the following asymptotic expansion holds

$$R_0(\zeta)=\sum_{k=-1}^{M}A_k\zeta^{k/2}+\mathcal{O}(\zeta^{(M+1)/2}),\quad \zeta\to 0,\quad \zeta\in\mathbb{C}\setminus[0,\infty) \tag{2.8}$$

in the norm of $\mathcal{L}(H^{-1}_\sigma;H^1_{-\sigma})$ with $\sigma>3/2+M+1$. Here

$$A_{-1}=\mathrm{Op}\Big[\frac{i}{2}\Big],\quad A_0=\mathrm{Op}\Big[-\frac{1}{2}|x-y|\Big], \tag{2.9}$$

and $A_k\in\mathcal{L}(H^{-1}_\sigma;H^1_{-\sigma})$ with $\sigma>3/2+k$ for $k=-1,0,1,....$
iv) The asymptotics (2.8) can be differentiated $M+2$ times: for $1\le r\le M+2$,

$$\partial^r_\zeta R_0(\zeta)=\partial^r_\zeta\Big(\sum_{k=-1}^{M}A_k\zeta^{k/2}\Big)+\mathcal{O}(\zeta^{\frac{M+1}{2}-r}),\quad \zeta\to 0,\quad \zeta\in\mathbb{C}\setminus[0,\infty) \tag{2.10}$$

in the norm of $\mathcal{L}(H^{-1}_\sigma;H^1_{-\sigma})$ with $\sigma>3/2+M+1$.

Let us denote $\Gamma:=(-\infty,-m)\cup(m,\infty)$, and let $\mathcal{A}^{\pm}_{-1}$ be the operator with the integral kernel

$$\mathcal{A}^{\pm}_{-1}(x-y)=\frac{i}{2\sqrt{\pm 2m}}\begin{pmatrix}\pm m & i\\ -im^2 & \pm m\end{pmatrix}. \tag{2.11}$$

Then the properties i) – iv) and (2.6) imply the following lemma.

Lemma 2.2. *i) The resolvent $\mathcal{R}_0(\omega)$ is strongly analytic function of $\omega\in\mathbb{C}\setminus\overline{\Gamma}$ with the values in $\mathcal{L}(\mathcal{F}_0,\mathcal{F}_0)$.*
ii) For $\omega\in\Gamma$, the convergence holds $\mathcal{R}_0(\omega\pm i\varepsilon)\to\mathcal{R}_0(\omega\pm i0)$ as $\varepsilon\to 0+$ in $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ with $\sigma>1/2$, uniformly in $|\omega|\ge m+r$ for any $r>0$.
iii) For any $M\ge 0$, the following asymptotics hold

$$\mathcal{R}_0(\omega)=\sum_{k=-1}^{M}(\pm\omega-m)^{k/2}\mathcal{A}^{\pm}_k+\mathcal{O}(|\omega\mp m|^{(M+1)/2}),\quad \omega\to\pm m,\quad \omega\in\mathbb{C}\setminus\overline{\Gamma} \tag{2.12}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma;\mathcal{F}_{-\sigma})$ with $\sigma>3/2+M+1$. Here $\mathcal{A}^{\pm}_k\in\mathcal{L}(\mathcal{F}_\sigma;\mathcal{F}_{-\sigma})$ for $k=-1,0,1,...$ and $\sigma>3/2+k$;
iv) The asymptotics (2.12) can be differentiated $M+1$ times: for $1\le r\le M+1$,

$$\partial^r_\omega\mathcal{R}_0(\omega)=\partial^r_\omega\Big(\sum_{k=-1}^{M}(\pm\omega-m)^{k/2}\mathcal{A}^{\pm}_k\Big)+\mathcal{O}(|\omega\mp m|^{\frac{M+1}{2}-r}),\quad \omega\to\pm m,\quad \omega\in\mathbb{C}\setminus\overline{\Gamma} \tag{2.13}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ with $\sigma>3/2+M+1$.

Finally, we state the asymptotics of $\mathcal{R}_0(\omega)$ for large ω which follow from known Agmon-Jensen-Kato decay [1, (A.2')] and [11, Theorem 8.1] of the resolvent R_0 (see also [15]):

Lemma 2.3. *For any $\rho > 0$ the following bounds hold for $m = 0, 1$ and $l = -1, 0, 1$,*

$$\|R_0^{(k)}(\zeta)\|_{\mathcal{L}(H^m_\sigma, H^{m+l}_{-\sigma})} \le C(\rho, k)|\zeta|^{-\frac{1-l+k}{2}}, \qquad \zeta \in \mathbb{C} \setminus (0, \infty), \qquad |\zeta| \ge \rho \tag{2.14}$$

with $\sigma > 1/2 + k$ for any $k = 0, 1, 2, \ldots$.

Then for $\mathcal{R}_0(\omega)$ we obtain

Corollary 2.4. *For any $\rho > 0$ the bounds hold*

$$\|\mathcal{R}_0^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})} \le C(\rho, k) < \infty, \qquad \omega \in \mathbb{C} \setminus \Gamma, \qquad |\omega \mp m| \ge \rho \tag{2.15}$$

with $\sigma > 1/2 + k$ for $k = 0, 1, 2, \ldots$.

Proof. The bounds follow from representation (2.6) for $\mathcal{R}_0(\omega)$ and asymptotics (2.14) for $R_0(\zeta)$ with $\zeta = \omega^2 - m^2$. □

Corollary 2.5. *For $t \in \mathbb{R}$ and $\Psi_0 \in \mathcal{F}_\sigma$ with $\sigma > 1/2$, the group $\mathcal{G}(t)$ admits the integral representation*

$$\mathcal{G}(t)\Psi_0 = \frac{1}{2\pi i} \int_\Gamma e^{-i\omega t} \Big[\mathcal{R}_0(\omega + i0) - \mathcal{R}_0(\omega - i0)\Big] \Psi_0 \, d\omega \tag{2.16}$$

where the integral converges in the sense of distributions of $t \in \mathbb{R}$ with the values in $\mathcal{F}_{-\sigma}$.

Proof. Summing up the representations (2.4) and (2.5), and sending $\varepsilon \to 0+$, we obtain (2.16) by the Cauchy theorem, Lemma 2.2 and Corollary 2.4. □

2.2 Time decay

The estimates (2.15) do not allow obtain the decay of $\mathcal{G}(t)$ by partial integration in (2.16). We deduce the decay from explicit formulas. The matrix kernel of the dynamical group $\mathcal{G}(t)$ can be written as $\mathcal{G}(t, x - y)$, where

$$\mathcal{G}(t, z) = \begin{pmatrix} \dot{G}(t, z) & G(t, z) \\ \ddot{G}(t, z) & \dot{G}(t, z) \end{pmatrix}, \qquad z \in \mathbb{R} \tag{2.17}$$

and

$$G(t, z) = \frac{1}{2}\theta(t - |z|) J_0(m\sqrt{t^2 - z^2}) \tag{2.18}$$

where J_0 is the Bessel function, and θ is the Heaviside function. Well known asymptotics of the Bessel function imply that for any $z \in \mathbb{R}$,

$$\mathcal{G}(t, z) = \mathcal{G}_0(t, z) + \mathcal{G}_r(t, z), \quad \mathcal{G}_r(t, z) = \mathcal{O}(t^{-3/2}), \quad t \to \infty$$

and $\mathcal{G}_0(t, z)$ stands for the matrix kernel

$$\mathcal{G}_0(t, z) := \frac{1}{\sqrt{2m\pi t}} \begin{pmatrix} -m\sin(mt - \frac{\pi}{4}) & \cos(mt - \frac{\pi}{4}) \\ & \\ -m^2\cos(mt - \frac{\pi}{4}) & -m\sin(mt - \frac{\pi}{4}) \end{pmatrix} = \mathcal{O}(t^{-1/2}), \quad t \to \infty \tag{2.19}$$

Therefore, the free Klein-Gordon group $\mathcal{G}(t)$ decays like $t^{-1/2}$ that does not correspond to (1.6). On the other hand, $\mathcal{G}_0$ is only term responsible for the slow decay. More exactly, in the next section we will prove the following basic proposition

Proposition 2.6. *For the operator $\mathcal{G}_r(t)$ with the kernel $\mathcal{G}_r(t, x-y)$, the following asymptotics holds*

$$\mathcal{G}_r(t) = \mathcal{O}(t^{-3/2}), \quad t \to \infty \tag{2.20}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$ with $\sigma > 5/2$.

The following key observation is that the "bad term" $\mathcal{G}_0(t, z)$ does not contribute to the high energy component of the total group $\mathcal{G}(t)$ since (2.19) contains just two frequencies $\pm m$ which are the end points of the continuous spectrum. This suggests that the high energy component of the group $\mathcal{G}(t)$ decays like $t^{-3/2}$.

More precisely, let us introduce the following *low energy* and *high energy* components of $\mathcal{G}(t)$:

$$\mathcal{G}_l(t) = \frac{1}{2\pi i}\int\limits_\Gamma e^{-i\omega t} l(\omega)\Big[\mathcal{R}_0(\omega + i0) - \mathcal{R}_0(\omega - i0)\Big]\, d\omega \tag{2.21}$$

$$\mathcal{G}_h(t) = \frac{1}{2\pi i}\int\limits_\Gamma e^{-i\omega t} h(\omega)\Big[\mathcal{R}_0(\omega + i0) - \mathcal{R}_0(\omega - i0)\Big]\, d\omega \tag{2.22}$$

where $l(\omega) \in C_0^\infty(\mathbb{R})$ is an even function, supp $l \in [-m-2\varepsilon, m+2\varepsilon]$, $l(\omega) = 1$ if $|\omega| \le m+\varepsilon$ with an $\varepsilon > 0$, and $h(\omega) = 1 - l(\omega)$.

Theorem 2.7. *Let $\sigma > 5/2$. Then the following asymptotics hold*

$$\mathcal{G}_h(t) = \mathcal{O}(t^{-3/2}), \quad t \to \infty \tag{2.23}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$.

Proof. We deduce asymptotics (2.23) from Proposition 2.6.
Step i) Let $\Psi(0) \in \mathcal{F}_\sigma$. Denote

$$\Psi^+(t) = \theta(t)\mathcal{G}(t)\Psi(0),\ \Psi_0^+(t) = \theta(t)\mathcal{G}_0(t)\Psi(0),\ \Psi_h^+(t) = \theta(t)\mathcal{G}_h(t)\Psi(0),\ \Psi_r^+(t) = \theta(t)\mathcal{G}_r(t)\Psi(0)$$

Then

$$\begin{aligned}
\Psi_h^+(t) &= \frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i\omega t} h(\omega)\mathcal{R}_0(\omega + i0)\Psi(0)d\omega \\
&= \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-i\omega t} h(\omega)\tilde\Psi^+(\omega)d\omega = \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-i\omega t} h(\omega)\Big[\tilde\Psi_0^+(\omega) + \tilde\Psi_r^+(\omega)\Big]\, d\omega \\
&= \Psi_r^+(t) + \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-i\omega t} h(\omega)\tilde\Psi_0^+(\omega)d\omega - \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-i\omega t} l(\omega)\tilde\Psi_r^+(\omega)d\omega
\end{aligned} \tag{2.24}$$

where the first term $\Psi_r^+(t)$ decays like (2.23) by (2.20).
Step ii) Let us consider the second summand in the RHS of (2.24). By (2.19) the matrix function $\tilde\Psi_0^+(\omega)$ is a smooth function for $|\omega| > m+\varepsilon$, and $\partial_\omega^k \tilde\Psi_0^+(\omega) = \mathcal{O}(\omega^{-1/2-k})$, $k = 0, 1, 2...$, $\omega \to \infty$. Hence partial integration implies that

$$\Big\| \int\limits_{\mathbb{R}} e^{-i\omega t} h(\omega)\tilde\Psi_0^+(\omega)d\omega \Big\|_{\mathcal{F}_{-\sigma}} = \mathcal{O}(t^{-N}), \quad \forall N \in \mathbb{N} \tag{2.25}$$

Step iii) Finally, let us consider the third summand in the RHS of (2.24).

$$\frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t} l(\omega)\tilde{\Psi}_r^+(\omega)d\omega = [L\star\Psi_r^+](t) = \mathcal{O}(t^{-3/2}), \quad l=\tilde{L} \tag{2.26}$$

in the norm of $\mathcal{F}_{-\sigma}$, since $L(t)=\mathcal{O}(t^{-N})$, $t\to\infty$ for any $N\in\mathbb{N}$, and $\|\Psi_r(t)\|_{\mathcal{F}_{-\sigma}}=\mathcal{O}(t^{-3/2})$ by (2.20). Finally, (2.24)- (2.26) imply (2.23). □

2.3 Proof of Proposition 2.6

For a fixed $0<\varepsilon<1$ we split the initial function $\Psi_0\in\mathcal{F}_\sigma$ in two terms, $\Psi_0=\Psi_{0,t}'+\Psi_{0,t}''$ such that

$$\|\Psi_{0,t}'\|_{\mathcal{F}_\sigma}+\|\Psi_{0,t}''\|_{\mathcal{F}_\sigma}\le C\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t\ge 1 \tag{2.27}$$

and

$$\Psi_{0,t}'(x)=0 \quad \text{for} \quad |x|>\frac{\varepsilon t}{2}, \quad \text{and} \quad \Psi_{0,t}''(x)=0 \quad \text{for} \quad |x|<\frac{\varepsilon t}{4} \tag{2.28}$$

We estimate $\mathcal{G}_r(t)\Psi_{0,t}'$ and $\mathcal{G}_r(t)\Psi_{0,t}''$ separately.

Step i) Let us consider $\mathcal{G}_r(t)\Psi_{0,t}''=\mathcal{G}(t)\Psi_{0,t}''-\mathcal{G}_0(t)\Psi_{0,t}''$. First we estimate $\mathcal{G}(t)\Psi_{0,t}''$ using energy conservation for the Klein-Gordon equation, and properties (2.28) and (2.27):

$$\|\mathcal{G}(t)\Psi_{0,t}''\|_{\mathcal{F}_{-\sigma}}\le\|\mathcal{G}(t)\Psi_{0,t}''\|_{\mathcal{F}_0}\le C\|\Psi_{0,t}''\|_{\mathcal{F}_0}\le C(\varepsilon)t^{-\sigma}\|\Psi_{0,t}''\|_{\mathcal{F}_\sigma}\le C_1(\varepsilon)t^{-5/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t\ge 1 \tag{2.29}$$

since $\sigma>5/2$. Second we estimate $\mathcal{G}_0(t)\Psi_{0,t}''$. By (2.19) we get $|\mathcal{G}_0(t)|\le C/\sqrt{t}$ for $t\ge 1$. Hence, for the second component $\pi_{0,t}''$ of vector-function $\Psi_{0,t}''$, we obtain by Cauchy inequality

$$\begin{aligned}|(\mathcal{G}_0^{i2}(t)\pi_{0,t}'')(y)| &= \Big|\mathcal{G}_0^{i2}(t)\int\pi_{0,t}''(x)dx\Big|\le\frac{C}{\sqrt{t}}\Big(\int|\pi_{0,t}''(x)|^2(1+x^2)^\sigma dx\Big)^{1/2}\Big(\int_{\varepsilon t/4}^{\infty}\frac{dx}{(1+x^2)^\sigma}\Big)^{1/2}\\ &\le \frac{C(\varepsilon)}{\sqrt{t}}t^{-\sigma+1/2}\|\pi_{0,t}''\|_{H_\sigma^0}\le C(\varepsilon)t^{-5/2}\|\pi_{0,t}''\|_{H_\sigma^0}, \quad i=1,2\end{aligned}$$

since $\sigma>5/2$. Hence

$$\|\mathcal{G}_0^{i2}(t)\pi_{0,t}''\|_{H_{-\sigma}^1}\le C(\varepsilon)t^{-5/2}\|\pi_{0,t}''\|_{H_\sigma^0}, \quad i=1,2$$

The first component of vector-function $\Psi_{0,t}''$ can be estimated similarly. Therefore,

$$\|\mathcal{G}_0(t)\Psi_{0,t}''\|_{\mathcal{F}_{-\sigma}}\le C(\varepsilon)t^{-5/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t\ge 1 \tag{2.30}$$

and (2.29)- (2.30) imply that

$$\|\mathcal{G}_r(t)\Psi_{0,t}''\|_{\mathcal{F}_{-\sigma}}\le C(\varepsilon)t^{-5/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t\ge 1 \tag{2.31}$$

Step ii) Now we consider $\mathcal{G}_r(t)\Psi_{0,t}'=\mathcal{G}(t)\Psi_{0,t}'-\mathcal{G}_0(t)\Psi_{0,t}'$. Let us split the operator $\mathcal{G}_r(t)$, for $t>1$, in two terms:

$$\mathcal{G}_r(t)=(1-\zeta)\mathcal{G}_r(t)+\zeta\mathcal{G}_r(t)$$

where ζ is the operator of multiplication by the function $\zeta(|x|/t)$ such that $\zeta=\zeta(s)\in C_0^\infty(\mathbb{R})$, $\zeta(s)=1$ for $|s|<\varepsilon/4$, $\zeta(s)=0$ for $|s|>\varepsilon/2$. Obviously, for any k, we have

$$|\partial_x^k\zeta(|x|/t)|\le C(\varepsilon)<\infty,\qquad t\ge 1$$

Furthermore, $1-\zeta(|x|/t)=0$ for $|x|<\varepsilon t/4$, then

$$||(1-\zeta)\mathcal{G}(t)\Psi'_{0,t}||_{\mathcal{F}_{-\sigma}}\le C(\varepsilon)t^{-\sigma}||(1-\zeta)\mathcal{G}(t)\Psi'_{0,t}||_{\mathcal{F}_0}\le C_1(\varepsilon)t^{-\sigma}||\mathcal{G}(t)\Psi'_{0,t}||_{\mathcal{F}_0}$$

Hence, by the energy conservation and (2.27), we obtain

$$||(1-\zeta)\mathcal{G}(t)\Psi'_{0,t}||_{\mathcal{F}_{-\sigma}}\le C_2(\varepsilon)t^{-\sigma}||\Psi'_{0,t}||_{\mathcal{F}_0}\le C_3(\varepsilon)t^{-\sigma}||\Psi'_{0,t}||_{\mathcal{F}_\sigma}\le C_4(\varepsilon)t^{-5/2}||\Psi_0||_{\mathcal{F}_\sigma},\qquad t\ge 1 \tag{2.32}$$

since $\sigma>5/2$.
Further, for the second component $\pi'_{0,t}$ of vector-function $\Psi'_{0,t}$, we obtain by Cauchy inequality

$$|(\mathcal{G}_0^{i2}(t)\pi'_{0,t})(y)|\le\frac{C}{\sqrt{t}}\|\pi'_{0,t}\|_{H^0_\sigma},\quad i=1,2$$

Hence,

$$\|(1-\zeta)\mathcal{G}_0^{i2}(t)\pi'_{0,t}\|_{H^1_{-\sigma}}\le\frac{C}{\sqrt{t}}\|\pi'_{0,t}\|_{H^0_\sigma}\Big(\int\limits_{\varepsilon t/4}^{\infty}\frac{dy}{(1+y^2)^\sigma}\Big)^{1/2}\le C(\varepsilon)t^{-5/2}\|\pi'_{0,t}\|_{H^0_\sigma},\quad i=1,2$$

The first component of vector-function $\Psi'_{0,t}$ can be estimate similarly. Therefore,

$$||(1-\zeta)\mathcal{G}_0(t)\Psi'_{0,t}||_{\mathcal{F}_{-\sigma}}\le C(\varepsilon)t^{-5/2}||\Psi_0||_{\mathcal{F}_\sigma},\qquad t\ge 1 \tag{2.33}$$

and then (2.32) and (2.33) imply

$$||(1-\zeta)\mathcal{G}_r(t)\Psi'_{0,t}||_{\mathcal{F}_{-\sigma}}\le C(\varepsilon)t^{-5/2}||\Psi_0||_{\mathcal{F}_\sigma},\qquad t\ge 1 \tag{2.34}$$

Step iii) Finally, let us estimate $\zeta\mathcal{G}_r(t)\Psi'_{0,t}$. Let $\chi_{\varepsilon t/2}$ be the characteristic function of the ball $|x|\le\varepsilon t/2$. We will use the same notation for the operator of multiplication by this characteristic function. By (2.28), we have

$$\zeta\mathcal{G}_r(t)\Psi'_{0,t}=\zeta\mathcal{G}_r(t)\chi_{\varepsilon t/2}\Psi'_{0,t} \tag{2.35}$$

The matrix kernel of the operator $\zeta\mathcal{G}_r(t)\chi_{\varepsilon t/2}$ is equal to

$$\mathcal{G}'_r(t,x-y)=\zeta(|x|/t)\mathcal{G}_r(t,x-y)\chi_{\varepsilon t/2}(y)$$

Well known asymptotics of the Bessel function imply the following lemma, which we prove in Appendix.

Lemma 2.8. *For any $\varepsilon\in(0,1)$ the bounds hold*

$$|\partial_z^k\mathcal{G}_r(t,z)|\le C(\varepsilon)|z|^2t^{-3/2},\quad |z|\le\varepsilon t,\quad t\ge 1,\quad k=0,1 \tag{2.36}$$

Since $\zeta(|x|/t) = 0$ for $|x| > \varepsilon t/2$ and $\chi_{\varepsilon t/2}(y) = 0$ for $|y| > \varepsilon t/2$, the estimate (2.36) implies that

$$|\partial_x^k \mathcal{G}_r'(t,z)| \le C(\varepsilon)|x-y|^2 t^{-3/2}, \quad k = 0,1, \quad t \ge 1 \tag{2.37}$$

The norm of the operator $\zeta \mathcal{G}_r(t)\chi_{\varepsilon t/2} : \mathcal{F}_\sigma \to \mathcal{F}_{-\sigma}$ is equivalent to the norm of the operator

$$\langle x\rangle^{-\sigma}\zeta \mathcal{G}_r(t)\chi_{\varepsilon t/2}(y)\langle y\rangle^{-\sigma} : \mathcal{F}_0 \to \mathcal{F}_0$$

The norm of the later operator does not exceed the sum in k, $k = 0,1$ of the norms of operators

$$\partial_x^k[\langle x\rangle^{-\sigma}\zeta \mathcal{G}_r(t)\chi_{\varepsilon t/2}(y)\langle y\rangle^{-\sigma}] : L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \tag{2.38}$$

The estimates (2.37) imply that operators (2.38) are Hilbert-Schmidt operators since $\sigma > 5/2$, and their Hilbert-Schmidt norms do not exceed $C(\varepsilon)t^{-3/2}$. Hence, (2.27) and (2.35) imply that

$$||\zeta \mathcal{G}_r(t)\Psi_{0,t}'||_{\mathcal{F}_{-\sigma}} \le C(\varepsilon)t^{-3/2}||\Psi_{0,t}'||_{\mathcal{F}_\sigma} \le C_1(\varepsilon)t^{-3/2}||\Psi_0||_{\mathcal{F}_\sigma}, \quad t \ge 1 \tag{2.39}$$

Finally, the estimates (2.39), (2.34) imply

$$||\mathcal{G}_r(t)\Psi_{0,t}'||_{\mathcal{F}_{-\sigma}} \le C(\varepsilon)t^{-3/2}||\Psi_0||_{\mathcal{F}_\sigma}, \quad t \ge 1 \tag{2.40}$$

Proposition 2.6 is proved.

3 Perturbed Klein-Gordon equation

To prove the long time decay for the perturbed Klein-Gordon equation, we first establish the spectral properties of the generator.

3.1 Spectral properties

According [20, formula (3.1)], let us introduce a generalized eigenspace $\mathbf{M}$ for the perturbed Schrödinger operator $H = -\Delta + V$:

$$\mathbf{M} = \{\psi \in H^1_{-1/2-0} : \ (1 + A_0 V)\psi \in \Re(A_{-1}),\ A_{-1}V\psi = 0\}$$

Where A_{-1} and A_0 are defined in (2.9), and $\Re(A_{-1})$ is the range of A_{-1}.

Below we assume that

$$\mathbf{M} = 0 \tag{3.1}$$

The condition (3.1) corresponds to the "nonsingular case" in [20, Section 7]. Let us show that the condition (3.1) holds for "generic" potentials satisfying

$$\int V(x)dx \ne 0 \tag{3.2}$$

Let us denote by $\mathbf{V}$ the set of all potentials satisfying the conditions (1.4) and (3.2).

Definition 3.1. *We say that property holds for "generic" potentials $V \in \mathbf{V}$, if for any fixed potential $V \in \mathbf{V}$, the property holds for kV with $k \in \mathbb{R}$ except for a discrete set of $k \in \mathbb{R}$.*

Lemma 3.2. *The condition* (3.1) *holds for generic potentials* $V \in \mathbf{V}$.

Proof. i) First we prove that for *generic* $V \in \mathbf{V}$ the operator $1 + A_0 V$ is invertible in $H^1_{-\sigma}$ with $\sigma \in (3/2, \beta - 3/2)$. Consider a family $T(k) = 1 + kA_0V$ with a real parameter k. Let $\mathcal{N}(k)$ denote the null space of $T(k)$. Then $\mathcal{N}(k) = \{0\}$ except for a discrete set Σ of k, because A_0V is a compact operator in $H^1_{-\sigma}$. Thus $T(k) = 1 + kA_0V$ is invertible for $k \in \mathbb{R} \setminus \Sigma$.
ii) Now we suppose that $k \in \mathbb{R} \setminus \Sigma$. Let us consider $\psi \in \mathbf{M}$. Recall that $A_{-1} = \mathrm{Op}\{\frac{i}{2}\}$, and then $\Re(A_{-1})$ consists of the constant functions. Hence, $\psi = T^{-1}(k)C$, where C is the constant function, and

$$A_{-1}kV\psi = kA_{-1}VT^{-1}(k)C = kC\frac{i}{2}\langle V, T^{-1}(k)1\rangle = 0 \qquad (3.3)$$

Let us show that $\langle V, T^{-1}(k)1\rangle \neq 0$ for $k \in \mathbb{R} \setminus \Sigma_1$, where Σ_1 is also a discrete set. Note that $T(0) = 1$, hence, $\langle V, T^{-1}(0)1\rangle = \langle V, 1\rangle \neq 0$ by (3.2). Therefore, the meromorphic function $k \mapsto \langle V, T^{-1}(k)1\rangle$ does not vanish identically, and thus $\langle V, T^{-1}(k)1\rangle \neq 0$ for all $k \in \mathbb{R}$ outside a discrete set. Now (3.3) implies that $C = 0$. Hence, $\psi = 0$. □

Denote by $R(\zeta) = (H - \zeta)^{-1}$, $\zeta \in \mathbb{C} \setminus \mathbb{R}$, the resolvent of the Schrödinger operator H.

Remark 3.3. i) By [20, Theorem 7.2], the condition (3.1) is equivalent to the boundedness of the resolvent $R(\zeta)$ at $\zeta = 0$ in the norm of $\mathcal{L}(H^{-1}_{\sigma}, H^1_{-\sigma})$ with a suitable $\sigma > 0$.
ii) $N(H) \subset \mathbf{M}$ where $N(H)$ is the zero eigenspace of the operator H. The imbedding is obtained in [20, Lemma 3.2]. The functions from $\mathbf{M} \setminus N(H)$ are called *zero resonance functions*. Hence, the condition (3.1) means that $\lambda = 0$ is neither eigenvalue nor resonance for the operator H.

Let us collect the properties of $R(\zeta)$ which are obtained in [1, 11, 20] under conditions (1.4) and (3.1). Note, that in [11] is considered 3D case, but corresponding properties can be proved in 1D case similarly.

R1. $R(\zeta)$ is strongly meromorphic function of $\zeta \in \mathbb{C} \setminus [0, \infty)$ with the values in $\mathcal{L}(H^{-1}_0, H^1_0)$; the poles of $R(\zeta)$ are located at a finite set of eigenvalues $\zeta_j < 0$, $j = 1, ..., N$, of the operator H with the corresponding eigenfunctions $\psi_j(x) \in H^2_s$ with any $s \in \mathbb{R}$.
R2. For $\zeta > 0$, the convergence holds $R(\zeta \pm i\varepsilon) \to R(\zeta \pm i0)$ as $\varepsilon \to 0+$ in $\mathcal{L}(H^{-1}_{\sigma}, H^1_{-\sigma})$ with $\sigma > 1/2$, uniformly in $\zeta \geq \rho$ for any $\rho > 0$ (cf. [11, Lemma 9.1]).
R3. The following expansion holds ([20, Theorem 7.5]):

$$R(\zeta) = B_0 + \mathcal{O}(\zeta^{1/2}), \quad \zeta \to 0, \quad \zeta \in \mathbb{C} \setminus (0, \infty) \qquad (3.4)$$

in the norm of $\mathcal{L}(H^{-1}_{\sigma}, H^1_{-\sigma})$ with $\sigma > 3/2$.
R4. The expression (3.4) can be differentiated two times (cf. [11, (6.14)]):

$$R^{(k)}(\zeta) = \mathcal{O}(\zeta^{1/2-k}), \quad \zeta \to 0, \quad \zeta \in \mathbb{C} \setminus (0, \infty), \quad k = 1, 2 \qquad (3.5)$$

in the norm of $\mathcal{L}(H^{-1}_{\sigma}, H^1_{-\sigma})$ with $\sigma > 5/2$.

Remark 3.4. The statement **R4** can be deduced from (2.8), (2.10), and the identities

$$R' = (1 - RV)R_0'(1 - VR), \quad R'' = \Big[(1 - RV)R_0'' - 2R'VR_0'\Big](1 - VR)$$

Further, the resolvent $\mathcal{R}(\omega) = (\mathcal{H} - \omega)^{-1}$ can be expressed similarly to (2.6):

$$\mathcal{R}(\omega) = \begin{pmatrix} \omega R(\omega^2 - m^2) & iR(\omega^2 - m^2) \\ -i(1 + \omega^2 R(\omega^2 - m^2)) & \omega R(\omega^2 - m^2) \end{pmatrix} \tag{3.6}$$

Hence, the properties **R1** – **R4** imply the corresponding properties of $\mathcal{R}(\omega)$:

Lemma 3.5. *Let the potential V satisfy conditions (1.4) and (3.1). Then*
i) $\mathcal{R}(\omega)$ is strongly meromorphic function of $\omega \in \mathbb{C} \setminus \overline{\Gamma}$ with the values in $\mathcal{L}(\mathcal{F}_0, \mathcal{F}_0)$;
ii) The poles of $\mathcal{R}(\omega)$ are located at a finite set

$$\Sigma = \{\omega_j^{\pm} = \pm\sqrt{m^2 + \zeta_j},\ j = 1, ..., N\}$$

of eigenvalues of the operator $\mathcal{H}$ with the corresponding eigenfunctions $\begin{pmatrix} \psi_j(x) \\ i\omega_j^{\pm}\psi_j(x) \end{pmatrix}$;
iii) For $\omega \in \Gamma$, the convergence holds $\mathcal{R}(\omega \pm i\varepsilon) \to \mathcal{R}(\omega \pm i0)$ as $\varepsilon \to 0+$ in $\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$ with $\sigma > 1/2$, uniformly in $|\omega| \geq m + r$ for any $r > 0$.
iv) The following asymptotics hold

$$\mathcal{R}(\omega) = \mathcal{R}_{\pm} + \mathcal{O}(|\omega \mp m|^{1/2}), \quad \omega \to \pm m, \quad \omega \in \mathbb{C} \setminus \Gamma \tag{3.7}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$ with $\sigma > 3/2$.
v) The asymptotics (3.7) can be differentiated two times:

$$\mathcal{R}'(\omega) = \mathcal{O}(|\omega \mp m|^{-1/2}), \quad \mathcal{R}''(\omega) = \mathcal{O}(|\omega \mp m|^{-3/2}), \quad \omega \to \pm m, \quad \omega \in \mathbb{C} \setminus \Gamma \tag{3.8}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$ with $\sigma > 5/2$.

Now we obtain the asymptotics of $\mathcal{R}(\omega)$ for large ω.

Lemma 3.6. *Let the potential V satisfy (1.4). Then for $s = 0, 1$ and $l = -1, 0, 1$ with $s + l \in \{0, 1\}$ we have*

$$\|R^{(k)}(\zeta)\|_{\mathcal{L}(H^s_\sigma, H^{s+l}_{-\sigma})} = \mathcal{O}(\zeta|^{-\frac{1-l+k}{2}}), \quad |\zeta| \to \infty, \quad \zeta \in \mathbb{C} \setminus [0, \infty) \tag{3.9}$$

with $\sigma > 1/2 + k$ for $k = 0, 1, 2$.

Proof. The lemma follows from Lemma 2.3 by the arguments from the proof of Theorem 9.2 in [11], where the bounds are proved for $s = 0$ and $l = 0, 1$. □

Hence (3.6) implies

Corollary 3.7. *Let the potential V satisfy (1.4). Then the bounds hold*

$$\|\mathcal{R}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})} = \mathcal{O}(1), \quad |\omega| \to \infty, \quad \omega \in \mathbb{C} \setminus \Gamma \tag{3.10}$$

with $\sigma > 1/2 + k$ for $k = 0, 1, 2$.

Finally, let us denote by $\mathcal{V}$ the matrix

$$\mathcal{V} = \begin{pmatrix} 0 & 0 \\ -iV & 0 \end{pmatrix} \tag{3.11}$$

Then the vectorial equation (1.2) reads

$$i\dot{\Psi}(t) = (\mathcal{H}_0 + \mathcal{V})\Psi(t)$$

The resolvents $\mathcal{R}(\omega)$ and $\mathcal{R}_0(\omega)$ are related by the Born perturbation series

$$\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega) + \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega), \quad \omega \in \mathbb{C} \setminus [\overline{\Gamma} \cup \Sigma] \tag{3.12}$$

which follows by iteration of $\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega)$. An important role in (3.12) plays the product $\mathcal{W}(\omega) := \mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}$. We obtain the asymptotics of $\mathcal{W}(\omega)$ for large ω.

Lemma 3.8. *Let $k = 0, 1, 2$, and the potential V satisfy (1.4) with $\beta > 1/2 + k + \sigma$ where $\sigma > 0$. Then the asymptotics hold*

$$\|\mathcal{W}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_{-\sigma},\mathcal{F}_\sigma)} = \mathcal{O}(|\omega|^{-2}), \quad |\omega| \to \infty, \quad \omega \in \mathbb{C} \setminus \Gamma \tag{3.13}$$

Proof. Bounds (3.13) follow from the algebraic structure of the matrix

$$\mathcal{W}^{(k)}(\omega) = \mathcal{V}\mathcal{R}_0^{(k)}(\omega)\mathcal{V} = \begin{pmatrix} 0 & 0 \\ -iV\partial_\omega^k R_0(\omega^2 - m^2)V & 0 \end{pmatrix} \tag{3.14}$$

since (2.14) with $s = 1$ and $l = -1$ implies that

$$\|VR_0^{(k)}(\zeta)Vf\|_{H^0_\sigma} \le C\|R_0^{(k)}(\zeta)Vf\|_{H^0_{\sigma-\beta}} = \mathcal{O}(|\zeta|^{-1-\frac{k}{2}})\|Vf\|_{H^1_{\beta-\sigma}} = \mathcal{O}(|\zeta|^{-1-\frac{k}{2}})\|f\|_{H^1_{-\sigma}}$$

for $1/2 + k < \beta - \sigma$. □

3.2 Time decay

In this section we combine the spectral properties of the perturbed resolvent and time decay for the unperturbed dynamics using the (finite) Born perturbation series. Our main result is the following.

Theorem 3.9. *Let conditions (1.4) and (3.1) hold. Then*

$$\|e^{-it\mathcal{H}} - \sum_{\omega_j \in \Sigma} e^{-i\omega_j t} P_j\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(|t|^{-3/2}), \quad t \to \pm\infty \tag{3.15}$$

with $\sigma > 5/2$,where P_j are the Riesz projectors onto the corresponding eigenspaces.

Proof. Lemma 3.5 iii) and asymptotics (3.7) and (3.10) with $k = 0$ imply similarly to (2.16), that

$$\Psi(t) - \sum_{\omega_j \in \Sigma} e^{-i\omega_j t} P_j \Psi_0 = \frac{1}{2\pi i} \int_\Gamma e^{-i\omega t} \Big[\mathcal{R}(\omega + i0) - \mathcal{R}(\omega - i0)\Big] \Psi_0 \, d\omega = \Psi_l(t) + \Psi_h(t) \tag{3.16}$$

where P_j stands for the corresponding Riesz projector

$$P_j\Psi_0 := -\frac{1}{2\pi i}\int_{|\omega-\omega_j|=\delta}\mathcal{R}(\omega)\Psi_0 d\omega$$

with a small $\delta > 0$, and

$$\Psi_l(t) = \frac{1}{2\pi i}\int\limits_\Gamma l(\omega)e^{-i\omega t}\Big[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\Big]\Psi_0\ d\omega \tag{3.17}$$

$$\Psi_h(t) = \frac{1}{2\pi i}\int\limits_\Gamma h(\omega)e^{-i\omega t}\Big[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\Big]\Psi_0\ d\omega \tag{3.18}$$

where $l(\omega)$ and $h(\omega)$ are defined in Section 2.2. Further we analyze $\Psi_l(t)$ and $\Psi_h(t)$ separately.

3.2.1 Time decay of $\Psi_l(t)$

We consider only the integral over $(m, m+2\varepsilon)$. The integral over $(-m-2\varepsilon, -m)$ is dealt with in the same way. We prove the desired decay of $\Psi_l(t)$ using a special case of Lemma 10.2 from [11]. Denote by $\mathbf{B}$ a Banach space with the norm $\|\cdot\|$.

Lemma 3.10. *Let $F \in C([m,a], \mathbf{B})$, satisfy*

$$F(m) = F(a) = 0, \qquad \|F''(\omega)\| = \mathcal{O}(|\omega-m|^{-3/2}), \quad \omega\to m \tag{3.19}$$

Then

$$\int\limits_m^a e^{-it\omega}F(\omega)d\omega = \mathcal{O}(t^{-3/2}), \quad t\to\infty \tag{3.20}$$

Due to (3.7)-(3.8), we can apply Lemma 3.10 with $F = l(\omega)\big(\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\big)$, $\mathbf{B} = \mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$, $a = m+2\varepsilon$ with a small $\varepsilon > 0$ and $\sigma > 5/2$, to get

$$\|\Psi_l(t)\|_{\mathcal{F}_{-\sigma}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}}, \quad t\in\mathbb{R}, \quad \sigma > 5/2 \tag{3.21}$$

3.2.2 Time decay of $\Psi_h(t)$

Let us substitute the series (3.12) into the spectral representation (3.18) for $\Psi_h(t)$:

$$\begin{aligned}
\Psi_h(t) &= \frac{1}{2\pi i}\int\limits_\Gamma e^{-i\omega t}h(\omega)\Big[\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\Big]\Psi_0\ d\omega \qquad (3.22)\\
&+ \frac{1}{2\pi i}\int\limits_\Gamma e^{-i\omega t}h(\omega)\Big[\mathcal{R}_0(\omega+i0)\mathcal{V}\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\mathcal{V}\mathcal{R}_0(\omega-i0)\Big]\Psi_0\ d\omega\\
&+ \frac{1}{2\pi i}\int\limits_\Gamma e^{-i\omega t}h(\omega)\Big[\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega+i0)-\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega-i0)\Big]\Psi_0\ d\omega\\
&= \Psi_{h1}(t)+\Psi_{h2}(t)+\Psi_{h3}(t), \qquad t\in\mathbb{R}
\end{aligned}$$

Further we analyze each term Ψ_{hk}, $k = 1, 2, 3$ separately.
Step i) The first term $\Psi_{h1}(t) = \mathcal{G}_h(t)\Psi_0$ by (2.22). Hence, Theorem 2.7 implies that

$$\|\Psi_{h1}(t)\|_{\mathcal{F}_{-\sigma}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}}, \quad t\in\mathbb{R}, \quad \sigma > 5/2 \tag{3.23}$$

Step ii) Now we consider the second term $\Psi_{h2}(t)$. Denote $h_1(\omega) = \sqrt{h(\omega)}$ and let

$$\Phi_{h1} = \frac{1}{2\pi i}\int\limits_\Gamma e^{-i\omega t}h_1(\omega)\Big[\mathcal{R}_0(\omega+i0) - \mathcal{R}_0(\omega-i0)\Big]\Psi_0\ d\omega$$

It is obvious that for Φ_{h1} the inequality (3.23) also holds. Namely,

$$\|\Phi_{h1}(t)\|_{\mathcal{F}_{-\sigma}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}}, \quad t\in\mathbb{R}, \quad \sigma > 5/2 \tag{3.24}$$

Now the second term $\Psi_{h2}(t)$ can be rewritten as a convolution.

Lemma 3.11. *The convolution representation holds*

$$\Psi_{h2}(t) = i\int\limits_0^t \mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\ d\tau, \qquad t\in\mathbb{R} \tag{3.25}$$

where the integral converges in $\mathcal{F}_{-\sigma}$ with $\sigma > 5/2$.

Proof. Then the term $\Psi_{h2}(t)$ can be rewritten as

$$\Psi_{h2}(t) = \frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i\omega t}h_1^2(\omega)\Big[\mathcal{R}_0(\omega+i0)\mathcal{V}\mathcal{R}_0(\omega+i0) - \mathcal{R}_0(\omega-i0)\mathcal{V}\mathcal{R}_0(\omega-i0)\Big]\Psi_0\ d\omega \tag{3.26}$$

Let us integrate the first term in the right hand side of (3.26), denoting

$$\mathcal{G}^{\pm}_{h1}(t) := \theta(\pm t)\mathcal{G}_{h1}(t), \quad \Phi^{\pm}_{h1}(t) := \theta(\pm t)\Phi_{h1}(t), \qquad t\in\mathbb{R}$$

We know that $h_1(\omega)\mathcal{R}_0(\omega+i0)\Psi_0 = i\tilde\Phi^+_{h1}(\omega)$, hence integrating the first term in the right hand side of (3.26), we obtain that

$$\begin{aligned}
\Psi^+_{h2}(t) &= \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-i\omega t}h_1(\omega)\mathcal{R}_0(\omega+i0)\mathcal{V}\tilde\Phi^+_{h1}(\omega)\ d\omega \\
&= \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-i\omega t}h_1(\omega)\mathcal{R}_0(\omega+i0)\mathcal{V}\Big[\int_{\mathbb{R}} e^{i\omega\tau}\Phi^+_{h1}(\tau)d\tau\Big]d\omega \\
&= \frac{1}{2\pi}(i\partial_t+i)^2\int\limits_{\mathbb{R}}\frac{e^{-i\omega t}}{(\omega+i)^2}\ h_1(\omega)\mathcal{R}_0(\omega+i0)\mathcal{V}\Big[\int_{\mathbb{R}} e^{i\omega\tau}\Phi^+_{h1}(\tau)d\tau\Big]d\omega
\end{aligned} \tag{3.27}$$

The last double integral converges in $\mathcal{F}_{-\sigma}$ with $\sigma > 5/2$ by (3.24), Lemma 2.2 ii), and (2.15) with $k = 0$. Hence, we can change the order of integration by the Fubini theorem. Then we obtain that

$$\Psi_{h2}^{+}(t) = i\int_{\mathbb{R}} \mathcal{G}_{h1}^{+}(t-\tau)\mathcal{V}\Phi_{h1}^{+}(\tau)d\tau = \begin{cases} i\int_0^t \mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)d\tau &, \ t>0 \\ 0 &, \ t<0 \end{cases} \tag{3.28}$$

since

$$\begin{aligned} \mathcal{G}_{h1}^{+}(t-\tau) &= \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega(t-\tau)}h_1(\omega)\mathcal{R}_0(\omega+i0)\,d\omega \\ &= \frac{1}{2\pi i}(i\partial_t+i)^2\int_{\mathbb{R}} \frac{e^{-i\omega(t-\tau)}}{(\omega+i)^2}\,h_1(\omega)\mathcal{R}_0(\omega+i0)\,d\omega \end{aligned}$$

by (2.4). Similarly, integrating the second term in the right hand side of (3.26), we obtain

$$\Psi_{h2}^{-}(t) = i\int_{\mathbb{R}} \mathcal{G}_{h1}^{-}(t-\tau)\mathcal{V}\Phi_{h1}^{-}(\tau)d\tau = \begin{cases} 0 &, \ t>0 \\ i\int_0^t \mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)d\tau &, \ t<0 \end{cases} \tag{3.29}$$

Now (3.25) follows since $\Psi_{h2}(t)$ is the sum of two expressions (3.28) and (3.29). □

Further, let us choose $\sigma > 5/2$ and $\sigma_1 \in (5/2,\ \min\{\sigma,\beta/2\})$. Then applying Theorem 2.7 with h_1 instead of h to the integrand in (3.25), we obtain that

$$\|\mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\|_{\mathcal{F}_{-\sigma}} \le \|\mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\|_{\mathcal{F}_{-\sigma_1}} \le \frac{C\|\mathcal{V}\Phi_{h1}(\tau)\|_{\mathcal{F}_{\sigma_1}}}{(1+|t-\tau|)^{3/2}}$$

$$\le \frac{C\|\Phi_{h1}(\tau)\|_{\mathcal{F}_{-\sigma_1}}}{(1+|t-\tau|)^{3/2}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_{\sigma_1}}}{(1+|t-\tau|)^{3/2}(1+|\tau|)^{3/2}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_{\sigma}}}{(1+|t-\tau|)^{3/2}(1+|\tau|)^{3/2}}$$

Therefore, integrating here in τ, we obtain by (3.25) that

$$\|\Psi_{h2}(t)\|_{\mathcal{F}_{-\sigma}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_{\sigma}}}{(1+|t|)^{3/2}}, \quad t\in\mathbb{R}, \quad \sigma > 5/2 \tag{3.30}$$

Step iii) Finally, let us rewrite the last term $\Psi_{h3}(t)$ as

$$\Psi_{h3}(t) = \frac{1}{2\pi i}\int_{\Gamma} e^{-i\omega t}h(\omega)\mathcal{N}(\omega)\Psi_0\,d\omega,$$

where $\mathcal{N}(\omega) := \mathcal{M}(\omega+i0) - \mathcal{M}(\omega-i0)$ for $\omega\in\Gamma$, and

$$\mathcal{M}(\omega) := \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega) = \mathcal{R}_0(\omega)\mathcal{W}(\omega)\mathcal{R}(\omega), \qquad \omega\in\mathbb{C}\setminus[\overline{\Gamma}\cup\Sigma] \tag{3.31}$$

Now we obtain the asymptotics of $\mathcal{N}(\omega)$ and its derivatives for large ω.

Lemma 3.12. *For $k = 0,1,2$ the asymptoticss hold*

$$\|\mathcal{M}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_{\sigma},\mathcal{F}_{-\sigma})} = \mathcal{O}(|\omega|^{-2}), \quad |\omega|\to\infty, \quad \omega\in\mathbb{C}\setminus\Gamma, \quad \sigma > 1/2+k \tag{3.32}$$

Proof. The asymptotics (3.32) follow from the asymptotics (2.15), (3.10) and (3.13) for $\mathcal{R}_0^{(k)}(\omega)$, $\mathcal{R}^{(k)}(\omega)$ and $\mathcal{W}^{(k)}(\omega)$. For example, let us consider the case $k=2$. Differentiating (3.31), we obtain

$$\mathcal{M}''=\mathcal{R}_0''\mathcal{W}\mathcal{R}+\mathcal{R}_0\mathcal{W}''\mathcal{R}+\mathcal{R}_0\mathcal{W}\mathcal{R}''+2\mathcal{R}_0'\mathcal{W}'\mathcal{R}+2\mathcal{R}_0'\mathcal{W}\mathcal{R}'+2\mathcal{R}_0\mathcal{W}'\mathcal{R}' \tag{3.33}$$

For a fixed $\sigma>5/2$, let us choose $\sigma'\in(5/2,\min\{\sigma,\beta-1/2\})$. Then for the first term in (3.33) we obtain by (3.10) and (3.13)

$$\begin{aligned}&\|\mathcal{R}_0''(\omega)\mathcal{W}(\omega)\mathcal{R}(\omega)f\|_{\mathcal{F}_{-\sigma}}\le\|\mathcal{R}_0''(\omega)\mathcal{W}(\omega)\mathcal{R}(\omega)f\|_{\mathcal{F}_{-\sigma'}}\le C\|\mathcal{W}(\omega)\mathcal{R}(\omega)f\|_{\mathcal{F}_{\sigma'}}\\ &=\mathcal{O}(|\omega|^{-2})\|\mathcal{R}(\omega)f\|_{\mathcal{F}_{-\sigma'}}=\mathcal{O}(|\omega|^{-2})\|f\|_{\mathcal{F}_{\sigma'}}=\mathcal{O}(|\omega|^{-2})\|f\|_{\mathcal{F}_{\sigma}},\quad |\omega|\to\infty,\quad \omega\in\mathbb{C}\setminus\Gamma\end{aligned}$$

Other terms can be estimated similarly choosing an appropriate value of σ'. Namely, $\sigma'\in(1/2,\min\{\sigma,\beta-5/2\})$ for the second term, $\sigma'\in(5/2,\min\{\sigma,\beta-1/2\})$ for the third, $\sigma'\in(3/2,\min\{\sigma,\beta-3/2\})$ for the forth and sixth terms, and $\sigma'\in(3/2,\min\{\sigma,\beta-1/2\})$ for the fifth term. ☐

Now we prove the decay of $\Psi_{h3}(t)$. By Lemma 3.12

$$(h\mathcal{N})''\in L^1((-\infty,-m-\varepsilon)\cup(m+\varepsilon,\infty);\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma}))$$

with $\sigma>5/2$ and $r=\varepsilon$. Hence, two times partial integration implies that

$$\|\Psi_{h3}(t)\|_{\mathcal{F}_{-\sigma}}\le\frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^2},\quad t\in\mathbb{R}$$

This completes the proof of Theorem 3.9. ☐

Corollary 3.13. *The asymptotics* (3.15) *imply* (1.6) *with the projector*

$$\mathcal{P}_c=1-\mathcal{P}_d,\quad \mathcal{P}_d=\sum_{\omega_j\in\Sigma}P_j \tag{3.34}$$

4 Application to the asymptotic completeness

We apply the obtained results to prove the asymptotic completeness which follows by standard Cook's argument. Let us note that the asymptotic completeness is proved in [17, 25, 30] by another methods for more general Klein-Gordon equations with an external Maxwell field. Our results give some refinement to the estimate of the remainder term.

Theorem 4.1. *Let conditions* (1.4) *and* (3.1) *hold. Then*
i) For solution to (1.2) *with any initial function* $\Psi(0)\in\mathcal{F}_0$*, the following long time asymptotics hold,*

$$\Psi(t)=\sum_{\omega_j\in\Sigma}e^{-i\omega_jt}\Psi_j+\mathcal{U}_0(t)\Phi_\pm+r_\pm(t) \tag{4.1}$$

where Ψ_j *are the corresponding eigenfunctions,* $\Phi_\pm\in\mathcal{F}_0$ *are the scattering states, and*

$$\|r_\pm(t)\|_{\mathcal{F}_0}\to0,\qquad t\to\pm\infty \tag{4.2}$$

ii) Furthermore,

$$\|r_\pm(t)\|_{\mathcal{F}_0}=\mathcal{O}(|t|^{-1/2}) \tag{4.3}$$

if $\Psi(0)\in\mathcal{F}_\sigma$ *with* $\sigma>5/2$.

Proof. First let us denote $\mathcal{X}_d := \mathcal{P}_d\mathcal{F}_0$ and $\mathcal{X}_c := \mathcal{P}_c\mathcal{F}_0$. For $\Psi(0) \in \mathcal{X}_d$ the asymptotics (4.1) obviously hold with $\Phi_\pm = 0$ and $r_\pm(t) = 0$. Hence, it remains to prove for $\Psi(0) \in \mathcal{X}_c$ the asymptotics

$$\Psi(t) = \mathcal{U}_0(t)\Phi_\pm + r_\pm(t) \tag{4.4}$$

with the remainder satisfying (4.2). Moreover, it suffices to prove the asymptotics (4.4), (4.3) for $\Psi(0) \in \mathcal{X}_c \cap \mathcal{F}_\sigma$ where $\sigma > 5/2$ since the space $\mathcal{F}_\sigma$ is dense in $\mathcal{X}_c$, while the group $\mathcal{U}_0(t)$ is unitary in $\mathcal{F}_0$ after a suitable modification of the norm. In this case Theorem 3.9 implies the decay

$$\|\Psi(t)\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-3/2}\|\Psi(0)\|_{\mathcal{F}_\sigma}, \quad t \to \pm\infty \tag{4.5}$$

The function $\Psi(t)$ satisfies the equation (1.2),

$$i\dot\Psi(t) = (\mathcal{H}_0 + \mathcal{V})\Psi(t)$$

Hence, the corresponding Duhamel equation reads

$$\Psi(t) = \mathcal{U}_0(t)\Psi(0) + \int_0^t \mathcal{U}_0(t-\tau)\mathcal{V}\Psi(\tau)d\tau, \qquad t \in \mathbb{R} \tag{4.6}$$

Finally, let us rewrite (4.6) as

$$\begin{aligned}\Psi(t) &= \mathcal{U}_0(t)\Big[\Psi(0) + \int_0^{\pm\infty} \mathcal{U}_0(-\tau)\mathcal{V}\Psi(\tau)d\tau\Big] \\ &\quad - \int_t^{\pm\infty} \mathcal{U}_0(t-\tau)\mathcal{V}\Psi(\tau)d\tau = \mathcal{U}_0(t)\Phi_\pm + r_\pm(t)\end{aligned} \tag{4.7}$$

It remains to prove that $\Phi_\pm \in \mathcal{F}_0$ and (4.3) holds. Let us consider the sign $''+''$ for the concreteness. The "unitarity" of $\mathcal{U}_0(t)$ in $\mathcal{F}_0$, the condition (1.4) and the decay (3.15) imply that for $\sigma' \in (5/2, \min\{\sigma, \beta\}]$

$$\begin{aligned}\int_0^\infty \|\mathcal{U}_0(-\tau)\mathcal{V}\Psi(\tau)\|_{\mathcal{F}_0}d\tau &\le C\int_0^\infty \|\mathcal{V}\Psi(\tau)\|_{\mathcal{F}_0}d\tau \le C_1\int_0^\infty \|\Psi(\tau)\|_{\mathcal{F}_{-\sigma'}}d\tau \\ &\le C_2\int_0^\infty (1+\tau)^{-3/2}\|\Psi(0)\|_{\mathcal{F}_\sigma}d\tau < \infty\end{aligned} \tag{4.8}$$

since $|V(x)| \le C\langle x\rangle^{-\beta} \le C\langle x\rangle^{-\sigma'}$. Hence, $\Phi_+ \in \mathcal{F}_0$. The estimate (4.3) follows similarly. □

A Appendix: Proof of Lemma 2.8

Here we prove Lemma 2.8. Differentiating (2.18), we obtain for $|z| < t$

$$\dot G(t,z) = -\frac{mt}{2\sqrt{t^2-z^2}}J_1(m\sqrt{t^2-z^2})$$

$$\ddot{G}(t,z)=\frac{m}{2}\frac{z^2}{\sqrt{(t^2-z^2)^3}}J_1(m\sqrt{t^2-z^2})+\frac{m^2t^2}{2(t^2-z^2)}J_2(m\sqrt{t^2-z^2})$$

The following asymptotics of Bessel functions are well-known ([29], p.195):

$$J_\nu(z)=\Big(\frac{2}{\pi z}\Big)^{1/2}\cos(z-\frac{\pi\nu}{2}-\frac{\pi}{4})+\mathcal{O}(z^{-3/2}),\quad z\to\infty.$$

Hence, (2.17) imply

$$\mathcal{G}(t,z)=\tilde{\mathcal{G}}_0(t,z)+\tilde{\mathcal{G}}_r(t,z)$$

where

$$\tilde{\mathcal{G}}_0(t,z):=\frac{\theta(t-|z|)}{\sqrt{2m\pi}}\left(\begin{array}{cc} -\dfrac{mt\sin(m\sqrt{t^2-z^2}-\frac{\pi}{4})}{\sqrt[4]{(t^2-z^2)^3}} & \dfrac{\cos(m\sqrt{t^2-z^2}-\frac{\pi}{4})}{\sqrt[4]{t^2-z^2}}\\[3ex] -\dfrac{m^2t^2\cos(m\sqrt{t^2-z^2}-\frac{\pi}{4})}{\sqrt[4]{(t^2-z^2)^5}} & -\dfrac{mt\sin(m\sqrt{t^2-z^2}-\frac{\pi}{4})}{\sqrt[4]{(t^2-z^2)^3}}\end{array}\right)$$

and for $\varepsilon\in(0,1)$ we have

$$|\partial_z^k\tilde{\mathcal{G}}_r(t,z)|\le C(\varepsilon)t^{-3/2},\quad |z|\le\varepsilon t,\quad k=0,1$$

It remains to prove the bounds of type (2.36) for the difference $Q(t,z)=\tilde{\mathcal{G}}_0(t,z)-\mathcal{G}_0(t,z)$. Let us consider the entry $Q^{12}(t,z)$. Applying the Lagrange formula, we obtain

$$|Q^{12}(t,z)|=\frac{1}{\sqrt{2\pi m}}\Big|\frac{\cos(m\sqrt{t^2-z^2}-\frac{\pi}{4})}{\sqrt[4]{t^2-z^2}}-\frac{\cos(mt-\frac{\pi}{4})}{\sqrt{t}}\Big|\le C(\varepsilon)z^2t^{-3/2},\quad |z|\le\varepsilon t \qquad \text{(A.1)}$$

Differentiating $Q^{12}(t,z)$, we obtain for $|z|\le\varepsilon t$

$$\partial_zQ^{12}(t,z)=\frac{z}{\sqrt{2\pi m}}\Big[\frac{1}{2\sqrt[4]{(t^2-z^2)^5}}\cos(m\sqrt{t^2-z^2}-\frac{\pi}{4})+\frac{m}{\sqrt[4]{(t^2-z^2)^3}}\sin(m\sqrt{t^2-z^2}-\frac{\pi}{4})\Big]$$

Hence,

$$|\partial_zQ^{12}(t,z)|\le C(\varepsilon)|z|\,t^{-3/2},\quad |z|\le\varepsilon t \qquad \text{(A.2)}$$

Other entries $Q^{ij}(t,z)$ also admit the estimates of type (A.1) and (A.2). Hence, the lemma follows since $\mathcal{G}_r(t)=\tilde{\mathcal{G}}_r(t)+Q(t,z)$.

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