On asymptotic stability of solitary waves for Schödinger equation coupled to nonlinear oscillator, II

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Abstract

The long-time asymptotics is analyzed for finite energy solutions of the 1D Schrödinger equation coupled to a nonlinear oscillator; mathematically the system under study is a nonlinear Schrödinger equation, whose nonlinear term includes a Dirac delta. The coupled system is invariant with respect to the phase rotation group $U(1)$. This article, which extends the results of a previous one, provides a proof of asymptotic stability of solitary wave solutions in the case that the linearization contains a single discrete oscillatory mode satisfying a non-degeneracy assumption of the type known as the Fermi Golden Rule.

[1] Supported partly by Alexander von Humboldt Research Award, by FWF grant P19138-N13, and DFG grant 436 RUS 113/929/0-1.

[2] Supported partly by RFBR grants 06-01-00096 and 07-01-00018a, and RFBR-DFG grant 08-01-91950-NNIOa.

[3] Partially supported by EPSRC grant A00133/01

1 Introduction and statement of results

1.1 Introduction

In this article we continue the study of large time asymptotics for a model $U(1)$-invariant nonlinear Schrödinger equation

$$i\dot{\psi}(x,t) = -\psi''(x,t) - \delta(x)F(\psi(0,t)), \quad x \in \mathbb{R}, \tag{1.1}$$

which was initiated in [1]. Here $\psi(x,t)$ is a continuous complex-valued wave function and F is a continuous function, the dots stand for the derivatives in t and the primes in x; all derivatives and the equation are understood in the distribution sense. Our main focus is on the role that certain solitary waves (also referred to as nonlinear bound states, or solitons) play in the description of the solution for large times. These solitary waves are solutions of the form $e^{i\omega t}\psi_\omega(x)$, where ψ_ω solves a nonlinear eigenvalue problem (1.10). In [1] the asymptotic stability of these solitary waves was proved under a condition on the nonlinearity which ensures that the linearization about the solitary wave consists entirely of continuous spectrum, except for the two dimensional generalized null space which is always present due to the $U(1)$ symmetry of the equation. In this article this result is extended to the case that the spectrum of the linearization includes an additional discrete component, which satisfies a non-degeneracy condition related to the Fermi Golden Rule. In order to explain these results fully we will introduce our conditions on the nonlinearity F, in the remainder of this section. In the following section we will discuss the basic properties of the solitary waves. We will then be able to state our main theorem precisely as theorem 1.3. For a more lengthy discussion of our motivation, and of previous results in the literature ([2, 6, 7, 8, 10, 11]) we refer the reader to the introduction of [1].

It will be convenient to rewrite (1.1) in real form: we identify a complex number $\psi = \psi_1 + i\psi_2$ with the real two-dimensional vector $(\psi_1, \psi_2) \in \mathbb{R}^2$ and rewrite (1.1) in the vectorial form

$$j\dot{\psi}(x,t) = -\psi''(x,t) - \delta(x)\mathbf{F}(\psi(0,t)), \tag{1.2}$$

where

$$j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.3}$$

and $\mathbf{F}(\psi) \in \mathbb{R}^2$ is the real vector version of $F(\psi) \in \mathbb{C}$. We assume that the oscillator force $\mathbf{F}$ admits a real-valued potential

$$\mathbf{F}(\psi) = -\nabla U(\psi), \quad \psi \in \mathbb{R}^2, \quad U \in C^2(\mathbb{R}^2). \tag{1.4}$$

Then (1.2) is formally a Hamiltonian system with Hamiltonian

$$\mathcal{H}(\psi) = \frac{1}{2}\int |\psi'|^2 dx + U(\psi(0)) \tag{1.5}$$

which is conserved for sufficiently regular finite energy solutions. We assume that the potential $U(\psi)$ satisfies the inequality

$$U(z) \geq A - B|z|^2 \quad \text{with some} \quad A \in \mathbb{R}, \quad B > 0. \tag{1.6}$$

We also assume that $U(\psi) = u(|\psi|^2)$ with $u \in C^2(\mathbb{R})$. Therefore, by (1.4),

$$F(\psi) = a(|\psi|^2)\psi, \quad \psi \in \mathbb{C}, \quad a \in C^1(\mathbb{R}), \tag{1.7}$$

where $a(|\psi|^2)$ is real.Then $F(e^{i\theta}\psi) = e^{i\theta}F(\psi), \quad \theta \in [0, 2\pi]$, and $e^{i\theta}\psi(x,t)$ is a solution to (1.1) if $\psi(x,t)$ is. Therefore, equation (1.1) is $U(1)$-invariant and the Nöther theorem implies the conservation of the following *charge*:

$$\mathcal{Q}(\psi) = \int |\psi|^2 dx = \text{const}\,. \tag{1.8}$$

Under these conditions the existence of global solutions to the Cauchy problem for (1.1) was proved in [4]. We work in the space $H^1(\mathbb{R}) = H^1$, the Sobolev space of complex valued measurable functions with $\int (|\psi'|^2 + |\psi|^2)dx < \infty$, and $C_b(\mathbb{R}, X)$ is the space of bounded continuous functions $\mathbb{R} \to X$ into a Banach space X.

Theorem 1.1 ([4]). *Under conditions (1.4), (1.6) and (1.7), the following statements hold.*
i) For any $\psi_0(x) \in H^1$ there exists a unique solution $\psi(t) = \psi(\,\cdot\,,t) \in C_b(\mathbb{R}, H^1)$ to the equation (1.1) with initial condition $\psi(x,0) = \psi_0(x)$.
ii) The charge $\mathcal{Q}(\psi(t))$ and Hamiltonian $\mathcal{H}(\psi(t))$ are constant along the solution.
iii) There exists $\Lambda(\psi_0) > 0$ such that the following a priori bound holds:

$$\sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^1} \le \Lambda(\psi_0) < \infty. \tag{1.9}$$

Next, in §1.2 we describe all nonzero solitary waves, and then formulate the main theorem in §1.3.

1.2 Solitary waves

Equation (1.1) admits finite energy solutions of type $\psi_\omega(x)e^{i\omega t}$, called *solitary waves* or *nonlinear eigenfunctions*. The frequency ω and the amplitude $\psi_\omega(x)$ solve the following *nonlinear eigenvalue problem*:

$$-\,\omega\psi_\omega(x) = -\psi''_\omega(x) - \delta(x)F(\psi_\omega(0)), \quad x \in \mathbb{R}. \tag{1.10}$$

It is straightforward to check (see [1]) that the set of all nonzero solitary waves consists of functions $\psi_\omega(x)e^{i\theta}$, $\psi_\omega(x) = C(\omega)e^{-\sqrt{\omega}|x|}$, $C > 0$, $\omega > 0$, where

$$\sqrt{\omega} = a(C^2)/2 > 0,$$

and where $\theta \in [0, 2\pi]$ is arbitrary. This condition means that C is restricted to lie in a set which, in the case of polynomial F, is a finite union of one-dimensional intervals. Notice that $C = 0$ corresponds to the zero function $\psi(x) = 0$ which is always a solitary wave as $F(0) = 0$, and for $\omega \le 0$ only the zero solitary wave exists. The real form of the solitary wave is $e^{j\theta}\Psi_\omega$ where $\Psi_\omega = (\psi_\omega(x), 0)$. We will also need the following lemma from [1]:

Lemma 1.2. *For $C > 0$, $a > 0$ we have*

$$\partial_\omega \int |\psi_\omega(x)|^2 dx > 0 \quad \textit{if} \;\; 0 < a' < a/C^2.$$

Linearization at the solitary wave $e^{j\theta}\Psi_\omega$ leads to the operator

$$\mathbf{B} = -\frac{d^2}{dx^2} + \omega - \delta(x)[a(C^2) + 2a'(C^2)C^2 P_1] = \begin{pmatrix} \mathbf{D}_1 & 0 \\ 0 & \mathbf{D}_2 \end{pmatrix}, \tag{1.11}$$

where P_1 is the projector in $\mathbb{R}^2$ acting as $\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \mapsto \begin{pmatrix} \chi_1 \\ 0 \end{pmatrix}$,

$$\mathbf{D}_1 = -\frac{d^2}{dx^2} + \omega - \delta(x)[a + 2a'C^2], \quad \mathbf{D}_2 = -\frac{d^2}{dx^2} + \omega - \delta(x)a.$$

(see [1]). Let $\mathbf{C} = j^{-1}\mathbf{B}$. The continuous spectrum of $\mathbf{C}$ coincides with $\mathcal{C}_+ \cup \mathcal{C}_-$ where $\mathcal{C}_+ = [i\omega, i\infty)$, and $\mathcal{C}_- = (-i\infty, -i\omega]$. The discrete spectrum depend is described in [1]. Zero is always present in the discrete spectrum on account of the circular symmetry of the problem, and there is a corresponding generalized eigenspace of dimension at least two. If $a' > a/C^2$ there is a positive eigenvalue corresponding to linear instability of the solitary wave, while for $a' < a/C^2$ the discrete spectrum consists either only of zero, or contains in addition two pure imaginary eigenvalues. The presence of such imaginary discrete spectrum corresponds to the possibility of a periodic pulsation of the solitary wave at the linearized level, a possibility which has to be taken care of in the proof of asymptotic stability.

1.3 Statement of the main theorem

Previously, in [1], we considered the case when

$$a' \in (-\infty, 0) \cup (0, \frac{a}{\sqrt{2}C^2}), \tag{1.12}$$

in which case the operator $\mathbf{C}$ has no discrete spectrum except zero. Under this condition we proved asymptotic stability for initial data close to a solitary wave both in the energy norm and in the weighted Banach norm, L^p_β, defined by,

$$\|f\|_{L^p_\beta} = \|(1 + |x|)^\beta f(x)\|_{L^p}. \tag{1.13}$$

In the present paper we extend this understanding to allow for the presence of additional discrete spectrum in the linearization: to be precise we will consider the case when

$$a' \in (\frac{a}{\sqrt{2}C^2}, \frac{a\sqrt{2}(1+\sqrt{3})}{4C^2}). \tag{1.14}$$

In this case, there are, in addition to zero, 2 simple pure imaginary eigenvalues $\pm i\mu$, which satisfy the property $2\mu > \omega$. If assumption (1.14) is true for a fixed value ω_0, it also true for values of ω in a small interval centered at ω_0. Let $u(x,\omega) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be the eigenvector of $\mathbf{C}$ associated to $i\mu$. We choose the function $u_1(x)$ to be real, in which case $u_2(x)$ is purely imaginary. Then $u^* := \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}$, is the eigenvector associated to $-i\mu$ (see appendix A). We consider the initial value $\psi(x,0) = \psi_0(x)$ to be of the form

$$\psi_0(x) = e^{j\theta_0}\Psi_{\omega_0}(x) + z_0 u(x,\omega_0) + \overline{z}_0 u^*(x,\omega_0) + f_0(x), \tag{1.15}$$

where f_0 belongs to the eigenspace associated to the continuous spectrum of $\mathbf{C}(\omega_0)$. We assume that z_0 and f_0 are sufficiently small, and also assume a non-degeneracy condition which we now explain. Let $\langle\cdot,\cdot\rangle$ denote the Hermitian scalar product in L^2 of $\mathbb{C}^2$-valued functions, and $(u,v)=u_1v_1+u_2v_2$ for $u,v\in\mathbb{C}^2$. Let $E_2[f,f]$ be the quadratic terms coming from the Taylor expansion of the nonlinearity:

$$E_2[f,f]=\delta(x)[a'(C^2)(f,f)\Psi_\omega+2a''(C^2)(\Psi_\omega,f)^2\Psi_\omega+2a'(C^2)(\Psi_\omega,f)f],\ f\in\mathbb{C}^2. \tag{1.16}$$

The non-degeneracy condition has the form

$$\langle E_2[u(\omega_0),u(\omega_0)],\tau_+(2i\mu_0)\rangle\neq 0, \tag{1.17}$$

where $\tau_+(2i\mu_0)$ is the eigenfunction associated to $2i\mu_0=2i\mu(\omega_0)$. This condition should be thought of as a nonlinear version of the Fermi Golden Rule of quantum mechanics ([9, Section XII.6] or [5]), and will be referred to simply as the Fermi Golden Rule; see [10, 11] for the its introduction into nonlinear evolution equations. In appendix E we express (1.17) in terms of C and $a(C^2)$, and hence show that *the Fermi Golden Rule holds generically for polynomial nonlinearity.*

Our main theorem is following:

Theorem 1.3. *Let conditions (1.4), (1.6) and (1.7) hold, $\beta>2$ and $\psi(x,t)\in C(\mathbb{R},H^1)$ be the solution to the equation (1.2) with initial value $\psi_0(x)=\psi(x,0)\in H^1\cap L^1_\beta$ of the form (1.15) which is close to a solitary wave $e^{j\theta_0}\Psi_{\omega_0}$:*

$$|z(0)|\le\varepsilon^{1/2},\quad \|f_0\|_{L^1_\beta}\le c\varepsilon^{3/2}. \tag{1.18}$$

Assume further that the spectral condition (1.14) and the non-degeneracy condition (1.17) hold for the solitary wave with $C=C(\omega_0)=C_0$. Then for ε sufficiently small the solution admits the following scattering asymptotics in $C_b(\mathbb{R})\cap L^2(\mathbb{R})$:

$$\psi(x,t)=e^{j\varphi_\pm(t)}\Psi_{\omega_\pm}(x)+e^{j^{-1}Lt}\Phi_\pm+O(t^{-\nu}),\quad t\to\pm\infty, \tag{1.19}$$

with some $\nu>0$, where $L=-\frac{\partial^2}{\partial x^2}$, $\Phi_\pm\in C_b(\mathbb{R})\cap L^2(\mathbb{R})$ are the corresponding asymptotic scattering states and $\varphi_\pm(t)=\omega_\pm t+p_\pm\log(1+k_\pm t)+\varkappa_\pm$, for some constants $\omega_\pm$, $p_\pm$, $k_\pm$, $\varkappa_\pm$.

The asymptotics (1.19) can be rewritten in terms of the original complex notation as:

$$\psi(x,t)=e^{i\varphi_\pm(t)}\psi_{\omega_\pm}(x)+W(t)\phi_\pm+O(t^{-\nu}),\quad t\to\pm\infty, \tag{1.20}$$

where $W(t)$ is the dynamical group of the free Schrödinger equation, and $\phi_\pm=\Phi^1_\pm+i\Phi^2_\pm$ ($\Phi^k_\pm$, $k=1,2$, being the components of the vector-function $\Phi_\pm$). Thus the main conclusion is that asymptotically the dynamics decomposes into a nonlinear bound state $e^{i\varphi_\pm(t)}\psi_{\omega_\pm}$ (undergoing uniform phase rotation, modulo the logarithmic phase shift in $\varphi_{\pm(t)}$) and a solution of the free Schrödiniger equation $W(t)\phi_\pm$.

The proof is divided into the following three main steps which are carried out, respectively, in §3, §4 and §5.

Step 1 To decompose the solution $\psi(t)$ according to the spectral decomposition of the operator $\mathbf{C}$ given in §2.1, and obtain a system of equations in §3.1 equivalent to (1.1). This system is then put into a canonical form, in which certain non-resonant terms are excluded. This is carried out in §3.4, the final form of the equations being given in §3.4.5.

Step 2 To use the time decay for the linearized evolution on the continuous spectral subspace established in §2.2 to prove asymptotic stability of the solitary waves in §4.6.

Step 3 Finally, to construct the wave operator and obtain the scattering asymptotics (1.19), in §5.2.

2 Linearization

In this section we summarize the spectral properties of the operator $\mathbf{C}$ which appears in the linearization of (1.2) about a soliton, and then give some estimates for the linearized evolution. The proof of these properties can be found in [1], with the exception of proposition 2.3 which is proved in appendix C.

2.1 Spectral properties of the linearization

The linearized equation reads

$$\dot{\chi}(x,t) = \mathbf{C}\chi(x,t), \qquad \mathbf{C} := j^{-1}\mathbf{B} = \begin{pmatrix} 0 & \mathbf{D}_2 \\ -\mathbf{D}_1 & 0 \end{pmatrix}. \tag{2.1}$$

Theorem 1.1 generalizes to the equation (2.1): the equation admits unique solution $\chi(x,t) \in C_b(\mathbb{R}, H^1)$ for every initial function $\chi(x,0) = \chi_0 \in H^1$. Denote by $e^{\mathbf{C}t}$ the dynamical group of equation (2.1) acting in the space H^1; for $T > 0$ there exists $c_T > 0$ such that

$$\|e^{\mathbf{C}t}\chi_0\|_{H^1} \le c_T \|\chi_0\|_{H^1}, \qquad |t| \le T. \tag{2.2}$$

The resolvent $\mathbf{R}(\lambda) := (\mathbf{C} - \lambda)^{-1}$ is an integral operator with matrix valued integral kernel

$$\mathbf{R}(\lambda, x, y) = \Gamma(\lambda, x, y) + P(\lambda, x, y), \tag{2.3}$$

where

$$\Gamma(\lambda,x,y) = \begin{pmatrix} \dfrac{1}{4k_+} & -\dfrac{1}{4k_-} \\ \dfrac{i}{4k_+} & \dfrac{i}{4k_-} \end{pmatrix} \begin{pmatrix} e^{ik_+|x-y|} - e^{ik_+(|x|+|y|)} & -i(e^{ik_+|x-y|} - e^{ik_+(|x|+|y|)}) \\ e^{ik_-|x-y|} - e^{ik_-(|x|+|y|)} & i(e^{ik_-|x-y|} - e^{ik_-(|x|+|y|)}) \end{pmatrix}, \tag{2.4}$$

$$P(\lambda,x,y) = \frac{1}{2D} \begin{pmatrix} e^{ik_+|x|} & e^{ik_-|x|} \\ ie^{ik_+|x|} & -ie^{ik_-|x|} \end{pmatrix} \begin{pmatrix} i\alpha - 2k_- & i\beta \\ -i\beta & -i\alpha + 2k_+ \end{pmatrix} \begin{pmatrix} e^{ik_+|y|} & -ie^{ik_+|y|} \\ e^{ik_-|y|} & ie^{ik_-|y|} \end{pmatrix}. \tag{2.5}$$

As explained already in §1.2, the continuous spectrum consists of $\mathcal{C}_+ \cup \mathcal{C}_-$, and correspondingly $k_\pm(\lambda) = \sqrt{-\omega \mp i\lambda}$ is (respectively) the square root defined with a cut in the complex λ plane so that $k_\pm(\lambda)$ is analytic on $\mathbb{C} \setminus \mathcal{C}_\pm$ and $\operatorname{Im} k_\pm(\lambda) > 0$ for $\lambda \in \mathbb{C} \setminus \mathcal{C}_\pm$. The constants α, β and $D = D(\lambda)$ are given by the formulas

$$\alpha = a + a'C^2, \ \beta = a'C^2, \ D = 2i\alpha(k_+ + k_-) - 4k_+k_- + \alpha^2 - \beta^2.$$

In addition to this continuous spectrum, there is discrete spectrum, which appears in this formalism as the set of poles of $\mathbf{R}(\lambda)$; these poles in turn correspond to the roots of the determinant $D(\lambda)$. From the analysis in [1] we know that if $a' \in (a/\sqrt{2}C^2, a/C^2)$, then the determinant has the following roots: $\lambda = 0$ with the multiplicity 2 and two pure imaginary roots

$$\lambda = \pm i\mu = \pm i\frac{\beta}{2}\sqrt{a^2 - \beta^2}, \quad \mu < \omega. \tag{2.6}$$

Note that spectral condition (1.14) is more restrictive. It implies in addition that $2\mu > \omega$, as can be verified by a simple computation.

The generalised null space X^0 of the non-self-adjoint operator $\mathbf{C}$ is two dimensional, is spanned by $j\Psi_\omega, \partial_\omega\Psi_\omega$, and

$$\mathbf{C}j\Psi_\omega = 0 \qquad \mathbf{C}\partial_\omega\Psi_\omega = j\Psi_\omega.$$

The symplectic form Ω for the vectors ψ and η is defined by

$$\Omega(\psi, \eta) = \langle \psi, j\eta \rangle \tag{2.7}$$

By Lemma 1.2

$$\Omega(j\Psi_\omega, \partial_\omega\Psi_\omega) = -\frac{1}{2}\partial_\omega \int |\psi_\omega|^2 dx \neq 0. \tag{2.8}$$

Hence, the symplectic form Ω is nondegenerate on X^0, i.e. X^0 is a symplectic subspace. There exists a symplectic projection operator $\mathbf{P}^0$ from $L^2(\mathbb{R})$ onto X^0 represented by the formula

$$\mathbf{P}^0\psi = \frac{1}{\langle \Psi_\omega, \partial_\omega\Psi_\omega \rangle}[\langle \psi, j\partial_\omega\Psi_\omega \rangle j\Psi_\omega + \langle \psi, \Psi_\omega \rangle \partial_\omega\Psi_\omega]. \tag{2.9}$$

Remark 2.1. *Since $j\Psi_\omega, \partial_\omega\Psi_\omega$ lie in $H^1(\mathbb{R})$ the operator $\mathbf{P}^0$ extends uniquely to define a continuous linear map $H^{-1}(\mathbb{R}) \to X^0$. In particular this operator can be applied to the Dirac measure $\delta(x)$.*

Denote by X^1 the eigensubspace corresponding to eigenvectors u and u^*, and by $\mathbf{P}^1$ a symplectic projection operator from $L^2(\mathbb{R})$ onto X^1. It may be represented by the formula

$$\mathbf{P}^1\psi = \frac{\langle \psi, ju \rangle}{\langle u, ju \rangle}u + \frac{\langle \psi, ju^* \rangle}{\langle u^*, ju^* \rangle}u^* \tag{2.10}$$

since $\langle u, ju^* \rangle = 0$, and $\langle u, ju \rangle = \overline{\langle u^*, ju^* \rangle} \neq 0$ by (3.66). Finally, $\mathbf{P}^c = 1 - \mathbf{P}^0 - \mathbf{P}^1$ is the symplectic projector onto the continuous spectral subspace.

2.2 Estimates for the linearized evolution

We now recall from [1] some estimates on the group $e^{\mathbf{C}t}$ which will be needed in §4. First we recall a bound for the action of $e^{\mathbf{C}t}$ on the Dirac distribution $\delta = \delta(x)$ for small t. Lemma 8.1 from [1] gives the following small t behaviour:

$$\|e^{\mathbf{C}t}\delta\|_{L^\infty} = \mathcal{O}(t^{-1/2}), \qquad t \to 0. \tag{2.11}$$

Second we list the large time dispersive estimates. To do this let us introduce, for $\beta \geq 2$, a Banach space $\mathcal{M}_\beta$, which is the subset of distributions which are linear combinations of L^1_β functions and multiples of the Dirac distribution at the origin with the norm:

$$\|\psi + C\delta(x)\|_{\mathcal{M}_\beta} := \|\psi\|_{L^1_\beta} + |C|, \tag{2.12}$$

and a Banach space $\mathcal{B}_\beta = B(\mathcal{M}_\beta, L^\infty_{-\beta})$ as the space of continuous linear maps $\mathcal{M}_\beta \to L^\infty_{-\beta}$ for any $\beta \geq 2$.

Proposition 2.2. *(see [1]) Assume that the spectral condition (1.14) holds. Then for $h \in \mathcal{M}_\beta$ with $\beta \geq 2$ the following bounds hold:*

$$\|e^{\mathbf{C}t}\mathbf{P}^c h\|_{L^\infty_{-\beta}} \leq c(1+t)^{-3/2}\|h\|_{\mathcal{M}_\beta}, \tag{2.13}$$

$$\|e^{\mathbf{C}t}\mathbf{\Pi}^\pm h\|_{L^\infty_{-\beta}} \leq c(1+t)^{-3/2}\|h\|_{\mathcal{M}_\beta}, \tag{2.14}$$

$$\|e^{\mathbf{C}t}\mathbf{C}^{-1}\mathbf{\Pi}^\pm h\|_{L^\infty_{-\beta}} \leq c(1+t)^{-3/2}\|h\|_{\mathcal{M}_\beta}, \tag{2.15}$$

where $\mathbf{\Pi}^+$ (resp. $\mathbf{\Pi}^-$) is the spectral projection operator onto the spectral subspace associated to $\mathcal{C}_+$ (resp. $\mathcal{C}_-$), the positive (resp. negative) part of the continuous spectrum.

We shall also need the following bound, which is new.

Proposition 2.3. *Assume that the spectral condition (1.14) holds. Then for $h \in \mathcal{M}_\beta$ with $\beta > 2$ and $t > 1$ the following bounds hold:*

$$\|e^{\mathbf{C}t}(\mathbf{C} \mp 2i\mu - 0)^{-1}\mathbf{P}^c h\|_{L^\infty_{-\beta}} \leq c(1+t)^{-3/2}\|h\|_{\mathcal{M}_\beta}, \tag{2.16}$$

$$\|e^{\mathbf{C}t}(\mathbf{C} \mp 2i\mu - 0)^{-1}\mathbf{\Pi}^\pm h\|_{L^\infty_{-\beta}} \leq c(1+t)^{-3/2}\|h\|_{\mathcal{M}_\beta}. \tag{2.17}$$

We prove this proposition in appendix C.

3 Spectral decomposition and canonical forms

In this section we will use the spectral resolution associated to the operator $\mathbf{C}$ to decompose the solution ψ, obtaining evolution equations for the different spectral components. Then, following [2], we introduce normal coordinate transformations to transform the evolution equations into simpler canonical forms in which certain non-resonant terms are absent. The final form of these equations is given in §3.4.5.

3.1 Modulation equations

In this section we present the modulation equations which allow a construction of solutions $\psi(x,t)$ of the equation (1.1) close at each time t to a soliton i.e. to one of the functions

$$Ce^{i\theta - \sqrt{\omega}|x|}, \qquad C = C(\omega) > 0$$

in the set $\mathcal{S}$ described in §1.2 with time varying ("modulating") parameters $(\omega, \theta) = (\omega(t), \theta(t))$.

We look for a solution to (1.2) in the form

$$\psi(x,t) = e^{j\theta(t)}\Phi(x,t), \quad \Phi(x,t) = \Psi_{\omega(t)}(x) + \chi(x,t). \tag{3.1}$$

Since this is a solution of (1.2) as long as $\chi \equiv 0$ with $\dot{\theta} = \omega$ and $\dot{\omega} = 0$ it is natural to look for solutions in which χ is small and

$$\theta(t) = \int_0^t \omega(s)ds + \gamma(t)$$

with γ treated perturbatively. We look for $\chi = \mathrm{w}(x,t) + f(x,t)$ where $\mathrm{w} = zu + \overline{z}u^* \in X^1$ and $f \in X^c$. Now we give a system of coupled equations for $\omega(t)$, $\gamma(t)$, $z(t)$ and $f(x,t)$.

Lemma 3.1. *Given a solution of (1.2) in the form (3.1) with $f \in X^c$ as just described, the functions $\omega(t)$, $\gamma(t)$, $z(t)$ and $f(x,t)$ satisfy the system*

$$\dot{\omega} = \frac{\langle \mathbf{P}^0 \mathbf{Q}[\chi], \Phi \rangle}{\langle \partial_\omega \Psi_\omega - \partial_\omega \mathbf{P}^0 \chi, \Phi \rangle}, \tag{3.2}$$

$$\dot{\gamma} = \frac{\langle \mathbf{Q}[\chi], j(\partial_\omega \Psi_\omega - \partial_\omega \mathbf{P}^0 \chi) \rangle}{\langle \partial_\omega \Psi_\omega - \partial_\omega \mathbf{P}^0 \chi, \Phi \rangle}, \tag{3.3}$$

$$\langle u, ju \rangle (\dot{z} - i\mu z) = \langle \mathbf{Q}[\chi], ju \rangle - \langle \partial_\omega \mathrm{w} - \partial_\omega \mathbf{P}^1 f, ju \rangle \dot{\omega} - \langle \chi, u \rangle \dot{\gamma} \tag{3.4}$$

$$\dot{f} = \mathbf{C} f + \dot{\omega} \partial_\omega \mathbf{P}^c \chi - \dot{\gamma} \mathbf{P}^c (j\chi) + \mathbf{P}^c \mathbf{Q}[\chi], \tag{3.5}$$

where $\mathbf{Q}[\chi] = -\delta(x) j^{-1} \big(\mathbf{F}(\Psi_\omega + \chi) - \mathbf{F}(\Psi_\omega) - \mathbf{F}'(\Psi_\omega)\chi \big)$ *represents the nonlinear part of the interaction.*

Proof. This can be proved as Proposition 2.2 in [2]. □

3.2 Frozen spectral decomposition

The linear part of the evolution equation (3.5) for f is non-autonomous, due to the dependence of the operator $\mathbf{C}$ on $\omega(t)$. In order to make use of the dispersive properties obtained in §2.2, it is convenient (following [2]) to introduce a small modification of (3.5), which leads to an autonomous linearized equation. Let us fix an interval $[0, T]$ and decompose $f(t) \in X_t^c$ into the sum

$$f = g + h, \quad g \in X_T^1, \quad h \in X_T^c. \tag{3.6}$$

Here $X_T^d = \mathbf{P}_T^d X$ is the spectral space associated to the discrete spectrum at time T and $X_T^c = \mathbf{P}_T^c X$ is the spectral space associated to the continuous spectrum at time T, $\mathbf{P}_T^c = \mathbf{P}^c(\omega(T))$ and $\mathbf{P}_T^d = I - \mathbf{P}_T^c$. In the following, we denote $\omega_T = \omega(T)$ and $\mathbf{C}_T = \mathbf{C}(\omega_T)$. We will obtain estimates uniform in T, and later consider the limit $T \to +\infty$.

We now introduce a shorthand for the bounds we are about to prove: $\mathcal{R}(A, B, \dots)$ (resp. $\mathcal{R}(\omega, A, \dots)$) is a general notation for a positive function which remain bounded as $A, B, \dots$ approach zero (resp. if ω is close to ω_0 and $A, \dots$ approach zero); it could be unbounded and even infinite if ω is outside some vicinity of ω_0. The formula $f = \mathcal{R}g$ implies that $|f| \le \mathcal{R}g$. Introducing also the notation $\mathcal{R}_1(\omega) = \mathcal{R}(\|\omega - \omega_0\|_{C[0,T]})$, we get

Lemma 3.2. *The function g is estimated in terms of h as follows:*

$$\|g\|_{L^\infty_{-\beta}} = \mathcal{R}_1(\omega) |\omega - \omega_T| \|h\|_{L^\infty_{-\beta}} \tag{3.7}$$

Proof. Let us use the identities

$$\mathbf{P}^d(g + h) = 0, \ \mathbf{P}_T^d g = g, \ \mathbf{P}_T^d h = 0.$$

Then we get

$$g + (\mathbf{P}^d - \mathbf{P}_T^d) g + (\mathbf{P}^d - \mathbf{P}_T^d) h = 0,$$

and $\mathbf{P}^d - \mathbf{P}_T^d$ is a"small" finite dimensional operator:

$$|\mathbf{P}^d - \mathbf{P}_T^d| = |\mathbf{P}^d(\omega_t) - \mathbf{P}^d(\omega_T)| \le \max_{\omega^* \in (\omega, \omega_T)} |\partial_\omega \mathbf{P}^d(\omega^*)| |\omega - \omega_T|.$$

□

Applying the projection $\mathbf{P}_T^c$ to (3.5), we get

$$\dot{h} = \mathbf{C}_T h + \mathbf{P}_T^c [(\mathbf{C} - \mathbf{C}_T) f + \mathbf{P}^c \mathbf{Q}[\chi] + \dot{\omega} \partial_\omega \mathbf{P}^c \chi - \dot{\gamma} \mathbf{P}^c (j\chi)]. \tag{3.8}$$

3.3 Asymptotic expansion of dynamics

The preceding sections have provided a change of variables $\psi \mapsto (\omega, \gamma, z, h)$ under which (1.2) is mapped into the system comprising (3.2),(3.3), (3.4) and (3.8). Since we are interested in proving that for large times z, h are small it is necessary to expand the inhomogeneous terms in these equations in terms of z, h. This is carried out in this section, leading to the conclusion that the system (3.2),(3.3), (3.4) and (3.8) can be written in more detail as the system comprising (3.22),(3.29), (3.33) and (3.41).

3.3.1 Preliminaries

This section is devoted to some useful preliminary estimates. We start with a bound for the denominator $\langle \partial_\omega \Psi - \partial_\omega \mathbf{P}^0 \chi, \Phi \rangle$, where $\Psi = \Psi_\omega$, that appears in the equation of motion (3.2)-(3.3). We have, with $\Delta = \langle \partial_\omega \Psi, \Psi \rangle$,

$$\langle \partial_\omega \Psi - \partial_\omega \mathbf{P}^0 \chi, \Phi \rangle = \langle \partial_\omega \Psi, \Psi \rangle \Big(1 + \frac{\langle \partial_\omega \Psi, \chi \rangle - \langle \partial_\omega \mathbf{P}^0 \chi, \Phi \rangle}{\langle \partial_\omega \Psi, \Psi \rangle} \Big) = \Delta \Big(1 + \frac{\langle \partial_\omega \Psi, \chi \rangle - \langle \partial_\omega \mathbf{P}^0 \chi, \Phi \rangle}{\Delta} \Big) \tag{3.9}$$

with

$$\frac{\langle \partial_\omega \Psi, \chi \rangle - \langle \partial_\omega \mathbf{P}^0 \chi, \Phi \rangle}{\Delta} = \mathcal{R}(\omega) \Big(|z| + \|f\|_{L^\infty_{-\beta}} + \|f\|^2_{L^\infty_{-\beta}} \Big). \tag{3.10}$$

We also need to expand the nonlinear term $\mathbf{F}(\psi) = a(|\psi|^2)\psi$ near the solitary wave since the inhomogeneous terms all involve $E[\chi]$, the nonlinear part of $\delta(x)F$, defined using the Taylor expansion of $\delta(x)\mathbf{F}\psi$ near Ψ:

$$\delta(x)\mathbf{F}(\psi) = \delta(x) \Big(a(C^2)\Psi + a(C^2)\chi + 2a'(C^2)(\chi, \Psi)\Psi \Big) + E[\chi]. \tag{3.11}$$

Thus $E[\chi]$ contains all the higher order terms which are at least quadratic in χ, as $\chi \to 0$, and $\mathbf{Q}[\chi] = jE[\chi]$. We expand $E[\chi]$ in the form

$$E[\chi] = E_2 + E_3 + E_R, \tag{3.12}$$

where E_j is of order j in χ and E_R the remainder. It is easy to check that

$$E_2[\chi, \chi] = \delta(x) \Big[a'(C^2)|\chi|^2\Psi + 2a''(C^2)(\Psi, \chi)^2\Psi + 2a'(C^2)(\Psi, \chi)\chi \Big], \tag{3.13}$$

$$E_3[\chi, \chi, \chi] = \delta(x) \Big[a'(C^2)|\chi|^2\chi + 2a''(C^2)(\Psi, \chi)^2\chi + 2a''(C^2)(\Psi, \chi)|\chi|^2\Psi + \frac{4}{3}a'''(C^2)(\Psi, \chi)^3\Psi \Big] \tag{3.14}$$

For E_R we have

$$E_R = \mathcal{R}(\omega, |z|, |f(0)|)(|z|^4 + |f(0)|^4), \tag{3.15}$$

It also useful to define $E_2[\chi_1, \chi_2]$, (resp. $E_3[\chi_1, \chi_2, \chi_3]$) as a symmetric bilinear (resp. trilinear) form

$$E_2[\chi_1, \chi_2] = \delta(x) \Big[a'(C^2)(\chi_1, \chi_2)\Psi + 2a''(C^2)(\Psi, \chi_1)(\Psi, \chi_2)\Psi + a'(C^2) \Big((\Psi, \chi_2)\chi_1 + (\Psi, \chi_1)\chi_2 \Big) \Big] \tag{3.16}$$

$$\begin{aligned} E_3[\chi_1, \chi_2, \chi_3] \;=\; & \delta(x) \Big[\frac{1}{6}a'(|\Psi|^2) \sum (\chi_i, \chi_j)\chi_k + \frac{1}{3}a''(|\Psi|^2) \sum (\Psi, \chi_i)(\Psi, \chi_j)\chi_k \\ & + \frac{1}{3}a''(|\Psi|^2) \sum (\Psi, \chi_i)(\chi_j, \chi_k)\Psi + \frac{4}{3}a'''(|\Psi|^2)(\Psi, \chi_1)(\Psi, \chi_2)(\Psi, \chi_3)\Psi \Big]. \end{aligned} \tag{3.17}$$

Here summation is taken over all transposition of integers $1, 2, 3$. Notice also that

$$\langle E_2[X, Y], Z\rangle = \langle X, E_2[Y^*, Z]\rangle \tag{3.18}$$

where X, Y, Z, are complex valued vector functions and $Y^* = (\overline{Y}_1, \overline{Y}_2)$.

In the remaining part of the paper we shall prove the following asymptotics:

$$\|f(t)\|_{L^\infty_{-\beta}} \sim t^{-1}, \quad z(t) \sim t^{-1/2}, \quad \|\mathrm{w}(t)\|_{H^1} \sim t^{-1/2}, \quad t \to \infty. \tag{3.19}$$

Remark 3.3. *To justufy these asymptotics, we will separate leading terms and remainders in right hand side of equations (3.2)-(3.4), (3.8). Namely, we shall expand the expressions for $\dot\omega$, $\dot\gamma$ and $\dot z$ up to and including terms of the order $\mathcal{O}(t^{-3/2})$, and for $\dot h$ up to $\mathcal{O}(t^{-1})$ keeping in mind the asymptotics (3.19). This choice is necessary for applicaton of the method of majorants.*

3.3.2 The equation for ω

Using the equality $\mathbf{Q}[\chi] = jE[\chi]$, and the fact that $j(\mathbf{P}^0)^* = \mathbf{P}^0 j$ (where * means adjoint with respect to the Hermitian inner product $\langle\,\cdot\,,\,\cdot\,\rangle$), we rewrite

$$\langle \mathbf{P}^0\mathbf{Q}[\chi], \Phi\rangle = \langle \mathbf{P}^0 jE[\chi], \Phi\rangle = -\langle E[\chi], j(\mathbf{P}^0)^*\Phi\rangle = -\langle E[\chi], \mathbf{P}^0 j\Phi\rangle$$

with $\chi = \mathrm{w} + f$ and $\Phi = \Psi + \chi = \Psi + \mathrm{w} + f$. Then equation (3.2) for $\dot\omega$ can be expanded up to $\mathcal{O}(t^{-3/2})$, assuming (3.19), as follows:

$$\begin{aligned}\dot\omega = &-\frac{1}{\Delta}\Big[\langle E_2[\mathrm{w},\mathrm{w}] + 2E_2[\mathrm{w}, f] + E_3[\mathrm{w},\mathrm{w},\mathrm{w}], j\Psi\rangle + \langle E_2[\mathrm{w},\mathrm{w}], \mathbf{P}^0 j\mathrm{w}\rangle\Big] \\ &+ \frac{1}{\Delta^2}\langle E_2[\mathrm{w},\mathrm{w}], j\Psi\rangle\Big(\langle\partial_\omega\Psi, \mathrm{w}\rangle - \langle\partial_\omega\mathbf{P}^0\mathrm{w}, \Psi\rangle\Big) + \Omega_R\end{aligned} \tag{3.20}$$

where

$$\Omega_R = \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{-\beta}})(|z|^2 + \|f\|_{L^\infty_{-\beta}})^2 \tag{3.21}$$

Substituting $\mathrm{w} = zu + \overline{z}u^*$, we can write (3.20) in the form

$$\dot\omega = \Omega_{20}z^2 + \Omega_{11}z\overline{z} + \Omega_{02}\overline{z}^2 + \Omega_{30}z^3 + \Omega_{21}z^2\overline{z} + \Omega_{12}z\overline{z}^2 + \Omega_{03}\overline{z}^3 + z\langle f, \Omega'_{10}\rangle + \overline{z}\langle f, \Omega'_{01}\rangle + \Omega_R \tag{3.22}$$

Let us now display explicitly some important terms of this expansion. First we compute the quadratic terms in (3.20) which are of order t^{-1} according to (3.19). They are obtained from the term

$$\langle E_2[\mathrm{w},\mathrm{w}], j\Psi\rangle = z^2\langle E_2[u, u], j\Psi\rangle + \overline{z}^2\langle E_2[u^*, u^*], j\Psi\rangle + 2z\overline{z}\langle E_2[u^*, u], j\Psi\rangle. \tag{3.23}$$

in expression (3.2). Let us take into account definition of E_2, the identity $(\Psi, j\Psi) = 0$ and that $\Phi = (\phi, 0)$, $\mathrm{w} = zu + \overline{z}u^*$ with $u = (u_1, u_2)$ where u_1 real and u_2 pure imaginary. Then we obtain

$$\begin{aligned}\langle E_2[\mathrm{w},\mathrm{w}], j\Psi\rangle &= \langle\delta(x)2a'(C^2)(\Psi,\mathrm{w})\mathrm{w}, j\Psi\rangle = 2a'(C^2)(z+\overline{z})(u(0),\Psi(0))(z-\overline{z})(u(0), j\Psi(0)) \\ &= 2(z^2 - \overline{z}^2)a'(C^2)(u(0),\Psi(0))(u(0), j\Psi(0)).\end{aligned}$$

Therefore

$$\Omega_{20} = \overline{\Omega}_{02} = -\frac{\langle E_2[u, u], j\Psi\rangle}{\Delta} = -\frac{2}{\Delta}a'(C^2)(u(0),\Psi(0))(u(0), j\Psi(0)) \tag{3.24}$$

is purely imaginary and

$$\Omega_{11} = -2\frac{\langle E_2[u,u^*], j\Psi\rangle}{\Delta} = 0. \tag{3.25}$$

Using the property (3.18), we find that

$$\Omega'_{10} = \overline{\Omega}'_{01} = -2\frac{E_2[u^*, j\Psi]}{\Delta}. \tag{3.26}$$

Remark 3.4. *Since $f \in X_t^c$ then*

$$\langle f, \Omega'_{10}\rangle = \langle \mathbf{P}^c f, \Omega'_{10}\rangle = \langle f, j\mathbf{P}^c j^{-1}\Omega'_{10}\rangle.$$

Therefore we can substitute Ω'_{10} in (3.22) by their projection $j\mathbf{P}^c j^{-1}\Omega'_{10}$.

3.3.3 The equation for γ

Using again the equality $\mathbf{Q} = jE$ we get

$$\langle \mathbf{Q}[\chi], j(\partial_\omega\Psi - \partial_\omega\mathbf{P}^0\chi)\rangle = \langle E[\chi], \partial_\omega\Psi - \partial_\omega\mathbf{P}^0\chi\rangle$$

Therefore (3.3), (3.9), (3.10), (3.12), (3.15) imply

$$\begin{aligned} \dot\gamma \ &= \Delta^{-1}\Big[\langle E_2[\mathrm{w},\mathrm{w}] + 2E_2[\mathrm{w},f] + E_3[\mathrm{w},\mathrm{w},\mathrm{w}], \partial_\omega\Psi\rangle - \langle E_2[\mathrm{w},\mathrm{w}], \partial_\omega\mathbf{P}^0\mathrm{w}\rangle\Big] \\ &- \ \Delta^{-2}\langle E_2[\mathrm{w},\mathrm{w}], \partial_\omega\Psi\rangle\Big(\langle\partial_\omega\Psi, \mathrm{w}\rangle - \langle\partial_\omega\mathbf{P}^0\mathrm{w}, \Psi\rangle\Big) + \Gamma_R, \end{aligned} \tag{3.27}$$

where

$$\Gamma_R = \mathcal{R}(\omega, |z| + |f(0)|)(|z|^2 + |f(0)|)^2 = \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{-\beta}})(|z|^2 + \|f\|_{L^\infty_{-\beta}})^2 \tag{3.28}$$

since $|f(0)| \le \|f\|_{L^\infty_{-\beta}}$. Equation (3.27) can thus be represented in the form

$$\dot\gamma = \Gamma_{20}z^2 + \Gamma_{11}z\overline{z} + \Gamma_{02}\overline{z}^2 + \Gamma_{30}z^3 + \Gamma_{21}z^2\overline{z} + \Gamma_{12}z\overline{z}^2 + \Gamma_{03}\overline{z}^3 + z\langle f, \Gamma'_{10}\rangle + \overline{z}\langle f, \Gamma'_{01}\rangle + \Gamma_R, \tag{3.29}$$

where

$$\begin{aligned} \Gamma_{20} \ &= \ \frac{\langle E_2[u,u], \partial_\omega\Psi\rangle}{\Delta}, \quad \Gamma_{11} = 2\frac{\langle E_2[u,u^*], \partial_\omega\Psi\rangle}{\Delta}, \quad \Gamma_{02} = \frac{\langle E_2[u^*,u^*], \partial_\omega\Psi\rangle}{\Delta}, \\ \Gamma'_{10} \ &= \ 2\frac{E_2[u^*, \partial_\omega\Psi]}{\Delta}, \quad \Gamma'_{01} = 2\frac{E_2[u, \partial_\omega\Psi]}{\Delta}. \end{aligned} \tag{3.30}$$

3.3.4 The equation for z

Denote $\varkappa = \langle u, ju\rangle$ and rewrite (3.4) in the form:

$$\begin{aligned} \dot z - i\mu z \ &= \ \frac{\langle E_2[\mathrm{w},\mathrm{w}] + 2E_2[\mathrm{w},f] + E_3[\mathrm{w},\mathrm{w},\mathrm{w}], u\rangle}{\varkappa} \\ &+ \ \frac{\langle\partial_\omega\mathrm{w}, ju\rangle\langle E_2[\mathrm{w},\mathrm{w}], j\Psi\rangle - \langle\mathrm{w}, u\rangle\langle E_2[\mathrm{w},\mathrm{w}], \partial_\omega\Psi\rangle}{\varkappa\Delta} + Z_R, \end{aligned} \tag{3.31}$$

where

$$Z_R = \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{-\beta}})(|z|^2 + \|f\|_{L^\infty_{-\beta}})^2. \tag{3.32}$$

Equation (3.31) can be represented in the form

$$\begin{aligned} \dot{z} &= i\mu z + Z_{20}z^2 + Z_{11}z\overline{z} + Z_{02}\overline{z}^2 + Z_{30}z^3 + Z_{21}z^2\overline{z} + Z_{12}z\overline{z}^2 + Z_{03}\overline{z}^3 \\ &+ z\langle f, Z'_{10}\rangle + \overline{z}\langle f, Z'_{01}\rangle + Z_R, \end{aligned} \tag{3.33}$$

where, using the calculations in the previous two sections, we have in particular,

$$\begin{aligned} Z_{20} &= \frac{\langle E_2[u,u],u\rangle}{\varkappa}, \quad Z_{11} = 2\frac{\langle E_2[u,u^*],u\rangle}{\varkappa}, \quad Z_{02} = \frac{\langle E_2[u^*,u^*],u\rangle}{\varkappa}, \\ Z_{21} &= \frac{\langle 3E_3[u^*,u,u],u\rangle}{\varkappa}, \\ &+ \frac{\langle \partial_\omega u^*, ju\rangle\langle E_2[u,u], j\Psi\rangle - \langle u^*,u\rangle\langle E_2[u,u],\partial_\omega\Psi\rangle - \langle u,u\rangle\langle 2E_2[u^*,u],\partial_\omega\Psi\rangle}{\varkappa\Delta}, \\ Z'_{10} &= 2\frac{E_2[u^*,u]}{\overline{\varkappa}}, \quad Z'_{01} = 2\frac{E_2[u,u]}{\overline{\varkappa}}. \end{aligned} \tag{3.34}$$

3.3.5 The equation for h

We now turn to equation (3.8) for $\dot{h}$ that we rewrite in the form

$$\dot{h} = \mathbf{C}_T h + \mathbf{P}^c_T\Big[(\mathbf{C} - \mathbf{C}_T)f + \mathbf{P}^c jE_2[w,w] + \dot{\gamma}\mathbf{P}^c j^{-1}f + H_R\Big], \tag{3.35}$$

where remainder H_R is

$$\begin{aligned} H_R &= \mathbf{P}^c j(E[\chi] - E_2[\mathrm{w},\mathrm{w}]) + \dot{\omega}\partial_\omega\mathbf{P}^c\chi + \dot{\gamma}\mathbf{P}^c j^{-1}\mathrm{w} \\ &= \mathbf{P}^c j(E[\chi] - E_2[\mathrm{w},\mathrm{w}]) - \dot{\omega}\partial_\omega\mathbf{P}^d\chi + \dot{\gamma}j^{-1}\mathrm{w} - \dot{\gamma}\mathbf{P}^d j^{-1}\mathrm{w} \end{aligned} \tag{3.36}$$

For the H_R we have, recalling (2.12), the following estimate

$$\begin{aligned} \|H_R\|_{\mathcal{M}_\beta} &= \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{-\beta}})(|z|^3 + |z|\|f\|_{L^\infty_{-\beta}} + \|f\|^2_{L^\infty_{-\beta}}) + \mathcal{R}(\omega)|\dot{\omega}|(|z| + \|f\|_{L^\infty_{-\beta}}) \\ &+ \mathcal{R}(\omega)|\dot{\gamma}||z| = \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{-\beta}})(|z|^3 + |z|\|f\|_{L^\infty_{-\beta}} + \|f\|^2_{L^\infty_{-\beta}}) \end{aligned} \tag{3.37}$$

Now we continue the isolation the leading terms in the right hand site of (3.35). Note that, from the formulae in the discussion surrounding (1.11),

$$\mathbf{C} - \mathbf{C}_T = j^{-1}(\omega - \omega_T) + j^{-1}(V - V_T), \quad \text{where} \quad V = -\delta(x)[a + bP_1].$$

Also

$$\mathbf{P}^c_T\mathbf{P}^c = \mathbf{P}^c_T[\mathbf{P}^c_T + \mathbf{P}^d_T - \mathbf{P}^d] = \mathbf{P}^c_T + \mathbf{P}^c_T[\mathbf{P}^d_T - \mathbf{P}^d]$$

Therefore (3.35) becomes

$$\dot{h} = \mathbf{C}_T h + \sigma(t)\mathbf{P}^c_T j^{-1}h + \mathbf{P}^c_T jE_2[w,w] + H'_R \tag{3.38}$$

with $\sigma(t) = \omega - \omega_T + \dot{\gamma}$, and

$$H'_R = \mathbf{P}^c_T[H_R + \sigma(t)j^{-1}g + j^{-1}(V - V_T)f + (\mathbf{P}^d_T - \mathbf{P}^d)j(E_2[w,w] + \dot{\gamma}f)].$$

Using the identity $\mathbf{P}_T^c = 1 - \mathbf{P}_T^d$, we obtain

$$\|H_R'\|_{\mathcal{M}_\beta} \le \mathcal{R}_1(\omega, |z| + \|f\|_{L^\infty_{-\beta}})(|z|^3 + |z|\|f\|_{L^\infty_{-\beta}} + \|f\|^2_{L^\infty_{-\beta}} + |\omega - \omega_T|(|z|^2 + |\dot\gamma|\|f\|_{L^\infty_{-\beta}}). \quad (3.39)$$

Next we need an additional construction to combine first two terms in RHS of (3.38). Namely, lemma 3.5 below shows that the "main part" of the second term is $i\sigma(t)(\mathbf{\Pi}_T^+ - \mathbf{\Pi}_T^-)h$, where $\mathbf{\Pi}^+$ and $\mathbf{\Pi}^-$ are the spectral projection operators on the spectral space associated to the positive and negative part of the continuous spectrum respectively at time T; see the discussion preceding (2.14). Hence, we denote

$$\mathbf{C}_M(t) = \mathbf{C}_T + i\sigma(t)(\mathbf{\Pi}_T^+ - \mathbf{\Pi}_T^-) \quad (3.40)$$

and rewrite (3.38) as

$$\dot h = \mathbf{C}_M(t)h + \mathbf{P}_T^c jE_2[w,w] + \tilde H_R \quad (3.41)$$

where

$$\tilde H_R = H_R' + \sigma(t)[\mathbf{P}_T^c j^{-1} - i(\mathbf{\Pi}_T^+ - \mathbf{\Pi}_T^-)]h \quad (3.42)$$

Lemma 3.5. *For $h \in X_T^c$ we have*

$$\|[\mathbf{P}_T^c j^{-1} - i(\mathbf{\Pi}_T^+ - \mathbf{\Pi}_T^-)]h\|_{L^1_\beta} \le \|h\|_{L^\infty_{-\beta}}. \quad (3.43)$$

This lemma is proved in appendix D. Lemma 3.5 and the bound (3.39) imply

Proposition 3.6. *The remainder $\tilde H_R$ admits the bound*

$$\|\tilde H_R\|_{\mathcal{M}_\beta} \le \mathcal{R}_1(\omega, |z| + \|f\|_{L^\infty_{-\beta}})(|z|^3 + |z|\|f\|_{L^\infty_{-\beta}} + \|f\|^2_{L^\infty_{-\beta}} + |\omega - \omega_T|(|z|^2 + \|f\|_{L^\infty_{-\beta}})). \quad (3.44)$$

3.4 Canonical form of the equations

Our goal is to transform the evolution equations for (ω, γ, z, h) to a more simple, canonical form. We will use the idea of normal coordinates, trying to keep unchanged the estimates for the remainders as much as is possible. This means we observe that for our purposes the unknowns (ω, γ, z, h) lie in a neighbourhood of the point $(\omega_0, 0, 0, 0)$ in the space $\in \mathbb{R} \times \mathbb{R} \times \mathbb{C} \times L^\infty_{-\beta}$. We seek a change of variables

$$\Theta : (\omega, \gamma, z, h) \mapsto (\omega_1, \gamma_1, z_1, h_1)$$

such that Θ is a diffeomorphism between neighbourhoods of $(\omega_0, 0, 0, 0)$ in the space $\in \mathbb{R} \times \mathbb{R} \times \mathbb{C} \times L^\infty_{-\beta}$, and $D\Theta(\omega_0, 0, 0, 0)$ is the identity. This map Θ is obtained, as usual in the normal form method, by looking for $(\omega_1, \gamma_1, z_1, h_1)$ as a power series in (ω, γ, z, h), starting with the identity map at highest order - see (3.47)-(3.49),(3.57),(3.73) and (3.82) for the detailed expressions. The coefficients in these expressions are then chosen so as to put the equations for $(\omega_1, \gamma_1, z_1, h_1)$ in a simpler canonical form which is suitable for our subsequent work. This final canonical form for the system is summarised in section 3.4.5.

3.4.1 Canonical form of the equation for h

As a starting point we expand out the middle term on the right hand side of (3.41), obtaining

$$\dot h = \mathbf{C}_M(t)h + H_{20}z^2 + H_{11}z\overline{z} + H_{02}\overline{z}^2 + \tilde H_R. \quad (3.45)$$

Here, the coefficients H_{ij} are defined by

$$H_{20} = \mathbf{P}_T^c j E_2[u,u], \quad H_{11} = 2\mathbf{P}_T^c j E_2[u,u^*], \quad H_{12} = \mathbf{P}_T^c j E_2[u^*,u^*]. \tag{3.46}$$

We want to extract from h the term of order $z^2 \sim t^{-1}$. For this purpose we expand h as

$$h = h_1 + k + k_1, \tag{3.47}$$

where

$$k = a_{20}z^2 + a_{11}z\overline{z} + a_{02}\overline{z}^2, \tag{3.48}$$

with some $a_{ij} \equiv a_{ij}(\omega, x)$ satisfying $a_{ij} = \overline{a}_{ij}$, and

$$k_1 = -\exp\left(\int_0^t \mathbf{C}_M(\tau)d\tau\right)k(0). \tag{3.49}$$

Note that k_1 is just the solution of the corresponding homogeneous equation $\dot{k}_1 = \mathbf{C}_M k_1$, since the operators $\mathbf{C}_M(t)$ all commute for different values of t. It follows from $k_1(0) = -k(0)$ that $h_1(0) = h(0)$.

Lemma 3.7. *There exist $a_{ij} \in L^\infty_{-\beta}$ in (3.48) such that the equation for h_1 has the form*

$$\dot{h}_1 = \mathbf{C}_M(t)h_1 + \hat{H}_R \tag{3.50}$$

where $\hat{H}_R = \tilde{H}_R + H'$, with estimates as in (3.44), *and also*

$$\|k\|_{L^\infty_{-\beta}} = \mathcal{R}_1(\omega)|z|^2. \tag{3.51}$$

Proof. (cf. Section 4.2.2 in [2]) We substitute (3.48) into (3.41) and equate the coefficients of the quadratic powers of z. In addition we replace the discrete eigenvalue $\mu(t)$ by its value at time T, i.e. $\mu_T = \mu(\omega(T))$, and include the correction in the remainder. Then we get

$$\begin{aligned} H_{20} - 2i\mu_T a_{20} &= -\mathbf{C}_T a_{20} \\ H_{11} &= -\mathbf{C}_T a_{11} \\ H_{02} + 2i\mu_T a_{02} &= -\mathbf{C}_T a_{02} \end{aligned}$$

and

$$\hat{H}_R = \tilde{H}_R + H'$$

where H' is defined as

$$\begin{aligned} H' = &\sum \partial_\omega a_{ij}\mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{-\beta}})|z|^2(|z| + \|f\|_{L^\infty_{-\beta}})^2 \\ &+ \sum a_{ij}\mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{-\beta}})|z|(|z| + \|f\|_{L^\infty_{-\beta}})^2 + \sum a_{ij}\mathcal{R}(\omega)|z|^2|\mu_T - \mu| - i\sigma(\mathbf{\Pi}_T^+ - \mathbf{\Pi}_T^-)k. \end{aligned} \tag{3.52}$$

The dependency in x appears here through the coefficients $a_{ij} = a_{ij}(\omega, x)$. Notice, from (1.16) and the formuae for the projection operators in §2.1, that each $H_{ij} \in X_T^c$ is the sum of a multiple of $\delta(x)$ and a function exponentially decreasing at infinity. Hence, there exists a solution a_{11} in the form

$$a_{11} = -\mathbf{C}_T^{-1}H_{11}, \tag{3.53}$$

where $\mathbf{C}_T^{-1}$ stands for regular part of the resolvent $R(\lambda)$ at $\lambda = 0$ since the singular part of $R(\lambda)H_{11}$ vanishes for $H_{11} \in X_T^c$. The function a_{11} is exponentially decreasing at infinity.
For a_{20} and a_{02} we choose the following inverse operators:

$$a_{20} = -(\mathbf{C}_T - 2i\mu_T - 0)^{-1}H_{20}, \quad a_{02} = \overline{a}_{20} = -(\mathbf{C}_T + 2i\mu_T - 0)^{-1}H_{02}, \tag{3.54}$$

This choice is motivated by proposition 2.3, and putting $t = 0$ in that proposition we have the bound (3.51). The remainder H' can be written as

$$H' = \sum_m (\mathbf{C}_T - 2i\mu_T m - 0)^{-1}A_m, \quad m \in \{-1, 0, 1\} \tag{3.55}$$

with $A_m \in X_T^c$, satisfying the estimate

$$\|A_m\|_{L^1_\beta} = \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{-\beta}})|z|\Big(|z||\omega_T - \omega| + (|z| + \|f\|_{L^\infty_{-\beta}})^2\Big). \tag{3.56}$$

□

3.4.2 Canonical form of the equation for ω

We want to remove all terms in the right hand side of (3.22) except the remainder Ω_R. This is possible by methods of Buslaev and Sulem [2, Proposition 4.1] since $\Omega_{11} = 0$ by (3.25).

Lemma 3.8. *There exist coefficients $b_{ij}(\omega)$, $0 \le i, j \le 3$, and vector function $b'_{ij}(x, \omega)$, $0 \le i, j \le 1$, such that real-valued function ω_1 defined as*

$$\omega_1 = \omega + b_{20}z^2 + b_{02}\overline{z}^2 + b_{30}z^3 + b_{21}z^2\overline{z} + b_{12}z\overline{z}^2 + b_{03}\overline{z}^3 + z\langle f, b'_{10}\rangle + \overline{z}\langle f, b'_{01}\rangle \tag{3.57}$$

obeys a differential equation of the form

$$\dot{\omega}_1 = \hat{\Omega}_R, \tag{3.58}$$

where $\hat{\Omega}_R$ satisfies the same estimate (3.21) as Ω_R:

$$\hat{\Omega}_R = \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{-\beta}})(|z|^2 + \|f\|_{L^\infty_{-\beta}})^2 \tag{3.59}$$

Proof. The calculation follows the classical method of normal coordinates. Substituting $\dot{\omega}$, $\dot{z}$, and $\dot{f}$ from (3.22), (3.33), (3.5) into the equation for $\dot{\omega}_1$ and comparing the coefficients of z^2, zf, etc. leads to a system of equations for the coefficients b_{20}, b'_{10}, ets. (cf. [2, Proposition 4.1])

$$\begin{aligned}
&\Omega_{20} + 2i\mu b_{20} = 0, \\
&\Omega'_{10} + i\mu b'_{10} + \mathbf{C}^* b'_{10} = 0, \\
&\Omega_{21} + 2Z_{11}b_{20} + i\mu b_{21} + 2Z_{20}b_{02} + \langle F_{11}, b'_{10}\rangle + \langle F_{20}, b'_{01}\rangle = 0, \\
&\Omega_{30} + 2Z_{20}b_{20} + 3i\mu b_{30} + \langle F_{20}, b'_{10}\rangle = 0.
\end{aligned} \tag{3.60}$$

It should be noted that the resonant term $z\overline{z}$ in the equation for $\dot{\omega}_1$ is absent. From the first equation of (3.60) we obtain

$$b_{20} = \overline{b}_{02} = \frac{i}{2\mu}\Omega_{20}, \tag{3.61}$$

Multiply the second equation of (3.60) by j we get

$$j\Omega'_{10} + i\mu j b'_{10} - \mathbf{C} j b'_{10} = 0$$

since $j\mathbf{C}^* = -\mathbf{C}j$. Without loss of generality we can assume that $j\Omega'_{10} \in X_t^c$ by Remark 3.4. Therefore, there exists the solution b'_{10} in the form

$$b'_{10} = \overline{b'}_{01} = -j(\mathbf{C} - i\mu)^{-1} j\Omega'_{10}, \tag{3.62}$$

where $(\mathbf{C} - i\mu)^{-1}$ stands for regular part of the resolvent $R(\lambda)$ at $\lambda = i\mu$ since the singular part of $R(\lambda)j\Omega'_{10}$ vanishes for $j\Omega'_{10} \in X_T^c$. The functions b'_{10}, b'_{01} decrease exponentially at infinity, and the equations for $b_{21} = \overline{b}_{12}$, $b_{30} = \overline{b}_{03}$ can be easily solved. □

3.4.3 Canonical form of the equation for z

In this section we obtain a canonical form of the equation (3.33) for z, and carry out a computation of the coefficient of the resonant "$z^2\overline{z}$ " term, which gives the Fermi Golden Rule. Substituting (3.6) and (3.47) into (3.33) and putting the contribution of $g + h_1 + k_1$ in the remainder $\tilde{Z}_R$, we obtain

$$\begin{aligned} \dot{z} &= i\mu z + Z_{20}z^2 + Z_{11}z\overline{z} + Z_{02}\overline{z}^2 + Z_{30}z^3 + Z_{21}z^2\overline{z} + Z_{12}z\overline{z}^2 + Z_{03}\overline{z}^3 \\ &+ Z'_{30}z^3 + Z'_{21}z^2\overline{z} + Z'_{12}z\overline{z}^2 + Z'_{03}\overline{z}^3 + \tilde{Z}_R. \end{aligned} \tag{3.63}$$

We have by (3.47)-(3.48)

$$\begin{aligned} Z'_{30} &= \langle a_{20}, Z'_{10}\rangle, \quad Z'_{21} = \langle a_{11}, Z'_{10}\rangle + \langle a_{20}, Z'_{01}\rangle, \\ Z'_{03} &= \langle a_{02}, Z'_{01}\rangle, \quad Z'_{12} = \langle a_{02}, Z'_{10}\rangle + \langle a_{11}, Z'_{01}\rangle. \end{aligned} \tag{3.64}$$

We are particularly interested in the resonant term $Z'_{21}z^2\overline{z}$. Formulas (3.34), (3.46), (3.53), (3.54) imply

$$Z'_{21} = -\langle \mathbf{C}_T^{-1} 2\mathbf{P}_T^c j E_2[u,u^*], 2\frac{E_2[u,u^*]}{\overline{\varkappa}}\rangle - \langle (\mathbf{C}_T - 2i\mu_T - 0)^{-1}\mathbf{P}_T^c j E_2[u,u], 2\frac{E_2[u,u]}{\overline{\varkappa}}\rangle \tag{3.65}$$

For the coefficient $\varkappa = \varkappa(\omega)$ we get (see [2, Proposition 3.1])

$$\varkappa = \langle u, ju\rangle = i\delta, \quad \text{with} \quad \delta > 0. \tag{3.66}$$

Now we can prove

Lemma 3.9. *Suppose that the non-degeneracy condition (1.17) is satisfied, then*

$$\operatorname{Re} Z'_{21} < 0 \tag{3.67}$$

for ω in some vicinity of ω_0.

Proof. We first notice that the coefficient $\langle \mathbf{C}_T^{-1} 2\mathbf{P}_T^c j E_2[u,u^*], E_2[u,u^*]\rangle$ appearing in the expression (3.65) for Z'_{21} is real, since the operator $\mathbf{C}_T^{-1}2\mathbf{P}_T^c j$ is selfadjoint. Hence by (3.66) $\operatorname{Re} Z'_{21}$ reduces to

$$\begin{aligned} \operatorname{Re} Z'_{21} &= -\operatorname{Re} 2\frac{\langle (\mathbf{C}_T - 2i\mu_T - 0)^{-1}\mathbf{P}_T^c j E_2[u,u], E_2[u,u]\rangle}{\varkappa} \\ &= -\frac{2}{\delta}\operatorname{Im}\langle R(2i\mu + 0)\mathbf{P}_T^c j E_2[u,u], E_2[u,u]\rangle, \end{aligned}$$

where we denote

$$R(\lambda) = R_T(\lambda) = (\mathbf{C}_T - \lambda)^{-1},\ \mathrm{Re}\,\lambda > 0, \quad \text{and} \quad \mu = \mu_T.$$

Using that $\mathbf{P}^c_T$ commutes with $R(2i\mu+0)$, we have $R(2i\mu+0)\mathbf{P}^c_T = \mathbf{P}^c_T R(2i\mu+0)\mathbf{P}^c_T$. We have also that $(\mathbf{P}^c_T)^* = -j\mathbf{P}^c_T j$, hence

$$\mathrm{Re}\,Z'_{21} = \frac{2}{\delta}\mathrm{Im}\,\langle R(2i\mu+0)\alpha, j\alpha\rangle$$

with $\alpha = \mathbf{P}^c_T j E_2[u,u]$. The function $\lambda \mapsto \langle R(\lambda)\alpha, j\alpha\rangle$ is analytic in the region $\mathbb{C} \setminus (\mathcal{C}_+ \cup \mathcal{C}_-)$ since $\alpha \in X^c_T$. Hence by the Cauchy residue theorem we have

$$\langle R(2i\mu+0)\alpha, j\alpha\rangle = -\frac{1}{2\pi i}\int\limits_{\mathcal{C}_+\cup\mathcal{C}_-} d\lambda\, \frac{\langle (R(\lambda+0) - R(\lambda-0))\alpha, j\alpha\rangle}{\lambda - 2i\mu - 0} \tag{3.68}$$

Now we use the representation

$$R(\lambda+0) - R(\lambda-0) \;=\; -\frac{\tau_\pm(\lambda)\otimes\overline{\tau}_\pm(\lambda)}{8ik_\pm D\overline{D}}j - \frac{s_\pm(\lambda)\otimes\overline{s}_\pm(\lambda)}{2ik_\pm}j, \quad \lambda\in\mathcal{C}_\pm, \tag{3.69}$$

where $D = D(\lambda+0)$ and $k_+ = k_+(\lambda+0) < 0$ for $\lambda\in\mathcal{C}_+$, $k_- = k_-(\lambda+0) > 0$ for $\lambda\in\mathcal{C}_-$

$$\tau_\pm(\lambda) = (\overline{D}e^{ik_\pm|x|} - De^{-ik_\pm|x|})v_\pm + 4\beta ik_\pm e^{ik_\mp|x|}v_\mp, \quad s_\pm(\lambda) = \sin(k_\pm x)v_\pm \tag{3.70}$$

are the even and the odd eigenfunctions of the operator $\mathbf{C}_T$ corresponding to the point $\lambda\in\mathcal{C}_\pm$ of the continuous spectrum (see appendix B). The representation (3.69) can be checked by direct calculation using formulas (D. 2)-(D. 4) for the resolvent. Then equation (3.68) becomes (with $\lambda = i\nu$)

$$\langle R(2i\mu+0)\alpha, j\alpha\rangle = -\frac{1}{16\pi}\int\limits_{-\infty}^{-\omega}\frac{d\nu}{k_-|D|^2}\frac{\langle\tau_-, j\alpha\rangle\overline{\langle\tau_-, j\alpha\rangle}}{\nu - 2\mu} - \frac{1}{16\pi}\int\limits_{\omega}^{\infty}\frac{d\nu}{k_+|D|^2}\frac{\langle\tau_+, j\alpha\rangle\overline{\langle\tau_+, j\alpha\rangle}}{\nu - 2\mu + i0}$$

since the function α is even. Using that

$$\frac{1}{\nu + i0} = p.v.\frac{1}{\nu} - i\pi\delta(\nu)$$

where p.v. is the Cauchy principal value, we have

$$\langle R(2i\mu+0)\alpha, j\alpha\rangle = -\frac{1}{16\pi}\int\limits_{-\infty}^{-\omega}\frac{d\nu}{k_-|D|^2}\frac{\langle\tau_-, j\alpha\rangle\overline{\langle\tau_-, j\alpha\rangle}}{\nu - 2\mu} - \frac{1}{16\pi}p.v.\int\limits_{\omega}^{\infty}\frac{d\nu}{k_+|D|^2}\frac{\langle\tau_+, j\alpha\rangle\overline{\langle\tau_+, j\alpha\rangle}}{\nu - 2\mu}$$
$$+\ \frac{i}{16}\frac{\langle\tau_+(2i\mu), j\alpha\rangle\overline{\langle\tau_+(2i\mu), j\alpha\rangle}}{k_+(2i\mu+0)|D(2i\mu+0)|^2} \tag{3.71}$$

The integral terms in (3.71) is real. Thus,

$$\mathrm{Im}\,\langle R_T(2i\mu_T+0)\alpha, j\alpha\rangle = \frac{|\langle\tau_+(2i\mu_T), E_2[u,u]\rangle|^2}{16k_+(2i\mu_T+0)|D(2i\mu_T+0)|^2}$$

The non-degeneracy condition (1.17) implies that $\langle\tau_+(2i\mu_T), E_2[u,u]\rangle \neq 0$ in some vicinity of ω_0. Using also the inequality $k_+(2i\mu_T+0) < 0$, we deduce $\mathrm{Re}\,Z'_{21} < 0$. $\square$

We now need an estimate on the remainder $\tilde{Z}_R$.

Lemma 3.10. *The remainder $\tilde{Z}_R$ has the form*

$$\tilde{Z}_R = \mathcal{R}_1(\omega, |z|+\|f\|_{L^\infty_{-\beta}})\Big[(|z|^2+\|f\|_{L^\infty_{-\beta}})^2+|z||\omega_T-\omega|\|h\|_{L^\infty_{-\beta}}+|z|\|k_1\|_{L^\infty_{-\beta}}+|z|\|h_1\|_{L^\infty_{-\beta}}\Big]. \quad (3.72)$$

Proof. The remainder $\tilde{Z}_R$ is given by

$$\tilde{Z}_R = Z_R + z\langle f-k, Z'_{10}\rangle + \overline{z}\langle f-k, Z'_{01}\rangle$$

where Z_R satisfies estimate (3.32). Since $f-k = g+k_1+h_1$, we have by (3.7)

$$\begin{aligned}|\langle f-k, Z'_{10}\rangle| &\leq \mathcal{R}(\omega)(\|g\|_{L^\infty_{-\beta}}+\|k_1\|_{L^\infty_{-\beta}}+\|h_1\|_{L^\infty_{-\beta}})\\ &\leq \mathcal{R}_1(\omega)(|\omega_T-\omega|\|h\|_{L^\infty_{-\beta}}+\|k_1\|_{L^\infty_{-\beta}}+\|h_1\|_{L^\infty_{-\beta}})\end{aligned}$$

which implies (3.72. □

We can apply now the method of normal coordinates to equation (3.63).

Lemma 3.11. *(cf. [2, Proposition 4.9])*
There exist coefficients c_{ij} such that the new function z_1 defined by

$$z_1 = z + c_{20}z^2 + c_{11}z\overline{z} + c_{02}\overline{z}^2 + c_{30}z^3 + c_{12}z\overline{z}^2 + c_{03}\overline{z}^3, \quad (3.73)$$

satisfies an equation of the form

$$\dot{z}_1 = i\mu(\omega)z_1 + iK(\omega)|z_1|^2 z_1 + \hat{Z}_R \quad (3.74)$$

where $\hat{Z}_R$ satisfies estimates of the same type as $\tilde{Z}_R$, and

$$\mathrm{Re}\ iK = \mathrm{Re}\, Z'_{21} < 0. \quad (3.75)$$

Proof. Substituting z_1 in equation (3.63) for z and equating the coefficients, we get, in particular,

$$c_{20} = \frac{i}{\mu}Z_{20}, \quad c_{11} = -\frac{i}{\mu}Z_{11}, \quad c_{02} = -\frac{i}{3\mu}Z_{02} \quad (3.76)$$

and

$$iK = Z_{21} + Z'_{21} + c_{11}Z_{20} + 2c_{20}Z_{11} + c_{11}\overline{Z}_{11} + 2c_{02}\overline{Z}_{02} \quad (3.77)$$

It is easy to check that all of the coefficients Z_{11}, Z_{20}, Z_{02}, and Z_{21} defined in (3.34) are pure imaginary, and hence (3.75) follows immediately. □

Denoting $K_T = K(\omega_T)$, the equation for z_1 is rewritten as

$$\dot{z}_1 = i\mu z_1 + iK_T|z_1|^2 z_1 + \hat{\hat{Z}}_R \quad (3.78)$$

where

$$\begin{aligned}\hat{\hat{Z}}_R = \hat{Z}_R + \mathcal{R}_1(\omega, |z|+\|f\|_{L^\infty_{-\beta}})|z|^3|\omega_T-\omega| &= \mathcal{R}_1(\omega, |z|+\|f\|_{L^\infty_{-\beta}})\Big[(|z|^2+\|f\|_{L^\infty_{-\beta}})^2 \quad (3.79)\\ &+ \ |z||\omega_T-\omega|(\|h\|_{L^\infty_{-\beta}}+|z|^2)+|z|\|k_1\|_{L^\infty_{-\beta}}+|z|\|h_1\|_{L^\infty_{-\beta}}\Big].\end{aligned}$$

It is easier to deal with $y = |z_1|^2$, rather than z_1, because y decreases at infinity while z_1 is oscillating The equation satisfied by y is simply obtained by multiplying (3.74) by $\overline{z}_1$ and taking the real part:

$$\dot{y} = 2\mathrm{Re}\,(iK_T)y^2 + Y_R, \tag{3.80}$$

where

$$Y_R = \mathcal{R}_1(\omega, |z|+\|f\|_{L^\infty_{-\beta}})|z|\Big[(|z|^2+\|f\|_{L^\infty_{-\beta}})^2+|z||\omega_T-\omega|(\|h\|_{L^\infty_{-\beta}}+|z|^2)+|z|\|k_1\|_{L^\infty_{-\beta}}+|z|\|h_1\|_{L^\infty_{-\beta}}\Big]. \tag{3.81}$$

3.4.4 Canonical form of the equation for γ

The only difference between equations (3.22) and (3.29) for ω and γ is that, in general the coefficient $\Gamma_{11} \neq 0$. We can nevertheless perform the same change of variables as for ω, obtaining:

Lemma 3.12. *There exist coefficients $d_{ij}(\omega)$, $0 \leq i, j \leq 3$, and vector functions $d'_{ij}(x, \omega)$ such that the new function γ_1 defined as*

$$\gamma_1 = \gamma + d_{20}z^2 + d_{02}\overline{z}^2 + d_{30}z^3 + d_{12}z^2\overline{z} + d_{12}z\overline{z}^2 + d_{023}\overline{z}^3 + z\langle f, d'_{10}\rangle + \overline{z}\langle f, d'_{01}\rangle, \tag{3.82}$$

with $d_{ij} = \overline{d}_{ji}$, is a solution of the differential equation

$$\dot{\gamma}_1 = \Gamma_{11}(\omega)z\overline{z} + \hat{\Gamma}_R. \tag{3.83}$$

Furthermore $\hat{\Gamma}_R$ satisfies the same estimate (3.28) as Γ_R.

3.4.5 Summary of the equations in canonical form

We summarize the main formulas of §3.4.1-§3.4.3. First we recall that

$$f = g + h, \quad g = \mathbf{P}^d_T f, \quad h = \mathbf{P}^c_T f, \quad h = k + k_1 + h_1$$

where k and h_1 are defined in (3.48)-(3.49). The equations satisfied by h and h_1 are, respectively, (see (3.38) and (3.50))

$$\dot{h} = \mathbf{C}_M h - \mathbf{P}^c_T jE_2[w, w] + \tilde{H}_R, \tag{3.84}$$

and

$$\dot{h}_1 = \mathbf{C}_M h_1 + \hat{H}_R. \tag{3.85}$$

The remainder $\tilde{H}_R$ is estimated in (3.44) and $\hat{H}_R = \tilde{H}_R + H'$, where H' is estimated in (3.55) and (3.56).
The second equation, given in Lemma 3.8, determines the evolution of ω_1:

$$\dot{\omega}_1 = \hat{\Omega}_R, \tag{3.86}$$

where $\hat{\Omega}_R$ is estimated in 3.59, and ω_1 and ω are related by (see (3.57))

$$\omega_1 - \omega = \mathcal{R}(\omega)|z|(|z| + \|f\|_{L^\infty_{-\beta}}). \tag{3.87}$$

The third equation describes the evolution of z_1 (see (3.74)):

$$\dot{z}_1 = i\mu z_1 + iK_T|z_1|^2 z_1 + \hat{\hat{Z}}_R, \tag{3.88}$$

with the estimate (3.79) for $\hat{\hat{Z}}_R$. ¿From (3.63), z and z_1 are related by that

$$z_1 - z = \mathcal{R}(\omega)|z|^2. \tag{3.89}$$

The fourth equation is for the evolution of $y = |z_1|^2$:

$$\dot{y} = 2\mathrm{Re}\,(iK_T)y^2 + Y_R, \tag{3.90}$$

where

$$|Y_R| \le |z_1||\hat{\hat{Z}}_R|. \tag{3.91}$$

The negativity of $\mathrm{Re}\,iK_T$ is a key point in the analysis and was proved in Lemma 3.9. It is clear from the last equation that this condition gives a nonlinear damping effect in the evolution of the amplitude of the discrete mode - this is the crucial dynamical consequence of the Fermi Golden Rule, with obvious relevance to the large time behaviour of the solutions.

3.5 A bound for $|\omega_T - \omega|$ and the initial conditions

We will need a uniform bound for $|\omega_T - \omega(t)|$ on the interval $[0, T]$. This can be obtained by comparison with the function $\omega_1(t)$ as follows:

$$\begin{aligned} &|\omega_T - \omega(t)| \le |\omega_{1T} - \omega_1(t)| + |\omega_{1T} - \omega_T| + |\omega_1(t) - \omega(t)| \qquad (3.92)\\ &\le \int_t^T |\dot{\omega}_1(\tau)|d\tau + \mathcal{R}(\omega_T, |z_T| + \|f_T\|_{L^\infty_{-\beta}})(|z_T| + \|f_T\|_{L^\infty_{-\beta}})^2 + \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{-\beta}})(|z| + \|f\|_{L^\infty_{-\beta}})^2 \\ &\le \max_{0\le t\le T} \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{-\beta}})\Big[\int_t^T (|z|^2 + \|f\|_{L^\infty_{-\beta}})^2 d\tau + (|z_T| + \|f_T\|_{L^\infty_{-\beta}})^2 + (|z| + \|f\|_{L^\infty_{-\beta}})^2\Big], \end{aligned}$$

where $z_T = z(T)$, $f_T = f(T)$. Using $|\omega| \le |\omega_0| + |\omega_0 - \omega_T| + |\omega - \omega_T|$, we have

$$\max_{0\le t\le T} \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{-\beta}}) = \mathcal{R}(\max_{0\le t\le T} |\omega - \omega_T|, \max_{0\le t\le T}(|z| + \|f\|_{L^\infty_{-\beta}})).$$

We denote such quantities by the symbol $\mathcal{R}_2(\omega, |z| + \|f\|_{L^\infty_{-\beta}})$. Then (3.92) becomes

$$|\omega_T - \omega| \le \mathcal{R}_2(\omega, |z| + \|f\|_{L^\infty_{-\beta}})\Big[\int_t^T (|z|^2 + \|f\|_{L^\infty_{-\beta}})^2 d\tau + (|z_T| + \|f_T\|_{L^\infty_{-\beta}})^2 + (|z| + \|f\|_{L^\infty_{-\beta}})^2\Big]. \tag{3.93}$$

As in (1.18), we suppose the smallness condition:

$$|z(0)| \le \varepsilon^{1/2}, \quad \|f(0)\|_{L^1_\beta} \le c\varepsilon^{3/2}, \tag{3.94}$$

where $\varepsilon > 0$ is sufficiently small. Equation (3.89) implies $|z_1|^2 \le |z|^2 + \mathcal{R}(\omega, z)|z|^3$. Therefore

$$y(0) = |z_1(0)|^2 \le \varepsilon + \mathcal{R}(\omega, |z_0|)\varepsilon^{3/2}. \tag{3.95}$$

¿From the formula $h = \mathbf{P}^c_T f = f + (\mathbf{P}^d - \mathbf{P}^d_T)f$, we see that

$$\|h(0)\|_{L^1_\beta} \le c\varepsilon^{3/2} + \mathcal{R}_1(\omega)|\omega_T - \omega|\|f(0)\|_{L^\infty_{-\beta}}. \tag{3.96}$$

3.6 A bound for k_1

Lemma 3.13. *The function k_1 defined in (3.49) satisfies the following bound:*

$$\|k_1\|_{L^\infty_{-\beta}} \le c|z(0)|^2 \frac{1}{(1+t)^{3/2}} \le c\frac{\varepsilon}{(1+t)^{3/2}}. \tag{3.97}$$

Proof. Equalities (3.40) and (3.49) imply

$$k_1 = -e^{\int_0^t \mathcal{C}_M(\tau)d\tau} k(0) = -e^{\mathbf{C}_T t + i\int_0^t \beta(\tau)d\tau(\mathbf{\Pi}_T^+ - \mathbf{\Pi}_T^-)} k(0) \tag{3.98}$$

Denoting $\nu = \int_0^t \beta(\tau)d\tau$, we obtain by expanding the exponential and using the idempotency of projections (the Euler trick):

$$e^{i\nu\mathbf{\Pi}_T^\pm} = \mathbf{\Pi}_T^\pm e^{i\nu} + \mathbf{\Pi}_T^\mp + \mathbf{P}_T^d.$$

Therefore

$$e^{i\nu(\mathbf{\Pi}_T^+ - \mathbf{\Pi}_T^-)} = (\mathbf{\Pi}_T^+ e^{i\nu} + \mathbf{\Pi}_T^- + \mathbf{P}_T^d)(\mathbf{\Pi}_T^- e^{-i\nu} + \mathbf{\Pi}_T^+ + \mathbf{P}_T^d) = \mathbf{\Pi}_T^+ e^{i\nu} + \mathbf{\Pi}_T^- e^{-i\nu} + \mathbf{P}_T^d.$$

Note that $\mathbf{C}_T$ commutes with $P_T^\pm$, hence

$$e^{\int_0^t \mathbf{C}_M(\tau)d\tau} = e^{\mathbf{C}_T t}(e^{i\nu}\mathbf{\Pi}_T^+ + e^{-i\nu}\mathbf{\Pi}_T^- + \mathbf{P}_T^d). \tag{3.99}$$

Since β is a real function, both exponentials are bounded. Further, by (3.48) we have $k(0) = a_{20}z^2(0) + a_{11}z(0)\overline{z}(0) + a_{02}\overline{z}^2(0)$ with a_{ij} defined in (3.53), (3.54). Therefore, the bounds (2.15), (2.16), (2.17) and assumption (3.94) imply (3.97). □

4 Large time asymptotics

In this section we will make use of the dispersive estimates given in §2.2 to prove the asymptotic representation for the solution of (1.2) with initial data as in theorem 1.3. The idea is to fix an interval $[0, T]$ and carry out the frozen spectral decomposition relative to the operator $\mathbf{C}_T = \mathbf{C}(\omega_T)$ at time T, as described in §3.2. We then obtain bounds for certain majorants on this interval which are uniform in T, and thus make it possible to obtain the asymptotics for the solution in the limit $T \to +\infty$. We will use the $\mathcal{R}$ notation explained prior to (3.7) and (3.93) to express estimates and bounds concisely.

4.1 Definition of majorants

We define the quantities

$$\mathbb{M}_0(T) = \max_{0\le t\le T} |\omega_T - \omega| \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{-1} \tag{4.1}$$

$$\mathbb{M}_1(T) = \max_{0\le t\le T} |z(t)| \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{-1/2} \tag{4.2}$$

$$\mathbb{M}_2(T) = \max_{0\le t\le T} \|h_1\|_{L^\infty_{-\beta}} \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{-3/2} \tag{4.3}$$

which will be refered to in the following as "majorants", and denote $\mathbb{M}$ the 3-dimensional vector $(\mathbb{M}_0, \mathbb{M}_1, \mathbb{M}_2)$. Observe that (3.7), (3.51) and (3.97) imply that f can be bounded in terms of these:

$$\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{-1} \|f\|_{L^\infty_{-\beta}} = \mathcal{R}_1(\omega)(\mathbb{M}_1^2 + \sqrt{\epsilon}\mathbb{M}_2), \tag{4.4}$$

so that control of $\mathbb{M}$ will allow complete control of the asymptotic behaviour of the solution. The goal of this section is to prove that if ε is sufficiently small, $\mathbb{M}$ is bounded uniformly in T. This is done by first bounding the initial data, and inhomogeneous terms, in the equations in section 3.4.5 in terms of the $\mathbb{M}_j$, and then using the estimates for the homogeneous evolutions to self-consistently bound the $\mathbb{M}_j$ in terms of themselves, uniformly in $T > 0$ and $\epsilon \ll 1$.

4.2 Estimate of the remainders and initial data

Lemma 4.1. *The remainder Y_R defined in (3.81) satisfies the estimate*

$$|Y_R| = \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\frac{\varepsilon^{5/2}}{(1+\varepsilon t)^2\sqrt{\varepsilon t}}(1+|\mathbb{M}|)^5. \tag{4.5}$$

Proof. Using again the equality $f = g + h = g + k + k_1 + h_1$, lemma 3.97 and the definitions of the $\mathbb{M}_j$, the remainder Y_R is bounded as follows:

$$Y_R = \mathcal{R}_2(\omega, |z| + \|f\|_{L^\infty_{-\beta}})|z|\Big[(|z|^2 + \|k_1\|_{L^\infty_{-\beta}} + \|h_1\|_{L^\infty_{-\beta}})^2 + |z||\omega_T - \omega|(|z|^2 + \|k_1\|_{L^\infty_{-\beta}} + \|h_1\|_{L^\infty_{-\beta}})$$

$$+|z|(\|k_1\|_{L^\infty_{-\beta}} + \|h_1\|_{L^\infty_{-\beta}})\Big] = \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\mathbb{M}_1\Bigg[\Big(\frac{\varepsilon}{1+\varepsilon t}\mathbb{M}_1^2 + \frac{\varepsilon}{(1+t)^{3/2}} + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\mathbb{M}_3\Big)^2$$

$$+\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\mathbb{M}_0\mathbb{M}_1\Big(\frac{\varepsilon}{1+\varepsilon t}\mathbb{M}_1^2 + \frac{\varepsilon}{(1+t)^{3/2}} + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\mathbb{M}_3\Big)$$

$$+\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\mathbb{M}_1\Big(\frac{\varepsilon}{(1+t)^{3/2}} + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\mathbb{M}_3\Big)\Bigg] = \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\frac{\varepsilon^{5/2}}{(1+\varepsilon t)^2\sqrt{\varepsilon + \varepsilon t}}(1+|\mathbb{M}|)^5,$$

establishing (4.1). □

Let us turn now to the remainder $\hat{H}_R = \tilde{H}_R + H'$ in equation (3.85) for h_1.

Lemma 4.2. *The first summand $\tilde{H}_R$ satisfies*

$$\|\tilde{H}_R\|_{\mathcal{M}_\beta} = \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\Big((1+\mathbb{M}_1)^3 + \varepsilon^{1/2}(1+|\mathbb{M}|)^4\Big). \tag{4.6}$$

Proof. It follows from (3.44)

$$\|\tilde{H}_R\|_{\mathcal{M}_\beta} = \mathcal{R}_2(\omega, |z| + \|f\|_{L^\infty_{-\beta}})\Big[|z|^3 + (|z| + |\omega_T - \omega|)(|z|^2 + \|k_1\|_{L^\infty_{-\beta}} + \|h_1\|_{L^\infty_{-\beta}})$$

$$+(|z|^2 + \|k_1\|_{L^\infty_{-\beta}} + \|h_1\|_{L^\infty_{-\beta}})^2 = \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\Bigg(\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\mathbb{M}_1^3 + \Big(\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\mathbb{M}_1 + \frac{\varepsilon}{1+\varepsilon t}\mathbb{M}_0\Big)$$

$$\Big(\frac{\varepsilon}{1+\varepsilon t}\mathbb{M}_1^2 + \frac{\varepsilon}{(1+t)^{3/2}} + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\mathbb{M}_2\Big) + \Big(\frac{\varepsilon}{1+\varepsilon t}\mathbb{M}_1^2 + \frac{\varepsilon}{(1+t)^{3/2}} + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\mathbb{M}_2\Big)^2$$

which implies (4.6). □

The second summand H' is represented as in (3.55) where the A_m are estimated in (3.56). For the A_m we now obtain:

Lemma 4.3.

$$\|A_m\|_{\mathcal{M}_\beta} = \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\Big(\mathbb{M}_1^3 + \varepsilon^{1/2}(1+|\mathbb{M}|)^3\Big). \tag{4.7}$$

Proof. Estimate (3.56) implies

$$\|A_m\|_{\mathcal{M}_\beta} = \mathcal{R}_2(\omega, |z| + \|f\|_{L^\infty_{-\beta}})|z|\Big(|z||\omega_T - \omega| + (|z| + \|k_1\|_{L^\infty_{-\beta}} + \|h_1\|_{L^\infty_{-\beta}})^2\Big)$$

$$= \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\mathbb{M}_1\left[\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\mathbb{M}_0\mathbb{M}_1 + \Big(\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\mathbb{M}_1 + \frac{\varepsilon}{(1+t)^{3/2}} + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\mathbb{M}_2\Big)^2\right]$$

which implies (4.7). □

Now we estimate the initial data. Referring to the formulas at the end of §3.5, we have

$$y(0) \le \varepsilon + \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\varepsilon^{3/2} = \varepsilon(1 + \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\varepsilon^{1/2}) \tag{4.8}$$

$$\|h(0)\|_{\mathcal{M}_\beta} \le c\varepsilon^{3/2} + \mathcal{R}_1(\omega)|\omega_T - \omega|\|f(0)\|_{L^\infty_{-\beta}} \le c\varepsilon^{3/2} + \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\varepsilon^2\mathbb{M}_0(1 + \mathbb{M}_1^2 + \varepsilon^{1/2}\mathbb{M}_2). \tag{4.9}$$

4.3 Integral inequalities and decay in time

This section is devoted to a study of the system:

$$\dot{y} = 2\mathrm{Re}\,(iK_T)y^2 + Y(t), \tag{4.10}$$

$$\dot{h}_1 = \mathbf{C}_M h_1 + H(x,t), \tag{4.11}$$

under some assumptions on the initial data, and on the inhomogeneous (or source) terms Y and H. Equation (4.10) for y is of Ricatti type, and is similar to (3.90), while (4.11) is similar to (3.85). First, for the initial data, we assume

$$y(0) \le \varepsilon y_0, \quad \|h_1(0)\|_{\mathcal{M}_\beta} \le \varepsilon^{3/2}h_0 \tag{4.12}$$

with some constant y_0 and $h_0 > 0$. As for the source terms, we assume that

$$|Y(t)| \le \overline{Y}\frac{\varepsilon^{5/2}}{(1+\varepsilon t)^2\sqrt{\varepsilon t}} \tag{4.13}$$

and that $H(x,t) = H_1(x,t) + H_2(x,t)$, where $H_2 = \sum\limits_m (\mathbf{C}_T - 2i\mu_T m - 0)^{-1}\mathcal{A}_m$, $\mathcal{A}_m \in X_T^c$ with the following bounds:

$$\|H_1\|_{\mathcal{M}_\beta} \le \overline{H}_1\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}, \tag{4.14}$$

$$\|\mathcal{A}_m\|_{\mathcal{M}_\beta} \le \overline{A}_m\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}, \tag{4.15}$$

where the quantities $\overline{Y}$, $\overline{H}_1$, $\overline{A}_m$ are supposed to be given positive constants. All these assumptions are motivated by the estimates of the remainders in §4.2, and by the final estimates we intend to prove on ω, z, h and h_1. Equation (4.10) corresponds to equation (3.90) and the assumption (4.13) on the source term has the form of estimate (4.5) for the remainder Y_R. Similarly, equation (4.11) corresponds to equation (3.85) and assumptions (4.14)-(4.15) correspond to the inequalities (4.6)- (4.7). Finally, corresponding to (3.75), we work under the assumption

$$\mathrm{Re}\ iK_T = -\mathrm{Im}\,K_T < 0.$$

Lemma 4.4. *([2, Proposition 5.6]) The solution of (4.10), with initial condition and source term satisfying (4.12) and (4.13) respectively, is bounded as follows for $t > 0$:*

$$|y(t) - \frac{y(0)}{1 + 2\mathrm{Im}\, K_T y(0)t}| \leq c\overline{Y}\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}, \quad c = c(y_0, \mathrm{Im}\, K_T). \tag{4.16}$$

Let us consider equation (4.11) for h_1.

Lemma 4.5. *The solution of (4.11), with initial condition and source term satisfying (4.12), (4.14) and (4.15), is bounded as follows:*

$$\|h_1\|_{L^\infty_{-\beta}} \leq c(\omega_T)\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\Big(h_0 + \overline{H}_1 + \sum_m \overline{A}_m\Big). \tag{4.17}$$

Proof. The function $h_1(x,t)$ can be expressed as:

$$h_1 = e^{\int_0^t \mathbf{C}_M(\tau)d\tau} h_1(0) + \int_0^t e^{\int_s^t \mathbf{C}_M(\tau)d\tau} H(s)ds,$$

To establish (4.17) we use the representation (3.99) and the bounds (2.14), (2.15) and (2.17) to deduce that

$$\|h_1\|_{L^\infty_{-\beta}} \leq \frac{c(\omega_T)}{(1+t)^{3/2}}\|h_1(0)\|_{\mathcal{M}_\beta} + \int_0^t \frac{c(\omega_T)}{(1+(t-s))^{3/2}}(\|H_1(s)\|_{\mathcal{M}_\beta} + \|A_m(s)\|_{\mathcal{M}_\beta})ds$$

$$\leq c(\omega_T)\Bigg[h_0\Big(\frac{\varepsilon}{1+t}\Big)^{3/2} + \int_0^t \frac{ds}{(1+(t-s))^{3/2}}\Bigg(\overline{H}_1\Big(\frac{\varepsilon}{1+\varepsilon s}\Big)^{3/2} + \sum_m \overline{A}_m\Big(\frac{\varepsilon}{1+\varepsilon s}\Big)^{3/2}\Bigg)\Bigg]$$

$$\leq c(\omega_T)\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\Big(h_0 + \overline{H}_1 + \sum_m \overline{A}_m\Big),$$

since $\int_0^t (1+t-s)^{-3/2}(1+\epsilon s)^{-3/2}ds \leq c(1+\epsilon t)^{-3/2}$ by [2, lemma 5.3]. $\square$

4.4 Inequalities for the majorants

In this section we estimate in turn the three majorants $\mathbb{M}_0, \mathbb{M}_1, \mathbb{M}_2$.

Lemma 4.6. *The majorants $\mathbb{M}_0(T)$, $\mathbb{M}_1(T)$, and $\mathbb{M}_2(T)$ satisfy*

$$\mathbb{M}_0(T) = \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\Big[(1+\mathbb{M}_1)^4 + \varepsilon(1+|\mathbb{M}|)^2\Big], \tag{4.18}$$

$$\mathbb{M}_1^2 = \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\Big(1 + \varepsilon^{1/2}(1+|\mathbb{M}|)^5\Big) \tag{4.19}$$

$$\mathbb{M}_2 = \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\Bigg[(1+\mathbb{M}_1)^3 + \varepsilon^{1/2}(1+|\mathbb{M}|)^4\Bigg]. \tag{4.20}$$

Proof. *Step i)* Using the equality $f = g + h = g + k + k_1 + h_1$ and bound (3.97) for k_1 we have, using the notation defined prior to (3.93)

$$\begin{aligned}
|z|^2 + \|f\|_{L^\infty_{-\beta}} &= \mathcal{R}_2(\omega, |z| + \|f\|_{L^\infty_{-\beta}})(\|k_1\|_{L^\infty_{-\beta}} + |z|^2 + \|h_1\|_{L^\infty_{-\beta}}) \\
&= \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\left(\frac{\varepsilon}{(1+t)^{3/2}} + \left(\frac{\varepsilon}{1+\varepsilon t}\right)\mathbb{M}_1^2 + \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}\mathbb{M}_2\right) \\
&= \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\left(\frac{\varepsilon}{1+\varepsilon t}\right)\left(1 + \mathbb{M}_1^2 + \varepsilon^{1/2}\mathbb{M}_2\right),
\end{aligned}$$

so that

$$|z|^2 + \|f\|_{L^\infty_{-\beta}} = \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\frac{\varepsilon}{1+\varepsilon t}\left(1 + \mathbb{M}_1^2 + \varepsilon^{1/2}\mathbb{M}_2\right). \tag{4.21}$$

Then (3.93) and (4.1) imply (4.18).

Step ii) Recall $y = |z_1|^2$ satisfies (4.10) with $Y = Y_R$, and Y_R satisfies the inequality (4.5) which is exactly the condition (4.13) with $\overline{Y} = \mathcal{R}(\varepsilon^{1/2}\mathbb{M})(1 + |\mathbb{M}|)^5$. Using (4.16) as well as (4.8) to bound the initial condition $y(0)$, it follows that

$$y \le \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\left[\frac{\varepsilon}{1+\varepsilon t} + \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}(1 + |\mathbb{M}|)^5\right].$$

Therefore

$$|z|^2 \le y + \mathcal{R}(\omega)|z|^3 \le \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\left[\frac{\varepsilon}{1+\varepsilon t} + \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}(1 + |\mathbb{M}|)^5 + \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}\mathbb{M}_1^3\right],$$

from which (4.19) follows.

Step iii) Let us now consider h_1, the solution of (3.85). It has the form (4.11) with $H = \hat{H}_R = \tilde{H}_R + H'$, where $\tilde{H}_R$ and H' identify respectively to H_1 and H_2. More precisely, using (4.6) and (4.7), we have

$$\overline{H_1} = \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\Big((1 + \mathbb{M}_1)^3 + \varepsilon^{1/2}(1 + |\mathbb{M}|)^4)\Big)$$

$$\overline{A}_m = \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\Big(\mathbb{M}_1^3 + \varepsilon^{1/2}(1 + |\mathbb{M}|)^3\Big).$$

Concerning the initial conditions, we know that $h_1(0) = h(0)$. Thus by (4.9)

$$h_0 = c + \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\varepsilon^{1/2}\mathbb{M}_0(1 + \mathbb{M}_1^2 + \varepsilon^{1/2}\mathbb{M}_2)$$

Applying Lemma 4.5, we deduce that

$$\|h_1\|_{L^\infty_{-\beta}} = \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}\left[(1 + \mathbb{M}_1)^3 + \varepsilon^{1/2}(1 + |\mathbb{M}|)^4\right],$$

which implies (4.20). □

4.5 Uniform bounds for majorants

The aim of this section is to prove that if ε is sufficiently small, all the $\mathbb{M}_i$ are bounded uniformly in T and ε.

Lemma 4.7. *For ε sufficiently small, there exists a constant M independent of T and ε, such that,*

$$|\mathbb{M}(T)| \leq M. \tag{4.22}$$

Proof. Combining the inequalities (4.18)-(4.20) for the $\mathbb{M}_i$, one get a estimate of the form

$$\mathbb{M}^2 \leq \mathcal{R}(\varepsilon^{1/2}\mathbb{M})\Big[(1+\mathbb{M}_1)^8 + \varepsilon^{1/2}(1+|\mathbb{M}|)^8\Big]$$

Replacing $\mathbb{M}_1^2$ in the right-hand by its bound (4.19), we get an inequality in the form

$$\mathbb{M}^2 \leq \mathcal{R}(\varepsilon^{1/2}\mathbb{M})(1+\varepsilon^{1/2}F(\mathbb{M}))$$

where $F(\mathbb{M})$ is an appropriate function . From this inequality it follows that $\mathbb{M}$ is bounded independent of $\varepsilon \ll 1$, since $\mathbb{M}(0)$ is small, and $\mathbb{M}(t)$ is a continuous function of t. □

Corollary 4.8. *The function $\omega(t)$ has a limit ω_+ as $t \to \infty$. Furthermore, the following estimates hold for all $t > 0$:*

$$|\omega_+ - \omega(t)| \leq M\frac{\varepsilon}{1+\varepsilon t}, \tag{4.23}$$

$$|z(t)| \leq M\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}, \tag{4.24}$$

$$\|h_1\|_{L^\infty_{-\beta}} \leq M\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}, \tag{4.25}$$

$$\|f\|_{L^\infty_{-\beta}} \leq c(M)\frac{\varepsilon}{1+\varepsilon t}. \tag{4.26}$$

Proof. Since $|\omega_T - \omega(t)| \leq \mathbb{M}_0 \dfrac{\varepsilon}{1+\varepsilon t}$, then applying this result to $|\omega(t_1) - \omega(t_2)|$, we see that $\omega(t)$ is a Cauchy sequence. It thus has a limit, denoted ω_+ and (4.23) holds. The next two results follow immediately, while the final one is a consequence of (4.4). □

4.6 Large time behaviour of the solution

In this section we deduce from corollary 4.8 a theorem which describe a large time behaviour of the solution. Notice that in the decomposition $f = g + h = g + h_1 + k + k_1$, a fixed time T has been chosen, and all the components depend on $\omega(T)$. From the above proposition, we know that $\omega(t)$ has a limit ω_+ as $t \to \infty$. So we can reformulate the decomposition by choosing $T = \infty$ and $\omega_T = \omega_+$. Namely, let us denote $\mathbf{P}^c_\infty = \mathbf{P}^c(\omega_+)$ and $\mathbf{P}^d_\infty = 1 - \mathbf{P}^c_\infty$. We define $f = g + h$ where $g = \mathbf{P}^d_\infty f$ and $h = \mathbf{P}^c_\infty f$. We also decompose $h = h_1 + k + k_1$ where

$$k = a_{20}z^2 + a_{11}z\overline{z} + a_{02}\overline{z}^2,$$

$$k_1 = -\exp\Big(\int_0^t \mathbf{C}_+(\tau)d\tau\Big)k(0)$$

where $a_{ij} = a_{ij}(\omega_+, x)$ and $\mathbf{C}_+ = \mathbf{C}(\omega_+) + i\Big(\omega(t) - \omega_+ + \dot{\gamma}(t)\Big)\Big(\mathbf{\Pi}^+_\infty - \mathbf{\Pi}^-_\infty\Big)$. All the estimates previously obtained in §3.4.2-§4 for finite T can be extended to $T = \infty$ and $\omega_T = \omega_+$ without modification. Thus we have proved the following result:

Theorem 4.9. *Let the conditions of theorem 1.3 hold. Then, for ε sufficiently small, there exist C^1 functions $\omega(t), \gamma(t), z(t)$ as in lemma 3.1, and constants $\omega_+ \in \mathbb{R}$ and $M > 0$, such that for all $t \geq 0$:*

$$\psi(x,t) = e^{j(\int\limits_0^t \omega(s)ds + \gamma(t))} \Big(\Psi_\omega(x) + z(t)u(x,\omega) + \overline{z}(t)u^*(x,\omega) + f(x,t) \Big), \tag{4.27}$$

and

$$|\omega(t) - \omega_+| \leq M \frac{\varepsilon}{1+\varepsilon t}, \quad |z(t)| \leq M \Big(\frac{\varepsilon}{1+\varepsilon t} \Big)^{1/2}, \quad \|f(t)\|_{L^\infty_{-\beta}} \leq M \frac{\varepsilon}{1+\varepsilon t}, \tag{4.28}$$

so that $\omega_+ = \lim\limits_{t\to\infty} \omega(t) \in \mathbb{R}$. A corresponding statement also holds for $t \to -\infty$.

5 Scattering asymptotics

We have now obtained the representation (4.27) of the solution $\psi(x,t)$. In order to prove statement of theorem 1.3 it remains to:

- construct asymptotic expressions for $\omega(t)$, $z(t)$, $\gamma(t)$, which is done in section 5.1 following [2], and then
- to prove the existence of $\Psi_\pm$, and hence obtain the scattering asymptotics (1.19); this second stage amounts to the construction of the wave operator, and is carried out in §5.2 by the study of some oscillatory integrals.

5.1 Large time behavior of $z(t)$, $\omega(t)$ and $\gamma(t)$

We start with equation (3.74) for z_1, rewritten as

$$\dot{z}_1 = i\mu z_1 + iK_+|z_1|^2 z_1 + \widehat{\widehat{Z}}_R$$

with $K_+ = K(\omega_+)$; by (3.79) the inhomogeneous term $\widehat{\widehat{Z}}_R$ satisfies the estimate

$$\widehat{\widehat{Z}}_R = \mathcal{R}(\varepsilon^{1/2}M) \frac{\varepsilon^2}{(1+\varepsilon t)^{3/2}\sqrt{\varepsilon t}} (1+M^4) = O(t^{-2}),\ t \to \infty.$$

On the other hand, we have, from (4.8) and (4.16),

$$y = \frac{y(0)}{1 + 2\mathrm{Im}\, K_+ y(0) t} + O(t^{-3/2}),\ t \to \infty.$$

Given the estimate (4.24) for $|z|$, and obviously the same one for $|z_1|$, we have

$$\dot{z}_1 = i\mu z_1 + iK_+ \frac{y(0)}{1 + 2\mathrm{Im}\, K_+ y(0) t} z_1 + Z_1, \quad Z_1 = O(t^{-2}),\ t \to \infty. \tag{5.1}$$

Since by assumption in theorem 1.3 $y(0) = O(\epsilon)$ we can write $y(0) = \epsilon y_0$ with $y_0 = O(1)$. Let us denote $2\mathrm{Im}\, K_+ y_0 = \epsilon k_+$, $\delta = \frac{\mathrm{Re}\, K_+}{\mathrm{Im}\, K_+}$ so that $\epsilon K_+ y_0 = \frac{i}{2}\epsilon k_+(1 - i\delta)$. The solution z_1 of (5.1) is written in the form

$$z_1 = \frac{e^{i\int\limits_0^t \mu(t_1)dt_1}}{(1+\epsilon k_+ t)^{\frac{1}{2}(1-i\delta)}} \Big[z_1(0) + \int\limits_0^t e^{-i\int\limits_0^s \mu(t_1)dt_1} (1+\epsilon k_+ s)^{\frac{1}{2}(1-i\delta)} Z_1(s)ds \Big] = z_\infty \frac{e^{i\int\limits_0^t \mu(t_1)dt_1}}{(1+\epsilon k_+ t)^{\frac{1}{2}(1-i\delta)}} + z_R$$

where

$$z_\infty(\omega) = z_1(0) + \int_0^\infty e^{-i\int_0^s \mu(t_1)dt_1}(1+\epsilon k_+ s)^{\frac{1}{2}(1-i\delta)} Z_1(s)ds$$

and

$$z_R = -\int_t^\infty e^{i\int_s^t \mu(t_1)dt_1}\Big(\frac{1+\epsilon k_+ s}{1+\epsilon k_+ t}\Big)^{\frac{1}{2}(1-i\delta)} Z_1(s)ds.$$

Here $\mu(t_1) = \mu(\omega(t_1))$. From the bound (5.1) on Z_1 it follows that $z_R = O(t^{-1})$. Therefore $z(t) = z_1(t) + O(t^{-1})$ satisfies

$$z(t) = z_+ \frac{e^{i\int_0^t \mu(t_1)dt_1}}{(1+\epsilon k_+ t)^{\frac{1}{2}(1-i\delta)}} + O(t^{-1}),\ t\to\infty, \quad z_+ = z_\infty(\omega_+). \tag{5.2}$$

¿From these formulas for $z(t)$, the asymptotic behavior of $\omega(t)$ and $\gamma(t)$ can be deduced as in [2, Sections 6.1 and 6.2], leading to the following:

Lemma 5.1. *In the situation of theorem 4.9, the functions $\omega(t)$ and $\gamma(t)$ have the following asymptotic behavior as $t \to +\infty$:*

$$\omega(t) = \omega_+ + \frac{q_+}{1+\epsilon k_+ t} + \frac{b_+}{1+\epsilon k_+ t}\cos(2\mu_+ t + b_1\log(1+\epsilon k_+ t) + b_2) + O(t^{-3/2}), \tag{5.3}$$

$$\gamma(t) = \gamma_+ + c_+\log(1+\epsilon k_+ t) + O(t^{-1}), \tag{5.4}$$

where ω_+, k_+ are as defined above, $\mu_+ = \mu(\omega_+)$, and q_+, b_+, b_1,b_2,c_+ are constants.

5.2 Soliton asymptotics

Here we prove the statement (1.19) in our main theorem 1.3. To achieve this we look for the solution $\psi(x,t)$ to (1.1), in the corresponding complex form

$$\psi = s + \mathrm{v} + f, \tag{5.5}$$

where

$$s(x,t) = \psi_{\omega(t)}(x)e^{i\theta(t)}, \quad \dot\theta(t) = \omega(t) + \dot\gamma(t)$$

is the accompanying soliton, and

$$\mathrm{v}(x,t) = v(x,t)e^{i\theta(t)}, \quad v(x,t) = \big(z(t)+\overline{z}(t)\big)u_1(x,\omega(t)) + i\big(z(t)-\overline{z}(t)\big)u_2(x,\omega(t)). \tag{5.6}$$

We aim now to prove the complex form (1.20) of the scattering asymptotics, by analysing f.

Lemma 5.2. *If $\psi(x,t)$ is a solution of (1.1), and f is as in* (5.5) - (5.6), *then*

$$i\dot f = -f'' + R$$

where

$$\begin{aligned} R &= \dot\gamma(s+\mathrm{v}) - i\dot\omega\partial_\omega(s+\mathrm{v}) - i[(\dot z - i\mu z)(u_i + iu_2) + (\dot{\overline{z}} + i\mu\overline{z})(u_1 - iu_2)]e^{i\theta} \\ &\quad -\delta(x)[af + be^{i\theta}\mathrm{Re}\,(e^{-i\theta}f) + \mathcal{O}(|f+\mathrm{v}|^2)]. \end{aligned} \tag{5.7}$$

Proof. Multiply (1.10) by $e^{i\theta}$, we obtain $-\omega s = -s'' - \delta(x)F(s)$ that implies

$$i\dot{s} = -\omega s - \dot{\gamma}s + i\dot{\omega}\partial_\omega s = -s'' - \dot{\gamma}s + i\dot{\omega}\partial_\omega s - \delta(x)F(s), \tag{5.8}$$

By the equation $\mathbf{C}u = i\mu u$, we obtain for the components u_1 and u_2 of vector u,

$$\begin{cases} -u_2'' + \omega u_2 - \delta(x)au_2 = i\mu u_1 \\ -u_1'' + \omega u_1 - \delta(x)[a+b]u_1 = -i\mu u_2. \end{cases}$$

Therefore $-v'' + \omega v - \delta(x)[av + b\mathrm{Re}\, v] = -\mu(z - \overline{z})u_1 - i\mu(z + \overline{z})u_2$ and then

$$\begin{aligned} i\dot{\mathrm{v}} &= -(\omega + \dot{\gamma})ve^{i\theta(t)} + i\dot{\omega}\partial_\omega ve^{i\theta(t)} + \big(i(\dot{z} + \dot{\overline{z}})u_1 - (\dot{z} - \dot{\overline{z}})u_2\big)e^{i\theta} \qquad (5.9)\\ &= -\mathrm{v}'' - \dot{\gamma}\mathrm{v} + i\dot{\omega}\partial_\omega \mathrm{v} - \delta(x)[av + b\mathrm{Re}\, v]e^{i\theta} + i[(\dot{z} - i\mu z)(u_1 + iu_2) + (\dot{\overline{z}} + i\mu\overline{z})(u_1 - iu_2)]e^{i\theta}. \end{aligned}$$

¿From (1.1), (5.8), (5.9) we obtain for the remainder $f(x,t) = \psi(x,t) - s(x,t) - \mathrm{v}(x,t)$

$$\begin{aligned} i\dot{f} &= -f''(x,t) + \dot{\gamma}(s + \mathrm{v}) - i\dot{\omega}\partial_\omega(s + \mathrm{v}) - i[(\dot{z} - i\mu z)(u_i + iu_2) + (\dot{\overline{z}} + i\mu\overline{z})(u_1 - iu_2)]e^{i\theta} \\ &\quad - \delta(x)[af + be^{i\theta}\mathrm{Re}\,(e^{-i\theta}f) + \mathcal{O}(|f + \mathrm{v}|^2)] = -f'' + R, \qquad (5.10) \end{aligned}$$

with R as in (5.7). $\square$

The function $f(x,t)$ which is a solution of 5.10 can be expressed formally as

$$\begin{aligned} f(t) &= W(t)f(0) + \int_0^t W(t-\tau)R(\tau)d\tau \\ &= W(t)\Big(f(0) + \int_0^\infty W(-\tau)R(\tau)d\tau\Big) - \int_t^\infty W(t-\tau)R(\tau)d\tau \qquad (5.11)\\ &= W(t)\phi_+ + r_+(t), \qquad (5.12) \end{aligned}$$

where $W(t)$ is the dynamical group of the free Schrödinger equation. To establish the asymptotic behavior (1.20), it suffices to prove that

$$\phi_+ \in C_b(\mathbb{R}) \cap L^2(\mathbb{R}), \quad \text{and} \quad \|r_+(t)\|_{C_b(\mathbb{R})\cap L^2(\mathbb{R})} = O(t^{-\nu}),\ t \to \infty. \tag{5.13}$$

These assertions follow from the definition (5.7) of the function R, and the following two lemmas. The first lemma studies the contribution to $\phi_+(x)$ and $r_+(x,t)$ from the terms in (5.7) involving $\delta(x)$ which is $\mathcal{O}(t^{-1})$ as $t \to \infty$ by (4.26).

Lemma 5.3. *Let $\Pi(t)$ be a continuous bounded function of $t \geq 0$, with $|\Pi(t)| \leq L_0$ and $|t\Pi(t)| \leq L_1$. Then*

$$\phi(x) := \int_0^\infty W(-\tau)[\delta(\cdot)\Pi(\tau)]d\tau = \int_0^\infty \frac{e^{-ix^2/(4\tau)}}{\sqrt{-4\pi i\tau}}\Pi(\tau)d\tau \in C_b(\mathbb{R}) \cap L^2(\mathbb{R}) \tag{5.14}$$

and for $\nu \in (0, \frac{1}{4})$ there exists $C = C(\nu, L_0, L_1) > 0$ such that the function

$$r(x,t) := \int_t^\infty W(t-\tau)[\delta(\cdot)\Pi(\tau)]d\tau = \int_t^\infty \frac{e^{ix^2/4(t-\tau)}}{\sqrt{4\pi i(t-\tau)}}\Pi(\tau)d\tau \tag{5.15}$$

satisfies $\|r(\cdot,t)\|_{C_b(\mathbb{R})\cap L^2(\mathbb{R})} \leq C(1+t)^{-\nu}$.

Proof. The C_b-properties follow from formulas (5.14) and (5.15) (in fact with $\nu = 1/2$). To prove the L^2-properties, let us change the variable to $\tau = 1/u$ to get:

$$\phi(x) = \frac{1}{\sqrt{-4\pi i}} \int_0^\infty e^{-iux^2/4}\, \eta(u)\, du = \frac{1}{\sqrt{-2i}} \mathcal{F}_{u\to x^2/4}(\eta(u)), \qquad \eta(u) = \Pi(1/u)/u^{3/2}, \quad (5.16)$$

where $\mathcal{F}_{u\to\xi}(f(u)) = \hat{f}(\xi)$ indicates the Fourier transform with argument ξ. By the assumptions on Π we have $|\eta(u)| \le L_0 u^{-3/2}$ as $u \to \infty$, and $|\eta(u)| \le L_1 u^{-1/2}$ as $u \to 0$. Therefore $\eta(u) \in L^p(\mathbb{R})$ for $1 \le p < 2$. It follows from the Hausdorff-Young inequality for the Fourier transform that $\phi \in L^q(\mathbb{R})$ for $q > 2$ *as a function of* $y = x^2$, i.e.

$$\int_0^\infty |\phi(x)|^q x\, dx < \infty, \ \forall q > 2,$$

and hence $\phi \in L^2(\mathbb{R})$, since it is already known to be bounded and continuous. It remains to prove the decay rate of $r(x,t)$ in the norm $L^2(\mathbb{R})$. Let us represent the function r as $r(x,t) = W(t)\rho(x,t)$, where

$$\rho(x,t) = \int_t^\infty W(-\tau)[\delta(\cdot)\Pi(\tau)]d\tau = \frac{1}{\sqrt{-4\pi i}} \int_0^{1/t} e^{-iux^2/4}\, \eta(u)\, du = \frac{1}{\sqrt{-2i}} \mathcal{F}_{u\to x^2/4}(\zeta_t(u)\eta(u)).$$

Here $\zeta_t(u)$ is the characteristic function of the interval $(0, 1/t)$. As above ρ is bounded, but also since

$$\|\zeta_t(u)\eta(u)\|_{L^p} = \Big(\int_0^{1/t} |\eta(u)|^p du\Big)^{1/p} \le L_1 \Big(\int_0^{1/t} u^{-p/2} du\Big)^{1/p} \le C t^{-\frac{1-p/2}{p}}, \quad 1 \le p < 2, \quad t > 1.$$

the Hausdorff-Young inequality implies that for any $q > 2$, and in fact as $t \to \infty$:

$$\|\rho(x,t)(1+|x|)^{1/q}\|_{L^q} \le C(t^{-\frac{1-p/2}{p}}),$$

for some constant $C = C(L_1, p)$, for $q^{-1} + p^{-1} = 1$. The Young inequality then implies that

$$\|\rho(x,t)\|_{L^2} \le \|\rho(x,t)(1+|x|)^{1/q}\|_{L^q} \|(1+|x|)^{-1/q}\|_{L^r} = O(t^{-\frac{1-p/2}{p}}), \ \frac{1}{2} = \frac{1}{q} + \frac{1}{r},$$

if $r > q$. To have $r > q$, we must take $q < 4$, or equivalently $p > 4/3$. Hence, we have $\nu = \dfrac{1-p/2}{p} < 1/4$. ∎

The second lemma studies the contribution to $\phi_+(x)$ and $r_+(x,t)$ from terms without $\delta(x)$ in (5.7). Consider the expansions (3.22), (3.29), (3.33), for $\dot{\omega}(t)$, $\dot{\gamma}(t)$, and $\dot{z}(t) - i\mu z(t)$: the main (quadratic) parts of these contain the terms $z_+^2(t)$, $\overline{z}_+^2(t)$, $z_+(t)\overline{z}_+(t)$, which are $O(t^{-1})$ as $t \to \infty$. The remainders are $O(t^{-3/2})$, and it is straightforward (from the unitarity of W) to bound the contribution of these to ϕ_+ in $C_b \cap L^2$, and to check that these contribute $O(t^{-1/2})$ to r_+ in $C_b \cap L^2$. Thus, without loss of generality, we may replace $\dot{\omega}(t)$, $\dot{\gamma}(t)$, and $\dot{z}(t) - i\mu z(t)$ by the main quadratic parts. We first show how to treat these terms with the phase $\theta(t)$ in (5.7) replaced by $\varphi_+(t) \equiv \omega_+ t$, and then consider the general case in lemma 5.5.

Lemma 5.4. *Let $\Pi(t)$ be one of the functions $z_+^2(t)e^{i\varphi_+(t)}$, $|z_+(t)|^2e^{i\varphi_+(t)}$ or $\overline{z}_+^2(t)e^{i\varphi_+(t)}$, where $z_+(t) = \dfrac{e^{i\mu_+ t}}{(1+\epsilon k_+ t)^{1/2}}$, with $\epsilon k_+ > 0$, $\varphi_+(t) = \omega_+ t$, and let $\psi(x)$ be a bounded continuous function of $x \in \mathbb{R}$, such that $x^2\psi(x) \in L^2(\mathbb{R})$. Then*

$$\int_0^\infty \Pi(\tau)W(-\tau)\psi d\tau \in C_b(\mathbb{R}) \cap L^2(\mathbb{R}) \tag{5.17}$$

and for each $\nu \in (0, \frac{3-\sqrt{5}}{4})$ there exists $C_\nu > 0$ such that

$$\Big\|\int_t^\infty \Pi(\tau)W(t-\tau)\psi d\tau\Big\|_{C_b(\mathbb{R})\cap L^2(\mathbb{R})} \leq C_\nu t^{-\nu}. \tag{5.18}$$

Proof. Since $\|W(t)\psi\|_{C_b} = O(t^{-1/2})$ then C_b- properties are evident, in fact with $\nu = 1/2$. It remains to prove the L^2-properties. Since $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have

$$\begin{aligned} W(t)\psi &= \frac{1}{\sqrt{4\pi i t}}\int e^{i|x-y|^2/4t}\psi(y)dy \\ &= \frac{e^{ix^2/4t}}{\sqrt{4\pi i t}}\int e^{-ixy/2t}\psi(y)dy + \frac{e^{ix^2/4t}}{\sqrt{4\pi i t}}\int e^{-ixy/2t}(e^{iy^2/4t}-1)\psi(y)dy \\ &= \frac{e^{ix^2/4t}}{\sqrt{2it}}\hat{\psi}(x/2t) + \frac{e^{ix^2/4t}}{\sqrt{2it}}\hat{\psi}_t(x/2t), \end{aligned} \tag{5.19}$$

where $\psi_t(y) = (e^{iy^2/4t}-1)\psi(y)$. For $t > 1$ we have, using $|e^{i\theta}-1| \leq \theta$,

$$\frac{1}{\sqrt{2t}}\|\hat{\psi}_t(\cdot/2t)\|_{L^2} = \|\hat{\psi}_t(\cdot)\|_{L^2} = \|\psi_t(\cdot)\|_{L^2} = \Big(\int |(e^{iy^2/4t}-1)\psi(y)|^2 dy\Big)^{1/2} \leq \frac{1}{4t}\|y^2\psi(y)\|_{L^2} \leq \frac{c}{t}.$$

Therefore, using the fact that $|\Pi(\tau)| \leq (1+\epsilon k_+\tau)^{-1}$, we deduce that

$$\int_0^\infty \Pi(\tau)\frac{e^{-ix^2/4\tau}}{\sqrt{-2i\tau}}\hat{\psi}_\tau(-x/2\tau)d\tau \in L^2(\mathbb{R}), \quad \text{and} \quad \int_t^\infty \Pi(\tau)\frac{e^{-ix^2/4\tau}}{\sqrt{-2i\tau}}\hat{\psi}_\tau(-x/2\tau)d\tau = O(t^{-1})$$

in the norm of $L^2(\mathbb{R})$. Hence, to prove (5.17) it suffices to check that

$$\phi(x) := \int_0^\infty \Pi(\tau)\frac{e^{-ix^2/4\tau}}{\sqrt{-2i\tau}}\hat{\psi}(-x/2\tau)d\tau \in L^2(\mathbb{R}), \tag{5.20}$$

and to prove (5.18) it suffices to check that

$$\rho(x,t) := \int_t^\infty \Pi(\tau)\frac{e^{-ix^2/4\tau}}{\sqrt{-2i\tau}}\hat{\psi}(-x/2\tau)d\tau = O(t^{-\nu}), \tag{5.21}$$

for appropriate $\nu > 0$, in the norm of $L^2(\mathbb{R})$.

First, let us prove (5.20) and (5.21) for $\Pi(t) = z_+^2(t)e^{i\varphi_+(t)}$. Note that $|\partial_\tau \hat{\psi}(x/2\tau)| \le \frac{c|x|}{\tau^2}$, because $x\psi(x) \in L^1(\mathbb{R})$, and then $\widehat{\psi'}$ is bounded function. Therefore we obtain (5.20) with help of integration by parts:

$$\begin{aligned} |\phi(x)| &= \Big|\int_0^\infty \frac{\hat{\psi}(-x/2t)\ 4\tau^2}{\sqrt{2\tau}(1+\epsilon k_+\tau)(x^2+4\tau^2(\omega_+ + 2\mu_+))}\partial_\tau\big(e^{-ix^2/4\tau + i(\omega_+ + 2\mu_+)\tau}\big)d\tau\Big| \qquad (5.22)\\ &\le c\int_0^\infty \Big|\partial_\tau\Big(\frac{\hat{\psi}(-x/2t)\tau^{3/2}}{(1+\epsilon k_+\tau)(x^2+4\tau^2(\omega_+ + 2\mu_+))}\Big)\Big|\ d\tau \le \frac{c}{1+|x|} \in L^2(\mathbb{R}). \end{aligned}$$

Similarly, (5.21) follows with $\nu = 1/2$. For the function $\Pi(t) = |z_+(t)|^2 e^{i\varphi_+(t)}$ the proof of (5.20) and (5.21) is similar.

Second, we consider the case $\Pi(t) = \overline{z}_+^2(t)e^{i\varphi_+(t)}$ which is more difficult, because the factor $x^2 + 4\tau^2(\omega_+ - 2\mu_+)$ vanishes for $\tau = t^* := |x|/2\sqrt{2\mu_+ - \omega_+}$. We consider only positive values of x since the negative values can be considered similarly. Let us choose $1/2 < p < 1$ and $1 < q < 4p - 1$. Then for large x we have

$$0 < t^* - x^p < t^* < t^* + x^p < x^q.$$

Let us represent $(0,\infty) = J_1 \cup J_2 \cup J_3$, where $J_1 = (x^q, \infty)$, $J_2 = [t^* - x^p, t^* + x^p]$, and $J_3 = (0,\infty) \setminus J_1 \cup J_2$. Then $\phi = \phi_1 + \phi_2 + \phi_3$, where ϕ_i is obtained by integrating the integrand in (5.20) over $\tau \in J_i$, $i = 1, 2, 3$. For ϕ_1 and ϕ_2, it is immediate, without the need to integrate by parts, that

$$|\phi_1(x)| \le c\int_{x^q}^{\infty} \frac{d\tau}{\tau^{3/2}} = cx^{-q/2}, \quad |\phi_2(x)| \le \frac{c}{x^{3/2}}\int_{t^*-x^p}^{t^*+x^p} d\tau \le cx^{-(3/2-p)}.$$

Therefore $\phi_1, \phi_2 \in L^2(1,\infty)$ since $3/2 - p > 1/2$. Next observe that

$$|x - 2\tau\sqrt{2\mu_+ - \omega_+}| > c\,x^p \quad \text{and} \quad \tau < x^q \qquad \text{for all } \tau \in J_3. \qquad (5.23)$$

Therefore for large x we use integration by parts to obtain

$$\begin{aligned} |\phi_3(x)| &\le cx^{-p} + \int_{J_3} \Big|\partial_\tau\Big(\frac{c\tau^{3/2}}{(1+\epsilon k_+\tau)(x+2\tau\sqrt{2\mu_+ - \omega_+})(x - 2\tau\sqrt{2\mu_+ - \omega_+})}\hat{\psi}(-x/2\tau)\Big)\Big|\ d\tau \\ &\le cx^{-p} + c\int_{J_3} \frac{d\tau}{(1+\tau^{3/2})x^p} + c\int_{J_3} \frac{d\tau}{(1+\tau^{1/2})x^{2p}} \le cx^{-p} + cx^{q/2-2p}. \qquad (5.24) \end{aligned}$$

(The boundary terms arising in this integration by parts can be estimated to be $\le c\,\tau^{1/2}x^{-p}/(x+\tau) \le c\,t^{-1/2}x^{-p}$.) Since $2p - q/2 > 1/2$ it follows that $\phi_3 \in L^2(1,\infty)$.

Similarly, we can estimate ρ. Again we assume, without loss of generality, x to be positive. For bounded x, say $0 \le x \le l$, $l > 0$ we can just estimate the integral (5.21) directly as

$$|\rho(x,t)| \le \Big|\int_t^\infty \frac{c\,d\tau}{\sqrt{\tau}(1+\epsilon k_+\tau)}\Big| \le ct^{-\frac{1}{2}}, \quad x \le l.$$

The decay of $\|\rho(\cdot,t)\|_{L^2}$ follows from this together with a bound of the form $|\rho(x,t)| \le ct^{-\nu}|x|^{-\alpha}$, with $\alpha > \frac{1}{2}$, which is valid for large t and $|x| > l \gg 1$. To obtain such a bound we decompose the integral (5.21) as $\rho = \sum_{i=1}^{3} \rho_i$, where $\rho_i(x,t)$ is obtained by restricting the integral in (5.21) to $\tau \in J_{it} = J_i \cap [t,\infty)$.

To bound ρ_1 there are two cases. Firstly, if $x^q > t$ then $|\rho_1| \le c\int_{x^q}^{\infty} \tau^{-3/2} d\tau \le cx^{-q/2}$. Secondly, if $x^q \le t$ then $|\rho_1| \le c\int_t^{\infty} \tau^{-3/2} d\tau \le ct^{-1/2}$. In both cases $|\rho_1| \le ct^{-\nu}|x|^{-(\frac{1}{2}-\nu)q}$.

To bound ρ_2, we notice that $\tau \ge cx$ on J_{2t}, and also $\tau \ge t$, and then just estimate

$$|\rho_2(x,t)| \le c\int_{J_{2t}} \frac{d\tau}{(1+\epsilon k_+\tau)\sqrt{\tau}} \le c\min \tau^{-3/2}|x|^p \le ct^{-\nu}|x|^{-3/2+p+\nu}.$$

To bound ρ_3, notice that for $\tau \in J_{3t}$ the inequalities (5.23) hold so that it is possible to integrate by parts (as in (5.22) above) since the denominator which appears is bounded below. As with ϕ_3, the boundary terms arising in this integration by parts are $\le c\,\tau^{1/2}x^{-p}/(x+\tau) \le c\,t^{-1/2}x^{-p}$. Also, as in (5.24), the integral which remains after this integration by parts can be bounded as $\le ct^{-1/2}x^{-p} + cx^{q/2-2p} \le ct^{-1/2}x^{-p} + ct^{-\nu}x^{q/2-2p+\nu q}$, since $t \le \tau \le x^q$ in J_{3t}, so that we may assume $t \le x^q$ in the estimation of ρ_3. Therefore, in conclusion we have the following estimate for $x > l \gg 1$ and t large:

$$|\rho(x,t)| \le ct^{-\nu}(x^{-p} + x^{-3/2+p+\nu} + x^{-2p+(1/2+\nu)q} + x^{-(1/2-\nu)q}), \tag{5.25}$$

This shows that $\rho(x,t)$ is square integrable and (5.21) holds if

$$p > 1/2, \quad 3/2 - p - \nu > 1/2, \quad 2p - (1/2+\nu)q > 1/2, \quad (1/2-\nu)q > 1/2$$

The conditions on the exponents can be written equivalently as

$$\frac{1}{1-2\nu} < q < \frac{4p-1}{1+2\nu}, \quad 1/2 < p < 1-\nu. \tag{5.26}$$

Such q and p exist if $0 < \nu < (3-\sqrt{5})/4$, since then $\frac{3-4\nu}{1+2\nu} > \frac{1}{1-2\nu}$, so choosing $p = 1-\nu-\epsilon > \frac{1}{2}$ for ϵ small and positive, and q in the interval (5.26) will work. ∎

Lemma 5.5. *The conclusions of lemma 5.4 are valid if φ_+ is replaced by θ satisfying $\dot{\theta}(\tau) - \dot{\varphi}_+(\tau) = O(\tau^{-1})$ for large τ.*

Proof. The estimates in the proof of lemma 5.4 which involve estimating the integral of the absolute value are completely unaffected by change of phase, so it is only necessary to re-assess the argument involving integration by parts, i.e. the treatment of ϕ_3 and ρ_3. For example, in the more difficult case when $\Pi(t) = \overline{z}_+^2(t)e^{i\theta(t)}$, we proceed as follows in the estimation of ρ_3. Write $\tilde{\phi} = \theta - \phi_+$, and integrate by parts exactly as before, leaving along the $e^{i\tilde{\phi}}$ factor: this factor then carries throught to the integrand after the integration by parts, and we need to bound:

$$\int_{J_{3t}} |\partial_\tau\Big(\frac{c\tau^{3/2}e^{i\tilde{\phi}(\tau)}}{(1+\epsilon k_+\tau)(x+2\tau\sqrt{2\mu_+-\omega_+})(x-2\tau\sqrt{2\mu_+-\omega_+})}\hat{\psi}(-x/2\tau)\Big)|\, d\tau$$

(The treatment of the boundary terms is unaffected since $|e^{i\tilde{\phi}}| = 1$.) But since by assumption $\dot{\tilde{\phi}}(\tau) = O(\tau^{-1})$ the extra contribution to the integrand can clearly be estimated for large x, τ in the same way as the term arising from differentiation of $\tau^{3/2}$, and so (5.25) still holds, as required to complete the proof. ∎

Remark 5.6. *The $t \to -\infty$ case is handled in an identical way.*

A The eigenfunctions of the discrete spectrum

Here we find the function $u = u(\omega)$ satisfying $\mathbf{C}u = \lambda u$, where $\lambda = i\mu$. Using the definition of the operator $\mathbf{C}$, we obtain

$$\begin{pmatrix} -\lambda & -\dfrac{d^2}{dx^2} + \omega \\ \dfrac{d^2}{dx^2} - \omega & -\lambda \end{pmatrix} u = \delta(x) \begin{pmatrix} 0 & a \\ -a-b & 0 \end{pmatrix} u \tag{A. 1}$$

If $x \neq 0$, the equation (A. 1) takes the form

$$\begin{pmatrix} -\lambda & -\dfrac{d^2}{dx^2} + \omega \\ \dfrac{d^2}{dx^2} - \omega & -\lambda \end{pmatrix} u = 0, \qquad x \neq 0. \tag{A. 2}$$

General solution is a linear combination of exponential solutions of type $e^{ikx}v$. Substituting to (A. 2), we get

$$\begin{pmatrix} -\lambda & k^2 + \omega \\ -k^2 - \omega & -\lambda \end{pmatrix} v = 0. \tag{A. 3}$$

For nonzero vectors v, the determinant of the matrix vanishes: $\lambda^2 + (k^2 + \omega)^2 = 0$. Then $k_\pm^2 + \omega = \mp i\lambda$. Finally, we obtain four roots $\pm k_\pm(\lambda)$ with

$$k_\pm(\lambda) = \sqrt{-\omega \mp i\lambda}, \tag{A. 4}$$

where the square root is defined as an analytic continuation from a neighborhood of the zero point $\lambda = 0$ taking the positive value of $\operatorname{Im}\sqrt{-\omega}$ at $\lambda = 0$. We choose the cuts $\mathcal{C}_+$ in the complex plane λ from the branching points to infinity. Then $\operatorname{Im} k_\pm(\lambda) > 0$ for $\lambda \in \mathbb{C} \setminus \mathcal{C}_\pm$.
It remains to derive the vector $v = (v_1, v_2)$ which is solution to (A. 3):

$$v_2 = -\frac{k_\pm^2 + \omega}{\lambda} v_1 = \frac{\pm i\lambda}{\lambda} v_1 = \pm i v_1.$$

Therefore, we have two corresponding vectors $v_\pm = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$ and we get four linearly independent exponential solutions.

$$v_+ e^{\pm ik_+x} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{\pm ik_+x}, \qquad v_- e^{\pm ik_-x} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{\pm ik_-x}.$$

Now we find the solution of (A. 1) in the form

$$u = Ae^{ik_+|x|}v_+ + Be^{ik_-|x|}v_-. \tag{A. 5}$$

At the point $x = 0$ we have a jump:

$$u'(+0) - u'(-0) = -\begin{pmatrix} a+b & 0 \\ 0 & a \end{pmatrix} u(-0) \tag{A. 6}$$

Substituting (A. 5), we get

$$2ik_+Av_+ + 2ik_-Bv_- = -M(Av_+ + Bv_-), \quad M = \begin{pmatrix} a+b & 0 \\ 0 & a \end{pmatrix}. \tag{A. 7}$$

Note that

$$\begin{cases} Mv_+ = \alpha v_+ + \beta v_- \\ \\ Mv_- = \alpha v_- + \beta v_+ \end{cases}, \qquad \text{where} \quad \alpha = a + \frac{b}{2}, \quad \beta = \frac{b}{2}.$$

Then (A. 7) becomes

$$\begin{cases} (2ik_+ + \alpha)A + \beta B = 0 \\ \\ \beta A + (2ik_- + \alpha)B = 0 \end{cases}$$

The determinant $D = (2ik_+ + \alpha)(2ik_- + \alpha) - \beta^2$ vanishes for $\lambda = i\mu$ since $i\mu$ is spectral point. Therefore, set $A = 1$, and obtain

$$u = e^{ik_+|x|}v_+ - \frac{\beta}{2ik_- + \alpha}e^{ik_-|x|}v_- \tag{A. 8}$$

Note, that $2ik_- + \alpha \neq 0$. Indeed, if $2ik_- + \alpha = 0$, then $\beta = 0$, $\alpha = a$. and $2ik_- + \alpha = -2\sqrt{\omega + \mu} + a = -\sqrt{a^2 + 4\mu} + a < 0$.
Since both $k_+ = \sqrt{-\omega + \mu}$ and $k_- = \sqrt{-\omega - \mu}$ are purely imaginary, the first component u_1 is real, while the second one u_2 is imaginary. It is easy to prove that $u^* = (u_1, -u_2)$ is the eigenfunction associated to $\lambda = -i\mu$.

B The eigenfunctions of the continuous spectrum

Consider $\lambda = i\nu$ with some $\nu > \omega$.
I. First we find an even solution $u = \tau_+$ of equation (A. 1) in the form

$$\tau_+ = (Ae^{ik_+|x|} + Be^{-ik_+|x|})v_+ + Ce^{ik_-|x|}v_-. \tag{B. 1}$$

At the point $x = 0$ we have, similarly (A. 6) and (A. 7),

$$2ik_+(A - B)v_+ + 2ik_-Cv_- = -M(A + B)v_+ + Cv_-$$

which equivalent to the system

$$\begin{cases} 2ik_+(A - B) = -\alpha(A + B) - \beta C \\ \\ 2ik_-C = -\beta(A + B) - \alpha C \end{cases} \tag{B. 2}$$

If $2ik_- + \alpha = 0$, then $D = -\beta^2$, $\beta \neq 0$, $A + B = 0$, and $C = -\dfrac{2ik_+}{\beta}(A - B)$. Put $A = D$ we obtain $B = -D$, $C = 4\beta ik_+$.
If $2ik_- + \alpha \neq 0$, then from the second equation of (B. 2) we get

$$C = -\frac{\beta}{\alpha + 2ik_-}(A + B). \tag{B. 3}$$

Then, set $A = \overline{D}$, we obtain from the first equation of (B. 2).

$$2ik_+(\overline{D} - B) = \frac{-\alpha(\alpha + 2ik_-) + \beta^2}{\alpha + 2ik_-}(\overline{D} + B).$$

Solving this equation, we get $B = -D$ and then $C = 4\beta i k_+$. Finally, we obtain

$$\tau_+ = (\overline{D} e^{ik_+|x|} - De^{-ik_+|x|})v_+ + 4\beta i k_+ e^{ik_-|x|} v_-. \tag{B. 4}$$

II. It is easily to check that an odd solution $u = s$ of equation (A. 1) is

$$s = \frac{1}{2i}(e^{ik_+x} - e^{-ik_+x})v_+ = \sin(k_+x)v_+. \tag{B. 5}$$

For $\lambda = i\nu$ with $\nu < -\omega$ we have similarly

$$\tau_- = (\overline{D} e^{ik_-|x|} - De^{-ik_-|x|})v_- + 4\beta i k_- e^{ik_+|x|} v_+, \quad s = \sin(k_-x)v_-.$$

C Proof of Proposition 2.3

First we prove the following lemma

Lemma C.1. *Let* $\mathbf{K}(t)$ *be the integral operator with the kernel*

$$K(t,x,y) = \int\limits_{|\nu-\nu_0|<\delta} \frac{f(\nu,x,y) - f(\nu_0,x,y)}{\nu - \nu_0}\, d\nu, \tag{C. 1}$$

where f *is a smooth function,* $f(\nu,x,y) = 0$ *for* $|\nu - \nu_0| \geq \delta$*, and*

$$|\partial_\nu^N f(\nu,x,y)| \leq c(N)(|x| + |y|)^N, \quad N = 0,1,2,... \tag{C. 2}$$

Then for any $\sigma > N$

$$\|\mathbf{K}(t)\|_{\mathcal{B}_\sigma} = \mathcal{O}(t^{-N}), \quad t \to \infty, \quad \text{with} \quad \sigma > N. \tag{C. 3}$$

Proof. For $z = |x| + |y| > \delta$ we split the integral in the right hand site of (C. 1) into two part and obtain

$$|K(t,x,y)| \leq \int\limits_{|\nu-\nu_0|<\frac{1}{z}} |\frac{f(\nu,x,y) - f(\nu_0,x,y)}{\nu - \nu_0}|\, d\nu + \int\limits_{\frac{1}{z}<|\nu-\nu_0|<\delta} |\frac{f(\nu,x,y) - f(\nu_0,x,y)}{\nu - \nu_0}|\, d\nu$$

$$\leq \frac{1}{z}c(1)z + 2c(0)\int\limits_{1/z}^{\delta} \frac{d\nu}{\nu} = c(1) + 2c(0)[\log z + c].$$

Hence, (C. 3) follows with $N = 0$. Equality (C. 3) for arbitrary $N \geq 1$ can be obtained by similar way, using the integration by parts and inequalities (C. 2). $\square$

Proof of Proposition 2.3 The operator $e^{\mathbf{C}t}(\mathbf{C} - 2i\mu - 0)^{-1}$ admits the Laplace representation

$$e^{\mathbf{C}t}(\mathbf{C} - 2i\mu - 0)^{-1} = -\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} e^{\lambda t} R(\lambda + 0)\, d\lambda\, R(2i\mu + 0).$$

Let us apply the Hilbert identity for the resolvent:

$$R(\lambda_1)R(\lambda_2) = \frac{1}{\lambda_1 - \lambda_2}[R(\lambda_1) - R(\lambda_2)],\ \mathrm{Re}\,\lambda_k > 0,\ k = 1, 2$$

to $\lambda_1 = \lambda + 0$ and $\lambda_2 = 2i\mu + 0$. We obtain

$$e^{\mathbf{C}t}(\mathbf{C} - 2i\mu - 0)^{-1} = -\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} e^{\lambda t}\frac{R(\lambda+0) - R(2i\mu+0)}{\lambda - 2i\mu}\,d\lambda$$

$$= -\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} e^{\lambda t}\zeta(\lambda)\frac{R(\lambda+0) - R(2i\mu+0)}{\lambda - 2i\mu}\,d\lambda - \frac{1}{2\pi i}\int\limits_{\mathcal{C}_+\cup\mathcal{C}_-} e^{\lambda t}(1-\zeta(\lambda))\frac{R(\lambda+0) - R(2i\mu+0)}{\lambda - 2i\mu}\,d\lambda$$

$$-\frac{1}{2\pi i}\int\limits_{(-i\infty,i\infty)\setminus(\mathcal{C}_+\cup\mathcal{C}_-)} e^{\lambda t}(1-\zeta(\lambda))\frac{R(\lambda+0) - R(2i\mu+0)}{\lambda - 2i\mu}\,d\lambda = \mathbf{K}_1(t) + \mathbf{K}_2(t) + \mathbf{K}_3(t),$$

where $\zeta(\lambda) \in C_0^\infty(i\mathbb{R})$, $\zeta(\lambda) = 1$ for $|\lambda - 2i\mu| < \delta/2$ and $\zeta(\lambda) = 0$ for $|\lambda - 2i\mu| > \delta$, with $0 < \delta < 2\mu - \omega$. By Lemma C.1 with $N = 2$, we obtain that

$$\|\mathbf{K}_1(t)\|_{\mathcal{B}_\beta} = \mathcal{O}(t^{-2}), \quad t \to \infty,$$

since $\beta > 2$. The bounds (C. 2) for $f(\nu) = R(\lambda + 0)$ follow from formulas (2.3)- (2.5).
For the operator $\mathbf{K}_2(t)$ we can apply the arguments from the proof of proposition 2.2 and obtain

$$\|\mathbf{K}_2(t)\|_{\mathcal{B}_\beta} = \mathcal{O}(t^{-3/2}), \quad t \to \infty.$$

Further, the integrand in $\mathbf{K}_3(t)$ is an analytic function of $\lambda \neq 0, \pm i\mu$ with the values in $\mathcal{B}_\beta$ for $\beta \geq 0$. At the points $\lambda = 0$ and $\lambda = \pm i\mu$ the integrand has the poles of finite order. Hoverever, all the Laurent coefficients vanish when applied to $\mathbf{P}^c h \in X^c$. Hence for $\mathbf{K}_3(t)$ we obtain, twice integrating by parts,

$$\|\mathbf{K}_3(t)\mathbf{P}^c h\|_{L^\infty_{-\beta}} \leq c(1+t)^{-3/2}\|h\|_{\mathcal{M}_\beta},$$

completing the proof. □

D Proof of Lemma 3.5

We use the following representation (see [1]):

$$\mathbf{P}^c_T = \frac{1}{2\pi i}\int\limits_{\mathcal{C}_+\cup\mathcal{C}_-}(\mathbf{R}(\lambda+0) - \mathbf{R}(\lambda-0))\,d\lambda$$

$$= \frac{1}{2\pi i}\int\limits_{\mathcal{C}_+}(\mathbf{R}(\lambda+0) - \mathbf{R}(\lambda-0))d\lambda + \frac{1}{2\pi i}\int\limits_{\mathcal{C}_-}(\mathbf{R}(\lambda+0) - \mathbf{R}(\lambda-0))d\lambda = \mathbf{\Pi}^+_T + \mathbf{\Pi}^-_T. \quad \text{(D. 1)}$$

Let us decompose the resolvent given by (2.4), and (2.5)) as

$$\mathbf{R}(\lambda, x, y) = \Gamma(\lambda, x, y) + P(\lambda, x, y) = \sum_{k=1}^{6} A_k(\lambda, x, y)\tau_k, \quad \text{(D. 2)}$$

where

$$A_1 = \frac{e^{ik_+|x-y|} - e^{ik_+(|x|+|y|)}}{4k_+}, \quad A_2 = \frac{i\alpha - 2k_-}{2D} e^{ik_+(|x|+|y|)}, \quad A_3 = \frac{i\beta}{2D} e^{ik_+|x|} e^{ik_-|y|},$$

$$A_4 = -\frac{i\beta}{2D} e^{ik_-|x|} e^{ik_+|y|}, \quad A_5 = \frac{-i\alpha + 2k_+}{2D} e^{ik_-(|x|+|y|)}, \quad A_6 = -\frac{e^{ik_-|x-y|} - e^{ik_-(|x|+|y|)}}{4k_-}, \text{(D. 3)}$$

and

$$\tau_1 = \tau_2 = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \ \tau_3 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \ \tau_4 = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \ \tau_5 = \tau_6 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \quad \text{(D. 4)}$$

For the matrices τ_k we obtain

$$\tau_k j^{-1} = i\tau_k, \ k = 1, 2, 4, \quad \tau_k j^{-1} = -i\tau_k, \ k = 3, 5, 6.$$

The terms A_1 and A_6 disappear when substituting (D. 2)-(D. 4) into (D. 1) since

$$A_1(\lambda + 0) - A_1(\lambda - 0) = 0, \ \lambda \in \mathcal{C}_-; \quad A_6(\lambda + 0) - A_6(\lambda - 0) = 0, \ \lambda \in \mathcal{C}_+,$$

and we get

$$P_T^c j^{-1} - i(\Pi_T^+ - \Pi_T^-) = \frac{1}{2\pi i} \int_{\mathcal{C}_+} [2(A_3(\lambda+0) - A_3(\lambda-0))\tau_3 + 2(A_5(\lambda+0) - A_5(\lambda-0))\tau_5] \ d\lambda$$

$$+\frac{1}{2\pi i} \int_{\mathcal{C}_-} [2(A_2(\lambda+0) - A_2(\lambda-0))\tau_2 + 2(A_4(\lambda+0) - A_4(\lambda-0))\tau_4] \ d\lambda.$$

Let us consider only the integral over $\mathcal{C}_+$; the integral over $\mathcal{C}_-$ can be dealt with by an identical argument. For $\lambda \in \mathcal{C}_+$ we have: $k_+ = \sqrt{-\omega - i\lambda}$ is real, and $k_+(\lambda + 0) = -k_+(\lambda - 0)$ while $k_- = \sqrt{-\omega + i\lambda}$ is pure imaginary with $\operatorname{Im} k_- > 0$ and $k_-(\lambda + 0) = k_-(\lambda - 0)$.

Note that $A_5(\lambda, x, y)$ for $\lambda \in \mathcal{C}_+$ exponentially decay if $|x|$, $|y| \to \infty$ and smallest exponential rate of the decaying is equal to $(2\omega)^{1/2}$.

It remains to consider the integral over $\mathcal{C}_+$ with integrand $A_3(\lambda+0) - A_3(\lambda-0)$. We change variable: $\zeta = \sqrt{-\omega - i\lambda}$ for the first summand and $\zeta = -\sqrt{-\omega - i\lambda}$ for the second. Then we get

$$I(x, y) = \int_{\mathcal{C}_+} (A_3(\lambda + 0) - A_3(\lambda - 0)) \ d\lambda = \frac{i\beta}{2} \int_{i\omega}^{\infty} e^{-\sqrt{\omega - i\lambda}|y|} \left(\frac{e^{i\sqrt{-\omega - i\lambda}|x|}}{D_+} - \frac{e^{-i\sqrt{-\omega - i\lambda}|x|}}{D_-}\right) d\lambda$$

$$= -\beta \int_{-\infty}^{+\infty} \frac{e^{-\sqrt{2\omega + \zeta^2}|y|} e^{i\zeta|x|}}{D(\zeta)} \ \zeta d\zeta,$$

where

$$D_\pm = \alpha^2 - \beta^2 \pm 2i\alpha\sqrt{-\omega - i\lambda} - 2\alpha\sqrt{\omega - i\lambda} \mp 4i\sqrt{-\omega - i\lambda}\sqrt{\omega - i\lambda}$$

and

$$D(\zeta) = \alpha^2 - \beta^2 + 2i\alpha\zeta - 2\alpha\sqrt{2\omega + \zeta^2} - 4i\zeta\sqrt{2\omega + \zeta^2}.$$

Writing $e^{i\zeta|x|} \ d\zeta = \frac{1}{i|x|} de^{i\zeta|x|}$, and integrating by parts, we get that

$$|I(x, y)| \le Ce^{-\sqrt{2\omega}|y|}(1 + |x|)^{-m}, \ \forall m \in \mathbf{N},$$

completing the proof of lemma 3.5.

E The Fermi Golden Rule

In this section we study further the condition embodied in the Fermi Golden Rule (1.17) , showing that it holds generically in a certain sense: in particular, if $a(\cdot)$ is a polynomial function then, generically, the set of values of C for which (1.17) fails is isolated. To start with let us use the formulae in Appendices A and B to express the Fermi Golden Rule in terms of C and $a(C^2)$ and its derivatives. By (B. 4)

$$\begin{aligned} \tau_+(2i\mu)\mid_{x=0} &= (\overline{D}-D)v_+ + 4\beta ik_+v_- = -4ik_+(\alpha+2ik_-)v_+ + 4\beta ik_+v_- \qquad \text{(E. 1)} \\ &= \sigma(\kappa v_+ + v_-) = \sigma\begin{pmatrix} \kappa+1 \\ i(\kappa-1)\end{pmatrix}, \quad \sigma = 4\beta ik_+, \quad \kappa = -\frac{\alpha+2ik_-}{\beta}, \quad k_\pm = \sqrt{-\omega\pm 2\mu}. \end{aligned}$$

Represent $E_2[u,u] = \delta(x)\tilde{E}_2[u(0),u(0)]$, where

$$\tilde{E}_2[u(0),u(0)] = a'(C^2)(u(0),u(0))\Psi(0) + 2a''(C^2)(\Psi(0),u(0))^2\Psi(0) + 2a'(C^2)(\Psi(0),u(0))u(0)$$

By (A. 8)

$$u(0) = \rho v_+ + v_- = \begin{pmatrix} \rho+1 \\ i(\rho-1)\end{pmatrix}, \quad \rho = -\frac{2ik_-+\alpha}{\beta}, \quad k_- = \sqrt{-\omega-\mu}.$$

Therefore $(u(0),u(0)) = (\rho+1)^2 - (\rho-1)^2 = 4\rho$ and

$$\tilde{E}_2[u(0),u(0)] = a'(C^2)4\rho\begin{pmatrix} C \\ 0\end{pmatrix} + 2a''(C^2)C^2(\rho+1)^2\begin{pmatrix} C \\ 0\end{pmatrix} + 2a'(C^2)C(\rho+1)\begin{pmatrix} \rho+1 \\ i(\rho-1)\end{pmatrix} \qquad \text{(E. 2)}$$

Using (E. 1) and (E. 2), we obtain

$$\langle\tau_+(2i\mu), E_2[u,u]\rangle = \sigma(\kappa+1)\Big[a'4\rho C + 2a''C^3(\rho+1)^2 + 2a'C(\rho+1)^2\Big] + \sigma(\kappa-1)2a'C(\rho^2-1)$$

Therefore, the Fermi Golden Rule (1.17) is equivalent to the condition

$$a'\Big[(\kappa+1)2\rho + (\kappa+1)(\rho+1)^2 + (\kappa-1)(\rho^2-1)\Big] + a''(\kappa+1)C^2(\rho+1)^2 \neq 0$$

or

$$a'' \neq -\frac{2a'(2\kappa\rho+2\rho+\kappa\rho^2+1)}{C^2(\kappa+1)(1+\rho)^2} \qquad \text{(E. 3)}$$

with

$$\kappa = -\frac{a+\frac{b}{2}-2\sqrt{\omega+2\mu}}{b/2}, \quad \rho = -\frac{a+\frac{b}{2}-2\sqrt{\omega+\mu}}{b/2}, \quad \omega = \frac{a^2}{4}, \quad \mu = \frac{b}{4}\sqrt{a^2-\frac{b^2}{4}}$$

Let $\nu = \frac{b}{2a}$. Then $\mu = \frac{\nu a}{2}\sqrt{a^2-\nu^2a^2} = \frac{\nu a^2}{2}\sqrt{1-\nu^2}$ and

$$\kappa = -\frac{a+\nu a-2\sqrt{\frac{a^2}{4}+\nu a^2\sqrt{1-\nu^2}}}{\nu a} = -\frac{1+\nu-\sqrt{1+4\nu\sqrt{1-\nu^2}}}{\nu},$$

$$\rho = -\frac{a+\nu a-2\sqrt{\frac{a^2}{4}+\frac{\nu a^2}{2}\sqrt{1-\nu^2}}}{\nu a} = -\frac{1+\nu-\sqrt{1+2\nu\sqrt{1-\nu^2}}}{\nu}$$

are also functions of ν only. Thus we conclude that there is a function $c = c(\nu)$ such that the Fermi Golden Rule holds if and only if

$$a''(C^2) \neq \frac{c(\nu)a'(C^2)}{C^2}, \quad \nu = \frac{a'(C^2)C^2}{a(C^2)}.$$

Notice that the function $c(\nu)$ is algebraic. It follows from this that if the function $a(\cdot)$ is polynomial, or even real analytic, then generically the Fermi Golden Rule holds except possibly at a discrete set of values of C. To see this observe that if the set of points where $F(C^2) \equiv a''(C^2) - \frac{c(\nu)a'(C^2)}{C^2}$ vanishes has an accumulation point, then F must be identically zero since it is real analytic. But the condition $a''(C^2) = \frac{c(\nu)a'(C^2)}{C^2}$ is a second order ordinary differential equation which determines the function $a(C^2)$ given its value $a(C_0^2)$ and that of its first derivative $a'(C_0^2)$ at any point $C = C_0$. Clearly a generic polynomial function $a(C^2)$ will not satisfy this equation, and so the set of points where the Fermi Golden Rule fails cannot have any accumulation points generically. This is also true for real analytic a in the following sense. Fix any C_0, then there is a two parameter family of functions $a(C^2)$ for which $a''(C^2) = \frac{c(\nu)a'(C^2)}{C^2}$; (this family of exceptional functions is parameterized by $a(C_0), a'(C_0)$). If $a(C^2)$ is not one of these functions then the set of values for which the Fermi Golden Rule fails is at most a discrete set.

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