*Functional Analysis and Its Applications, Vol. 47, No. 1, pp. 21–26, 2013*
*Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 47, No. 1, pp. 26–32, 2013*


# On the Titchmarsh Convolution Theorem for Distributions on the Circle

**A. A. Komech and A. I. Komech**

Received January 9, 2011

Abstract. We prove a version of the Titchmarsh convolution theorem for distributions on the circle. We show that a certain "naïve" form of the Titchmarsh theorem can be violated, but only for the convolution of distributions with certain symmetry properties.

Key words: Titchmarsh convolution theorem, symmetry properties, periodic distributions.

**1. Introduction.** The Titchmarsh convolution theorem [6] states that, for any two compactly supported distributions $f, g \in \mathscr{E}'(\mathbb{R})$,

$$\inf\operatorname{supp} f * g = \inf\operatorname{supp} f + \inf\operatorname{supp} g \quad \text{and} \quad \sup\operatorname{supp} f * g = \sup\operatorname{supp} f + \sup\operatorname{supp} g. \tag{1}$$

Its higher-dimensional analogue due to Lions [4] states that, for $f, g \in \mathscr{E}'(\mathbb{R}^n)$, the convex hull of the support of $f * g$ is equal to the sum of the convex hulls of the supports of $f$ and $g$. Different proofs of the Titchmarsh convolution theorem are contained in [7, Chap. VI] (real-analytic), [1, Theorem 4.3.3] (harmonic-analytic), and [3, Lect. 16, Theorem 5] (complex-analytic).

In this note we generalize the Titchmarsh theorem to periodic distributions, which we consider as distributions on the circle (to be more precise, on the torus) $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$.

First, we note that there are zero divisors in the algebra of distributions on the circle under convolution. Indeed, for any two distributions $f, g \in \mathscr{E}'(\mathbb{T})$, one has

$$(f + S_\pi f) * (g - S_\pi g) = f * g + S_\pi(f * g) - S_\pi(f * g) - f * g = 0. \tag{2}$$

Here $S_y$, $y \in \mathbb{T}$, is the shift operator defined on $\mathscr{E}'(\mathbb{T})$ by the relation

$$(S_y f)(\omega) = f(\omega - y), \tag{3}$$

which is understood in the sense of distributions. Yet, the cases when the Titchmarsh convolution theorem "does not hold" (in some naïve form) can be explicitly described. This leads to a version of the Titchmarsh convolution theorem for distributions on the circle (Theorem 1 below).

Our interest in properties of a convolution on the circle is due to applications to the theory of attractors for finite difference approximations of nonlinear dispersive equations. In [2] we considered the weak attractor of finite energy solutions to the $\mathbf{U}(1)$-invariant Klein–Gordon equation in 1D coupled to a nonlinear oscillator. We proved that the global attractor of all finite energy solutions is formed by the set of all solitary waves $\phi_\omega(x)e^{-i\omega t}$ with $\omega \in \mathbb{R}$ and $\phi_\omega \in H^1(\mathbb{R})$. The general strategy of the proof was to consider the omega-limit trajectories of a finite energy solution $\psi(x,t) \in \mathbb{C}$, which are defined as solutions with Cauchy data at the omega-limit points of the set $\{(\psi(t), \dot\psi(t)) : t \geqslant 0\}$ in the local energy seminorms. To prove the convergence to the set of soliton solutions, we showed that the time spectrum of each omega-limit trajectory is inside the spectral gap. Applying the Titchmarsh convolution theorem to the equation satisfied by an omega-limit trajectory, we conclude that its time spectrum consists of at most one frequency, and hence any omega-limit trajectory is a solitary wave (or zero). We expect that this approach can be generalized to the Klein–Gordon equation in discrete space-time [5]. The main difference is that, in this case, the frequency domain is the circle (rather than the real line) and there are two (rather than one) spectral gaps in the continuous spectrum. Thus, the convolution theorem is to be applied to distributions with frequency support in the two spectral gaps, and analyzing the time spectrum of an omega-limit trajectory requires a version of the Titchmarsh convolution theorem for distributions supported on two intervals of the circle.

**Acknowledgments.** The authors are grateful to E. A. Gorin for his interest and support.

**2. Main results.** For $I \subset \mathbb{T}$ and $n \in \mathbb{N}$, we set

$$\mathscr{R}_n(I) = \bigcup_{k\in\mathbb{Z}_n} S_{2\pi k/n} I, \quad \text{where } \mathbb{Z}_n = \mathbb{Z} \bmod n.$$

Let $f, g \in \mathscr{E}'(\mathbb{T})$, and let $I, J \subset \mathbb{T}$ be two closed intervals such that $\operatorname{supp} f \subset \mathscr{R}_n(I)$ and $\operatorname{supp} g \subset \mathscr{R}_n(J)$; suppose also that there is no closed interval $I' \subsetneq I$ for which $\operatorname{supp} f \subset \mathscr{R}_n(I')$ and there is no closed interval $J' \subsetneq J$ for which $\operatorname{supp} g \subset \mathscr{R}_n(J')$.

**Remark 1.** For $f, g \in \mathscr{E}'(\mathbb{T})$, the intervals $I$ and $J$ are analogues of the convex hulls of supports.

**Theorem 1** (Titchmarsh theorem for distributions on the circle)**.** *Let $n \in \mathbb{N}$, $n \geqslant 2$. Assume that*

$$|I| + |J| < 2\pi/n. \tag{4}$$

*Let $K \subset I+J \subset \mathbb{T}$ be a closed interval such that $\operatorname{supp} f*g \subset \mathscr{R}_n(K)$. If $\lambda := \inf K - \inf I - \inf J > 0$, then there are $\alpha, \beta \in \mathbb{C}$ such that $\alpha^n = \beta^n = 1$, $\alpha \neq \beta$,*

$$\left.\begin{aligned} \Big(\sum_{k\in\mathbb{Z}_n} \alpha^k S_{2\pi k/n} f\Big)\Big|_{(\sup I - 2\pi/n, \inf I + \lambda)} &= 0, \\ \inf\operatorname{supp} \Big(\sum_{k\in\mathbb{Z}_n} \alpha^k S_{2\pi k/n} g\Big)\Big|_{(\sup J - 2\pi/n, \inf J + \lambda)} &= \inf J \end{aligned}\right\} \tag{5}$$

*and*

$$\left.\begin{aligned} \inf\operatorname{supp} \Big(\sum_{k\in\mathbb{Z}_n} \beta^k S_{2\pi k/n} f\Big)\Big|_{(\sup I - 2\pi/n, \inf I + \lambda)} &= \inf I, \\ \Big(\sum_{k\in\mathbb{Z}_n} \beta^k S_{2\pi k/n} g\Big)\Big|_{(\sup J - 2\pi/n, \inf J + \lambda)} &= 0. \end{aligned}\right\} \tag{6}$$

**Remark 2.** Relations (6) follow from (5) due to the symmetric role of $f$ and $g$. The conclusion $\alpha \neq \beta$ follows by comparing (5) and (6). Indeed, the first relation in (5) implies that

$$\inf\operatorname{supp} \Big(\sum_{k\in\mathbb{Z}_n} \alpha^k S_{2\pi k/n} f\Big)\Big|_I \geqslant \inf I + \lambda > \inf I,$$

which would contradict the first relation in (6) if we had $\alpha = \beta$.

Applying the reflection to $\mathbb{T}$, we also obtain the following result.

**Corollary 1.** *If $\rho := \sup I + \sup J - \sup K > 0$, then there are $\alpha, \beta \in \mathbb{C}$ such that $\alpha^n = \beta^n = 1$, $\alpha \neq \beta$,*

$$\left.\begin{aligned} \Big(\sum_{k\in\mathbb{Z}_n} \alpha^k S_{2\pi k/n} f\Big)\Big|_{(\sup I - \rho, \inf I + 2\pi/n)} &= 0, \\ \sup\operatorname{supp} \Big(\sum_{k\in\mathbb{Z}_n} \alpha^k S_{2\pi k/n} g\Big)\Big|_{(\sup J - \rho, \inf J + 2\pi/n)} &= \sup J \end{aligned}\right\} \tag{7}$$

*and*

$$\left.\begin{aligned} \sup\operatorname{supp}\Bigl(\sum_{k\in\mathbb{Z}_n}\beta^k S_{2\pi k/n}f\Bigr)\Bigr|_{(\sup I-\rho,\inf I+2\pi/n)} &= \sup I,\\ \Bigl(\sum_{k\in\mathbb{Z}_n}\beta^k S_{2\pi k/n}g\Bigr)\Bigr|_{(\sup J-\rho,\inf J+2\pi/n)} &= 0. \end{aligned}\right\} \tag{8}$$

That is, if $K \subsetneq I + J$ (informally, we could say that a certain naïve form of the Titchmarsh convolution theorem is not satisfied), then both $f$ and $g$ have certain symmetry properties on $\mathscr{R}_n(U)$ and $\mathscr{R}_n(V)$, where $U$ and $V$ are disjoint closed intervals, which can be chosen so that $U \cup K = V \supset I + J$.

In the case $n = 2$, we obtain the following result.

**Corollary 2.** *Suppose that* $n = 2$, $f, g \in \mathscr{E}'(\mathbb{T})$, *and* $I$, $J$, *and* $K$ *are the same as in Theorem* 1. *Then* $\lambda := \inf K - \inf I - \inf J > 0$ *if and only if there is an* $\alpha = \pm 1$ *such that*

$$(f + \alpha S_\pi f)|_{(\sup I-\pi,\inf I+\lambda)} = 0 \qquad \textit{and} \qquad (g - \alpha S_\pi g)|_{(\sup J-\pi,\inf J+\lambda)} = 0.$$

**Proof.** The "only if" part follows from Theorem 1. We check the "if" part by direct computation. Let $f \in \mathscr{E}'(I \cup S_\pi I)$, where $I \subset \mathbb{T}$, $|I| < \pi/2$, and let $g \in \mathscr{E}'(J \cup S_\pi J)$, where $J \subset \mathbb{T}$, $|J| < \pi/2$; assume that $f = \pm S_\pi f$ on $(\sup I - \pi, \inf I + \lambda)$ and $g = \mp S_\pi g$ on $(\sup J - \pi, \inf J + \lambda)$. Then, as in (2),

$$\begin{aligned} (f * g)|_{(\sup I+\sup J-2\pi,\inf I+\inf J+\lambda)} &= f|_{(\sup I-\pi,\inf I+\lambda)} * g|_{(\sup J-\pi,\inf J+\lambda)}\\ &\qquad + (S_\pi f)|_{(\sup I-\pi,\inf I+\lambda)} * (S_\pi g)|_{(\sup J-\pi,\inf J+\lambda)}\\ &= f|_{(\sup I-\pi,\inf I+\lambda)} * g|_{(\sup J-\pi,\inf J+\lambda)}\\ &\qquad - f|_{(\sup I-\pi,\inf I+\lambda)} * g|_{(\sup J-\pi,\inf J+\lambda)} = 0. \end{aligned}$$ □

We set $f^\sharp(\omega) = f(-\omega)$. Let $f \in \mathscr{E}'(\mathbb{T})$, and let $I \subset \mathbb{T}$ be a closed interval such that $\operatorname{supp} f \subset \mathscr{R}_2(I)$. Assume that there is no closed interval $I' \subsetneq I$ for which $\operatorname{supp} f \subset \mathscr{R}_2(I')$.

**Theorem 2.** *If* $I \subset (-\pi/2, \pi/2)$ *and* $|I| < \pi/2$, *then the inclusion* $\operatorname{supp} f * f^\sharp \subset \{0; \pi\}$ *implies that* $\operatorname{supp} f \subset \{\inf I; \sup I; \pi + \inf I; \pi + \sup I\}$. *Moreover, there are distributions* $\mu, \nu \in \mathscr{E}'(\mathbb{T})$, *each supported at a point, such that*

$$f = \mu + S_\pi \mu + \nu - S_\pi \nu. \tag{9}$$

**Remark 3.** Theorem 2 remains valid if we define $f^\sharp(\omega) = \overline{f(-\omega)}$. The proof does not change.

Finally, let us formulate the convolution theorem for powers of a distribution. Let $f \in \mathscr{E}'(\mathbb{T})$. Take a closed interval $I \subset \mathbb{T}$ such that $\operatorname{supp} f \subset \mathscr{R}_n(I)$ and there is no $I' \subsetneq I$ for which $\operatorname{supp} f \subset \mathscr{R}_n(I')$.

**Theorem 3** (Titchmarsh theorem for powers of a distribution on the circle)**.** *Assume that* $|I| < 2\pi/pn$ *for some* $p \in \mathbb{N}$. *Then the smallest closed interval* $K \subset pI$ *such that* $\operatorname{supp} f^{*p} \subset \mathscr{R}_n(K)$ *is* $K = pI$.

Above, we used the notations $pI = \underbrace{I + \dots + I}_{p}$ and $f^{*p} = \underbrace{f * \dots * f}_{p}$.

**3. Proofs.** First, we prove the following lemma.

**Lemma 1.** *Let* $f_j \in \mathscr{E}'(I)$, $j \in \mathbb{Z}_n$. *Then there is an* $\alpha \in \mathbb{C}$ *such that* $\alpha^n = 1$ *and*

$$\inf\operatorname{supp}\sum_{j\in\mathbb{Z}_n}\alpha^j f_j = \min_{j\in\mathbb{Z}_n}\inf\operatorname{supp} f_j. \tag{10}$$

**Proof.** We set $a := \min_{j\in\mathbb{Z}_n}\inf\operatorname{supp} f_j$. Let us assume that, contrary to the statement of the lemma, there is an $\epsilon > 0$ such that $\inf\operatorname{supp}\sum_{j\in\mathbb{Z}_n}\alpha^j f_j \geqslant a + \epsilon$ for any $\alpha = \gamma^m$, where

$\gamma = \exp(2\pi i/n)$ and $m \in \mathbb{N}$, $1 \leqslant m \leqslant n$. Then, for any test function $\varphi \in \mathscr{D}(\mathbb{R})$ with $\operatorname{supp}\varphi \subset (a-\epsilon, a+\epsilon)$, we have

$$0 = \Big\langle \varphi, \sum_{j\in\mathbb{Z}_n} \gamma^{jm} f_j \Big\rangle = \sum_{j\in\mathbb{Z}_n} \gamma^{jm} \langle \varphi, f_j \rangle, \qquad 1 \leqslant m \leqslant n. \tag{11}$$

Using the formula for the Vandermonde determinant, we obtain

$$\det \begin{bmatrix} 1 & \gamma & \gamma^2 & \cdots & \gamma^{n-1} \\ 1 & \gamma^2 & \gamma^4 & \cdots & \gamma^{2(n-1)} \\ 1 & \gamma^3 & \gamma^6 & \cdots & \gamma^{3(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \gamma^n & \gamma^{2n} & \cdots & \gamma^{n(n-1)} \end{bmatrix} = \prod_{1\leqslant j<k\leqslant n} (\gamma^k - \gamma^j) \neq 0. \tag{12}$$

Hence (11) implies that $\langle \varphi, f_j \rangle = 0$ for all $j \in \mathbb{Z}_n$. Due to the arbitrariness of $\varphi$, this leads to $f_j|_{(a-\epsilon,a+\epsilon)} = 0$ for all $j \in \mathbb{Z}_n$, which contradicts the definition of $a$. □

**Proof of Theorem 1.** One has $\operatorname{supp} f \subset \mathscr{R}_n(I)$, $\operatorname{supp} g \subset \mathscr{R}_n(J)$, and $\operatorname{supp} f * g \subset \mathscr{R}_n(K) \subset \mathscr{R}_n(I+J)$. Due to constraint (4), each of the sets $\mathscr{R}_n(I)$, $\mathscr{R}_n(J)$, and $\mathscr{R}_n(I+J)$ consists of $n$ disjoint intervals. For $j \in \mathbb{Z}_n$, we set $f_j = (S_{2\pi j/n} f)|_I \in \mathscr{E}'(I)$, $g_j = (S_{2\pi j/n} g)|_J \in \mathscr{E}'(J)$, and $h_j = (S_{2\pi j/n}(f*g))|_K \in \mathscr{E}'(I+J)$; then

$$h_j = (S_{2\pi j/n}(f*g))\big|_{I+J} = \sum_{\substack{k+l\equiv j \bmod n \\ k,l\in\mathbb{Z}_n}} (S_{2\pi k/n} f)|_I * (S_{2\pi l/n} g)|_J = \sum_{\substack{k+l\equiv j \bmod n \\ k,l\in\mathbb{Z}_n}} f_k * g_l, \qquad j \in \mathbb{Z}_n. \tag{13}$$

Using relation (13), we obtain

$$\Big(\sum_{k\in\mathbb{Z}_n} \alpha^k f_k\Big) * \Big(\sum_{l\in\mathbb{Z}_n} \alpha^l g_l\Big) = \sum_{j\in\mathbb{Z}_n} \alpha^j \Big[\sum_{\substack{k+l\equiv j \bmod n \\ k,l\in\mathbb{Z}_n}} f_k * g_l\Big] = \sum_{j\in\mathbb{Z}_n} \alpha^j h_j \tag{14}$$

for any $\alpha \in \mathbb{C}$ such that $\alpha^n = 1$. The application of the Titchmarsh convolution theorem (1) to this relation yields

$$\inf\operatorname{supp} \sum_{j\in\mathbb{Z}_n} \alpha^j f_j + \inf\operatorname{supp} \sum_{j\in\mathbb{Z}_n} \alpha^j g_j = \inf\operatorname{supp} \sum_{j\in\mathbb{Z}_n} \alpha^j h_j \geqslant \inf K; \tag{15}$$

we took into account the inequality $\min_{j\in\mathbb{Z}_n} \inf\operatorname{supp} h_j \geqslant \inf K$. By Lemma 1, there is an $\alpha \in \mathbb{C}$ with $\alpha^n = 1$ for which $\inf\operatorname{supp} \sum_{j\in\mathbb{Z}_n} \alpha^j g_j = \min_{j\in\mathbb{Z}_n} \inf\operatorname{supp} g_j = \inf J$; this is equivalent to the second relation in (5). For the same value of $\alpha$, (15) yields

$$\inf\operatorname{supp} \sum_{j\in\mathbb{Z}_n} \alpha^j f_j \geqslant \inf K - \inf J = \inf I + \lambda.$$

This is equivalent to the first relation in (5). According to Remark 2, this proves the required assertion. □

**Proof of Theorem 2.** If $I$ consists of one point, i.e., $I = \{p\} \subset (-\pi/2, \pi/2)$, then $\operatorname{supp} f = \mathscr{R}_2(p) = \{p; \pi + p\}$, and (9) holds with

$$\mu = \frac{f + S_\pi f}{2}\Big|_I \qquad \text{and} \qquad \nu = \frac{f - S_\pi f}{2}\Big|_I.$$

Now, we assume that $|I| > 0$. We set $J = -I$ and $K = \{0\} \subset I + J$. Then $\operatorname{supp} f^\sharp \subset \mathscr{R}_2(J)$, and there is no $J' \subsetneq J$ for which $\operatorname{supp} f^\sharp \subset \mathscr{R}_2(J')$. According to the conditions of the theorem, $\operatorname{supp} f * f^\sharp \subset \mathscr{R}_2(K)$; hence one has

$$\lambda := \inf K - \inf I - \inf J = \sup I - \inf I = |I| > 0. \tag{16}$$

Applying Theorem 1 to (16), we conclude that there is an $\alpha \in \{\pm 1\}$ such that

$$(f + \alpha S_\pi f)|_{(\sup I - \pi, \sup I)} = 0 \tag{17}$$

and also $\inf \operatorname{supp}(f^\sharp + \alpha S_\pi f^\sharp)|_{(-\pi/2,\pi/2)} = -\sup I$; this last relation implies that

$$\sup \operatorname{supp}(f + \alpha S_\pi f)|_{(-\pi/2,\pi/2)} = \sup I. \tag{18}$$

Similarly, by Theorem 1, there is a $\beta \in \{\pm 1\}$ such that

$$(f^\sharp + \beta S_\pi f^\sharp)|_{(-\inf I - \pi, -\inf I)} = 0;$$

hence

$$(f + \beta S_\pi f)|_{(\inf I, \inf I + \pi)} = 0. \tag{19}$$

Comparing (18) with (19), we conclude that $\alpha \neq \beta$; therefore, $\alpha = -\beta$; thus, (17) and (19) lead us to conclude that both $f$ and $S_\pi f$ vanish on $(\inf I, \sup I)$, whence

$$\operatorname{supp} f \subset \{\inf I; \sup I; \pi + \inf I; \pi + \sup I\}.$$

By (17) and (19), if $\alpha = 1$, then relation (9) holds with $\mu = f|_{(\inf I, \pi/2)}$ and $\nu = f|_{(-\pi/2, \sup I)}$, and if $\alpha = -1$, then relation (9) holds with $\mu = f|_{(-\pi/2, \sup I)}$ and $\nu = f|_{(\inf I, \pi/2)}$. □

Notice that the proof of Theorem 3 for the case $p = 2$ immediately follows from Theorem 1. (For example, relations (5) with $f = g$ are mutually contradictory unless $\lambda = 0$.) By induction, we obtain a proof for $p = 2^N$ with any $N \in \mathbb{N}$. This implies the assertion of Theorem 3 for any $p \leqslant 2^N$, but only under the condition $|I| < 2\pi/(2^N n)$, which is stronger than $|I| < 2\pi/(pn)$. Instead of trying to use Theorem 1, we give an independent proof.

**Proof of Theorem 3.** One has $\operatorname{supp} f^{*p} \subset \mathscr{R}_n(pI)$. Due to the smallness of $I$, each of $\mathscr{R}_n(I)$ and $\mathscr{R}_n(pI)$ is a collection of $n$ disjoint intervals. We set $f_j := (S_{2\pi j/n} f)|_I \in \mathscr{E}'(I)$ and $h_j := (S_{2\pi j/n}(f^{*p}))|_I \in \mathscr{E}'(I)$. Then

$$\begin{aligned} h_j = (S_{2\pi j/n}(f^{*p}))|_{pI} &= \sum_{\substack{j_1+\dots+j_p \equiv j \bmod n \\ j_1,\dots,j_p \in \mathbb{Z}_n}} (S_{2\pi j_1/n} f)|_I * \dots * (S_{2\pi j_p/n} f)|_I \\ &= \sum_{\substack{j_1+\dots+j_p \equiv j \bmod n \\ j_1,\dots,j_p \in \mathbb{Z}_n}} f_{j_1} * \dots * f_{j_p}, \qquad j \in \mathbb{Z}_n. \end{aligned} \tag{20}$$

Taking into account (20), we obtain

$$\Big(\sum_{j\in\mathbb{Z}_n} \alpha^j f_j\Big)^{*p} = \sum_{j\in\mathbb{Z}_n} \alpha^j \Big[\sum_{j_1+\dots+j_p = j \bmod n} f_{j_1} * \dots * f_{j_n}\Big] = \sum_{j\in\mathbb{Z}_n} \alpha^j h_j \tag{21}$$

for any $\alpha \in \mathbb{C}$ such that $\alpha^n = 1$. The application of the Titchmarsh convolution theorem to (21) yields

$$p \inf \operatorname{supp} \sum_{j\in\mathbb{Z}_n} \alpha^j f_j = \inf \operatorname{supp} \sum_{j\in\mathbb{Z}_n} \alpha^j h_j.$$

By Lemma 1, there is an $\alpha \in \mathbb{C}$ such that $\alpha^n = 1$ and

$$\inf \operatorname{supp} \sum_{j\in\mathbb{Z}_n} \alpha^j f_j = \min_{j\in\mathbb{Z}_n} \inf \operatorname{supp} f_j;$$

hence, for this value of $\alpha$,

$$p \min_{j\in\mathbb{Z}_n} \inf \operatorname{supp} f_j = \inf \operatorname{supp} \sum_{j\in\mathbb{Z}_n} \alpha^j h_j \geqslant \min_{j\in\mathbb{Z}_n} \inf \operatorname{supp} h_j.$$

On the other hand, (20) immediately yields the inequalities $\inf \operatorname{supp} h_j \geqslant p \min_{k\in\mathbb{Z}_n} \inf \operatorname{supp} f_k$ for any $j \in \mathbb{Z}_n$. It follows that

$$\min_{j\in\mathbb{Z}_n} \inf \operatorname{supp} h_j = p \min_{j\in\mathbb{Z}_n} \inf \operatorname{supp} f_j$$

and, similarly,

$$\max_{j\in\mathbb{Z}_n} \sup \operatorname{supp} h_j = p \max_{j\in\mathbb{Z}_n} \sup \operatorname{supp} f_j. \qquad \square$$

## References


[1] L. Hörmander, The Analysis of Linear Partial Differential Operators. I, Springer Study Edition, Springer-Verlag, Berlin, 1990.

[2] A. I. Komech and A. A. Komech, "Global attractor for a nonlinear oscillator coupled to the Klein–Gordon field," Arch. Ration. Mech. Anal., **185**:1 (2007), 105–142.

[3] B. Y. Levin, Lectures on Entire Functions, In collaboration with Yu. Lyubarskii, M. Sodin, and V. Tkachenko, Transl. Math. Monographs, vol. 150, Amer. Math. Soc., Providence, RI, 1996.

[4] J.-L. Lions, "Supports de produits de composition. I," C. R. Acad. Sci. Paris, **232** (1951), 1530–1532.

[5] W. Strauss and L. Vazquez, "Numerical solution of a nonlinear Klein–Gordon equation," J. Comput. Phys., **28**:2 (1978), 271–278.

[6] E. Titchmarsh, "The zeros of certain integral functions," Proc. London Math. Soc., **25** (1926), 283–302.

[7] K. Yosida, Functional Analysis, Grundlehren der Mathematischen Wissenschaften, vol. 123, Springer-Verlag, Berlin, 1980.



Texas A&M University
Institute for Information Transmission Problems, Moscow
e-mail: comech@math.tamu.edu

University of Vienna
Institute for Information Transmission Problems, Moscow
e-mail: alexander.komech@univie.ac.at


*Translated by A. A. Komech and A. I. Komech*