On invariants for the Poincaré equations and applications

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Abstract

A special version of the Noether theory of invariants is developed for the Poincaré equations. This theory is applied to the Maxwell–Lorentz equations coupled to the Abraham rotating extended electron with the configuration space $SO(3)$.

Keywords: Poincaré equations; conservation laws; Noether theory of invariants; Abraham's rotating extended electron; Maxwell–Lorentz equations; Hamilton's least action principle.

1 Introduction

Our main goal is the clarification of the theory of invariants for the Maxwell–Lorentz equations (2.3)–(2.6) with an extended rotating charged particle. The conservation laws for this system were established in [6] by direct differentiation using the dynamical equations. However, the relation of these invariants to the symmetry properties were not obvious.

Our main result is the derivation of these conservation laws from the symmetry properties of the Lagrangian functional. This derivation relies on the Poincaré equations, which are equivalent to the Hamilton variational principle [2, 8].

The crux is that the phase space of the Maxwell–Lorentz equations is not a linear space but rather the manifold, which is the direct product of the orthogonal group $SO(3)$ by a linear function space. In [5] we have shown that the *Lorentz torque equation* (2.6) is equivalent to the Poincaré equations on the group $SO(3)$ which follow from the Hamilton least action principle for the Maxwell–Lorentz equations.

A fairly general theory of invariants for variational equations on manifolds was developed in [3] and [4]. The last chapter of [3] concerns the Maxwell equations with a given current. In [4] a point particle in a given Maxwell field is considered.

On the other hand, the invariants of the coupled Maxwell–Lorentz equations (2.3)–(2.6) with an extended rotating particle were not obtained previously from the symmetry properties. This is why we develop a special much simpler self-contained version of the Noether theory of invariants for the Poincaré equations which is sufficient for our purposes.

We show that the corresponding 'Poincaré invariants' coincide with the classical known expressions obtained in [6]. We consider solutions for which all our formal differentiations and integration by parts hold true.

2 Maxwell–Lorentz equations

The Maxwell fields $E(x,t)$ and $B(x,t)$ are generated by motion of a rotating charge. External fields E^{ext} and B^{ext} are generated by the corresponding external charges and currents. For simplicity we assume that the mass distribution, $m\,\rho(x)$, and the charge distribution, $e\,\rho(x)$, are proportional to each other. Here m is the total mass, e is the total charge, and we use a system of units such that $m = 1$ and $e = 1$. The coupling function $\rho(x)$ is a sufficiently smooth radially symmetric function of fast decay as $|x| \to \infty$,

$$\rho(x) = \rho_r(|x|). \qquad (C)$$

2.1 The angular velocity

Let us denote by $\omega(t) \in \mathbb{R}^3$ the angular velocity 'in space' (in the terminology of [2]) of the charge. Namely, let us fix a 'center point' O of the rigid body. Then the trajectory of each fixed point of the body is described by

$$x(t) = q(t) + R(t)(x(0) - q(0)),$$

where $q(t)$ is the position of O at time t, and $R(t) \in SO(3)$. Respectively, the velocity reads

$$\dot x(t) = \dot q(t) + \dot R(t)(x(0) - q(0)) = \dot q(t) + \dot R(t)R^{-1}(t)(x(t) - q(t)) = \dot q(t) + \omega(t) \wedge (x(t) - q(t)), \quad (2.1)$$

where $\omega(t) \in \mathbb{R}^3$ corresponds to the skew-symmetric matrix $\dot R(t)R^{-1}(t)$ by the rule

$$\dot R(t)R^{-1}(t) = \mathcal{J}\omega(t) := \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix}. \quad (2.2)$$

We assume that x and q refer to a certain Euclidean coordinate system in $\mathbb{R}^3$, and the vector product $\wedge$ is defined in this system by standard formulas. The identification (2.2) of a skew-symmetric matrix and the corresponding angular velocity vector is true in any Euclidean coordinate system of the same orientation as the initial one.

2.2 Dynamical equations

Now the system of Maxwell-Lorentz equations with spin reads as follows, see [6, 9]

$$\dot{E} = \nabla \wedge B - (\dot{q} + \omega \wedge (x-q))\rho(x-q) \quad (a), \quad \dot{B} = -\nabla \wedge E \quad (b), \tag{2.3}$$

$$\nabla \cdot E(x,t) = \rho(x - q(t)) \quad (a), \quad \nabla \cdot B(x,t) = 0 \quad (b), \tag{2.4}$$

$$\ddot{q} = \int [E + E^{ext} + (\dot{q} + \omega \wedge (x-q)) \wedge (B + B^{ext})]\rho(x-q)\,dx, \tag{2.5}$$

$$I\,\dot{\omega} = \int (x-q) \wedge [E + E^{ext} + (\dot{q} + \omega \wedge (x-q)) \wedge (B + B^{ext})]\rho(x-q)\,dx, \tag{2.6}$$

where I is the moment of inertia defined by

$$I = \frac{2}{3}\int x^2 \rho(x)\,dx. \tag{2.7}$$

The equations (2.3) are Maxwell equations with the corresponding charge density and current; equations (2.4) are the constraints. The back reaction of the field onto the particle is given through the Lorentz force equation (2.5), and the Lorentz torque equation (2.6) deals with rotational degrees of freedom.

2.3 The variational Hamilton principle

Let us introduce the *electromagnetic potentials* $\mathcal{A} = (A_0, A)$, $\mathcal{A}^{ext} = (A_0^{ext}, A^{ext})$:

$$B = \nabla \wedge A, \quad E = -\nabla A_0 - \dot{A}. \tag{2.8}$$

$$B^{ext} = \nabla \wedge A^{ext}, \quad E^{ext} = -\nabla A_0^{ext} - \dot{A}^{ext}. \tag{2.9}$$

Next we define the Lagrangian

$$L(\mathcal{A}, q, R, \dot{\mathcal{A}}, \dot{q}, \dot{R}) = \frac{1}{2}\int \left(E^2(x) - B^2(x)\right)\,dx + \frac{1}{2}\dot{q}^2 + \frac{1}{2}I\omega^2$$
$$- \int [A_0(x) + A_0^{ext}(x)]\rho(x-q)\,dx + \int (\dot{q} + \omega \wedge (x-q)) \cdot [A(x) + A^{ext}(x)]\rho(x-q)\,dx, \tag{2.10}$$

where $E(x)$ and $B(x)$ are expressed in terms of $\mathcal{A}(x)$ and $\dot{\mathcal{A}}(x)$ according to (2.8), and $\omega = \mathcal{J}^{-1}\dot{R}R^{-1}$ by (2.2).

This Lagrangian functional depends on R only trough ω due to the spherical symmetry of the charge and mass distributions (C). Respectively, the dynamical equations (2.3)–(2.6) involve R only through ω. On the other hand, in the case of non-radial densities the Lagrangian and the equations involve R explicitly, and the moment of inertia I becomes a matrix with $x \otimes x$ instead of x^2 in (2.7). The corresponding action functional has the form

$$S = S(\mathcal{A}, q, R) := \int_{t_1}^{t_2} L(\mathcal{A}(t), q(t), R(t), \dot{\mathcal{A}}(t), \dot{q}(t), \dot{R}(t))\,dt \tag{2.11}$$

The Hamilton least action principle now reads

$$\delta S(\mathcal{A}, q, R) = 0, \tag{2.12}$$

where the variation is taken over $\mathcal{A}(t), q(t)$, $R(t)$ with the boundary conditions

$$(\delta\mathcal{A}, \delta q, \delta R)|_{t=t_1} = (\delta\mathcal{A}, \delta q, \delta R)|_{t=t_2} = 0. \tag{2.13}$$

Regular solutions and external potential. Everywhere below we consider *regular solutions* to the system (2.3)–(2.6). This means that $q \in C^2(\mathbb{R}, \mathbb{R}^3)$, $\omega \in C^1(\mathbb{R}, \mathbb{R}^3)$, and all the involved functions and fields/potentials are sufficiently smooth and have (with all the necessary derivatives) a sufficient decay as $|x| \to \infty$ so that the partial integrations below are allowed.

In [5, Theorem 2.1] we have shown that, for regular solutions, the Maxwell-Lorentz system (2.3)–(2.6) is equivalent to the least action principle (2.12)–(2.13). In detail,

$$\frac{\delta S}{\delta\mathcal{A}} = 0 \quad (a), \qquad \frac{\delta S}{\delta q} = 0 \quad (b), \qquad \frac{\delta S}{\delta R} = 0 \quad (c). \tag{2.14}$$

Here (2.14), (a) and (b) are equivalent respectively to the standard Euler-Lagrange equations

$$\frac{d}{dt}\frac{\delta L}{\delta\dot{\mathcal{A}}} = L_{\mathcal{A}} \quad (a) \qquad \frac{d}{dt}L_{\dot q} = L_q \quad (b) \tag{2.15}$$

for the Lagrangian (2.10). The equation (2.15), (a) is equivalent to the Maxwell equations (2.3) with the constraints (2.4), and the equation (2.15), (b) is equivalent to the Lorentz force equation (2.5). Note that the equations (2.14), (a), (b) are equivalent to the standard Euler–Lagrange equations (2.15), because the variables $\mathcal{A}$, $\dot{\mathcal{A}}$, q, and $\dot q$ vary in the corresponding linear spaces. So, we will call these variables the “Lagrange variables”.

On the other hand, $R \in SO(3)$, and so, the variational equation (2.14) (c) cannot be transformed to a Euler-Lagrange equation since $SO(3)$ is not a linear space. We have shown in [5, Theorem 2.1] that (2.14) (c) is equivalent to the Lorentz torque equation (2.6). This follows from the variational *Poincaré equations* with the Lagrangian L expressed in suitable coordinates on the tangent bundle to $SO(3)$.

The coordinates are defined in the basis of right-invariant vector fields on the group $SO(3)$. Namely, consider an orthonormal basis $\{e_k\}$ with the right orientation in $\mathbb{R}^3$. Then

$$e_1 \wedge e_2 = e_3, \;\; e_2 \wedge e_3 = e_1, \;\; e_3 \wedge e_1 = e_2, \tag{2.16}$$

and the angular velocity $\omega(t) = \mathcal{J}^{-1}\dot R(t)R^{-1}(t)$ can be expanded as

$$\omega(t) = \sum \omega_k(t)e_k. \tag{2.17}$$

Further, the algebra $so(3)$ of skew-symmetric 3×3 matrices with the matrix commutator is isomorphic to the algebra $\mathbb{R}^3$ with the vector product through the isomorphism $\mathcal{J}$ of (2.2):

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \mathcal{J}(\omega_1, \omega_2, \omega_3). \tag{2.18}$$

Namely, let $A, B \in so(3)$, $a, b \in \mathbb{R}^3$, and $A = \mathcal{J}a$, $B = \mathcal{J}b$. Then

$$AB - BA = \mathcal{J}(a \wedge b). \tag{2.19}$$

Further, $\dot{R}R^{-1} \in T_E SO(3)$ is the tangent vector $\dot{R}$ to $SO(3)$ at the point R translated to the unit E of $SO(3)$ by the right translation R^{-1}. By the linear isomorphism (2.18),

$$\dot{R}R^{-1} = \sum \omega_k \tilde{e}_k, \qquad \tilde{e}_k := \mathcal{J}e_k. \tag{2.20}$$

Hence,

$$\dot{R} = \dot{R}R^{-1}R = \sum \omega_k v_k(R), \qquad v_k(R) := \tilde{e}_k R. \tag{2.21}$$

As the result, $\dot{R}$ has the same coordinates w.r.t. the vector fields v_k at the point R as ω in the basis $\{e_k\}$. The fields $v_k(R)$ are right translations of $\tilde{e}_k$ and hence are right-invariant.

In [5, Lemma 6.1] it was shown that for the vector fields v_k on $SO(3)$ the following commutation relations hold:

$$[v_1, v_2] = -v_3, \;\; [v_2, v_3] = -v_1, \;\; [v_3, v_1] = -v_2. \tag{2.22}$$

We will identify vector fields with the corresponding opertors of differentiation. According to the Poincaré theory [2, 8], the equation (2.14) (c) is equivalent to the Poincaré equations

$$\frac{d}{dt}\hat{L}_{\omega_k}(Y(t)) = \sum_{ij} c_{ik}^j \omega_i(t) \hat{L}_{\omega_j}(Y(t)) + v_k \hat{L}(Y(t)), \;\; k = 1, 2, 3, \tag{2.23}$$

where $Y(t) := (\mathcal{A}(t), q(t), \dot{\mathcal{A}}(t), \dot{q}(t), \omega(t))$ and $\hat{L}(\mathcal{A}, q, \dot{\mathcal{A}}, \dot{q}, \omega)$ is defined as the right hand side of (2.10), and the constants c_{ik}^j arise from commutation relations

$$[v_i, v_k](R) = \sum c_{ik}^j v_j(R).$$

In Appendix A, we recall the calculation of the Poincaré equations (2.23). These calculations will be used througough the paper.

Note that the Lagrangian $\hat{L}$ does not depend explicitly on R, and hence $v_k(\hat{L}) = 0$, $k = 1, 2, 3$. Now the corresponding Poincaré equations read

$$\frac{d}{dt}\frac{\partial \hat{L}(\omega(t))}{\partial \omega_k} = \sum_{ij} c_{ik}^j \omega_i \frac{\partial \hat{L}(\omega(t))}{\partial \omega_j}, \quad k = 1, 2, 3. \tag{2.24}$$

In our case (2.22) and (A.7) imply

$$c_{21}^3 = c_{32}^1 = c_{13}^2 = 1, \;\; c_{31}^2 = c_{12}^3 = c_{23}^1 = -1, \;\; \text{all the rest} \;\; c_{ik}^j = 0.$$

Thus, we can rewrite (2.24) as

$$\frac{d}{dt}\frac{\partial \hat{L}(\omega(t))}{\partial \omega} = \omega \wedge \frac{\partial \hat{L}(\omega(t))}{\partial \omega}, \tag{2.25}$$

where $\dfrac{\partial \hat{L}}{\partial \omega}$ is the column vector with the components $\dfrac{\partial \hat{L}}{\partial \omega_k}$, $k = 1, 2, 3$.

We summarize the situation as follows (see [5]). The Lagrangian $\hat{L}$ depends on the two groups of variables: on the 'Lagrangian variables' $\mathcal{A}$, $\dot{\mathcal{A}}$, q, $\dot{q}$ and on the variables ω_k which we will call the 'Poincaré variables'. The variational equations (2.14) (a), (b) imply the Maxwell-Lorentz equations (2.3)–(2.5), while (2.14) (c) give the Lorentz torque equations (2.6).

3 Invariants for the Poincaré equations

When the external fields possess a symmetry with respect to the Lagrangian variables, the corresponding conservation laws are given by the Noether theorem on invariants [1]. In this section we extend the Noether theory to the Poincaré equations.

Let $v_1(g), \ldots, v_n(g)$ be vector fields on an n-dimensional manifold M, which are linearly independent at each point $g \in M$. In particular such vector fields exist for any open region $M \subset \mathbb{R}^n$. Then TM is isomorphic to $M \times \mathbb{R}^n$, and any function $L(g, \dot{g})$ on TM can be expressed in the *Poincaré variables* g, ω:

$$\hat{L}(g,\omega) := L(g,\dot{g}), \qquad \dot{g} = \sum \omega_k v_k(g). \tag{3.1}$$

In [8], Poincaré discovered that the corresponding Hamilton least action principle is equivalent to the equations

$$\frac{d}{dt}\hat{L}_{\omega_k}(g(t),\omega(t)) = \sum_{ij} c^j_{ik}(g)\omega_i \hat{L}_{\omega_j}(g,\omega) + v_k(g)\hat{L}(g,\omega), \ \ k = 1, ..., n. \tag{3.2}$$

where the 'structure constants' $c^j_{ik}(g)$ arise from the commutation relations

$$[v_i, v_j](g) = \sum c^k_{ij}(g) v_k(g), \qquad g \in M.$$

see the details in Appendix A. Here we develop the theory of invariant for the Poincaré equations (3.2). Let us start with the energy conservation.

Theorem 3.1 *The 'energy'*

$$E := \hat{L}_\omega \cdot \omega - \hat{L} = \sum_k \hat{L}_{\omega_k}\omega_k - \hat{L} \tag{3.3}$$

is conserved along the paths of the Poincaré equations (3.2).

Proof Let a smooth path $(g(t), \omega(t))$ satisfy the Poincaré equations (3.2). Using (3.2) we obatin

$$\begin{aligned} \frac{d}{dt}(\hat{L}_\omega \cdot \omega - \hat{L}) &= \frac{d}{dt}\hat{L}_\omega \cdot \omega + \hat{L}_\omega \cdot \dot{\omega} - \hat{L}_g \cdot \dot{g} - \hat{L}_\omega \cdot \dot{\omega} = \sum_k \frac{d}{dt}\hat{L}_{\omega_k}\omega_k - \hat{L}_g \cdot \dot{g} \\ &= \sum_k (\sum_{ij} c^j_{ik}\omega_i \hat{L}_{\omega_j} + v_k(\hat{L}))\omega_k - \hat{L}_g \cdot \dot{g}. \end{aligned} \tag{3.4}$$

Note that $\hat{L}_g \cdot \dot{g} = \hat{L}_g \cdot \sum \omega_k v_k = \sum \omega_k \hat{L}_g \cdot v_k = \sum v_k(\hat{L})\omega_k$. Thus, we obtain

$$\frac{d}{dt}(\hat{L}_\omega \cdot \omega - \hat{L}) = \sum_k (\sum_{ij} c^j_{ik}\omega_i \hat{L}_{\omega_j})\omega_k = \sum_j \hat{L}_{\omega_j} \sum_{ik} c^j_{ik}\omega_i\omega_k = 0, \tag{3.5}$$

since $\sum\limits_{ik} c^j_{ik}\omega_i\omega_k = 0$ by the skew-symmetry property (A.1) of the coefficients c^j_{ik}. □

Remark 3.2 *In the Lagrangian case (i.e., when M is a linear space and $\omega = \dot{g}$), the invariant (3.3) coincides with the standard energy functional.*

Now let us consider the general case of a one-parametric group of diffeomorphisms $h^s : M \to M$ (in particular, $h^0 = Id_M$). Let the Lagrangian L be invariant with respect to the diffeomorphisms h^s, i.e.,

$$L(h^s g, dh^s \dot{g}) = L(g, \dot{g}), \qquad (g, \dot{g}) \in TM, \qquad s \in \mathbb{R}. \tag{3.6}$$

Definition 3.3 *The corresponding Poincaré invariant I and the corresponding 'current' $w = (w_1, ..., w_n)$ are defined as*

$$I(g,\omega) := \sum \hat{L}_{\omega_k} w_k(g), \qquad \frac{dh^s g}{ds}\Big|_{s=0} = \sum w_k(g) v_k(g). \tag{3.7}$$

These definitions generalize the corresponding Noether formulas [1] to the case of Poincaré equations.

Theorem 3.4 *Let condition (3.6) hold. Then the function $I(g,\omega)$ is conserved along the paths of the Poincaré equations (3.2).*

Proof Let a smooth path $(g(t), \omega(t))$ satisfy the Poincaré equations (3.2). Setting $g(s,t) := h^s g(t)$, we obtain

$$\dot{g}(s,t) = dh^s \dot{g}(t) = \sum \omega_k(s,t) v_k(g(s,t)).$$

In particular, $g(0,t) = g(t)$ and $\dot{g}(t) := \sum \omega_k(t) v_k(g(t))$. By (3.6), the quantity

$$\hat{L}(g(s,t), \omega(s,t)) := L(g(s,t), \sum_k \omega_k(s,t) v_k(g(s,t))) = L(g(s,t), dh^s \dot{g}(t))$$

does not depend on s; here $\omega(s,t) = (\omega_1(s,t), ..., \omega_n(s,t))$. Denote by prime the derivative in s, and by dot the derivative in t. Then we obtain

$$0 = \frac{d}{ds}\hat{L}(g(s,t), \omega(s,t)) = \hat{L}_g \cdot g' + \sum_k \hat{L}_{\omega_k} \omega_k' = \hat{L}_g \cdot g' + \sum_k \hat{L}_{\omega_k} (\sum_{ij} c_{ij}^k \omega_i w_j + \dot{w}_k) =: S \tag{3.8}$$

by the formula (A.3) of Appendix A. First we change the order of summation on the right-hand side:

$$S = \hat{L}_g \cdot g' + \sum_k \hat{L}_{\omega_k} \dot{w}_k + \sum_j (\sum_{ik} c_{ij}^k \omega_i \hat{L}_{\omega_k}) w_j. \tag{3.9}$$

Next we wish to evaluate the term $\sum\limits_{ik} c_{ij}^k \omega_i \hat{L}_{\omega_k}$ for $s = 0$. Namely, $g(0,t) = g(t)$ together with $\omega(0,t) = \omega(t)$ satisfy the Poincaré equations (3.2). Hence, for $s = 0$

$$\begin{aligned} S &= \hat{L}_g \cdot g' + \sum_k \hat{L}_{\omega_k} \dot{w}_k + \sum_j \left(\frac{d}{dt} \hat{L}_{\omega_j} - v_j(\hat{L}) \right) w_j \\ &= \sum_k \hat{L}_{\omega_k} \dot{w}_k + \sum_j \frac{d}{dt} \hat{L}_{\omega_j} \cdot w_j + \hat{L}_g \cdot g' - \sum_j v_j(\hat{L}) w_j. \end{aligned} \tag{3.10}$$

However, the definition of the current w in (3.7) implies that

$$\hat{L}_g \cdot g' - \sum_j v_j(\hat{L}) w_j = \sum_j \hat{L}_g \cdot v_j w_j - \sum_j v_j(\hat{L}) w_j = 0$$

Therefore, (3.10) gives

$$S = \sum_k \hat{L}_{\omega_k} \dot{w}_k + \sum_k \frac{d}{dt} \hat{L}_{\omega_k} \cdot w_k = \frac{d}{dt} (\sum_k \hat{L}_{\omega_k} w_k) = \dot{I}(t). \tag{3.11}$$

The proof is complete, since $S = 0$ by (3.8). □

Remark 3.5 *Let $M = \mathbb{R}^n$ and $v_k = \nabla_{g_k}$ denote the commuting vector fields of the differentiations w.r.t. coordinates g_k. Then the Poincaré equations (3.2) read as the Euler-Lagrange equations, and the Poincaré invariant (3.7) coinsides with the Noether invariant $L_{\dot{g}} \cdot \frac{dh^s g}{ds}\Big|_{s=0}$.*

4 Invariants for the Lagrange-Poincaré equations

Here we generalize the theory of the previous section to systems with the configuration space $Y \times M$, where Y is a Hilbert space either of finite or infinite dimension, while M is a finite-dimensional manifold endowed with the vector fields $v_k(g)$ as above. Then $TY \simeq Y \times Y$ and $TM \simeq M \times \mathbb{R}^n$. Let $L(X, V, g, \dot{g})$ be a differentiable Lagrangian which is defined on $TY \times TM$. Let us define

$$\hat{L}(X, V, g, \omega) := L(X, V, g, \dot{g}), \qquad \dot{g} = \sum \omega_k v_k(g). \tag{4.1}$$

Next, let a smooth path $(X(t), V(t)), g(t), \omega(t)$ satisfy the standard Euler–Lagrange equations w.r.t. the variables (X, V) and the Poincaré equations w.r.t. the variables (g, ω):

$$\left\{ \begin{array}{rcl} \frac{d}{dt}\hat{L}_V & = & L_X, \\ \\ \frac{d}{dt}\hat{L}_{\omega_k} & = & \sum\limits_{ij} c^j_{ik}(g)\omega_i \hat{L}_{\omega_j} + v_k(g)\hat{L}, \quad k = 1, ..., n. \end{array} \right| \tag{4.2}$$

Theorem 4.1 *Let (4.2) hold. Then the energy*

$$E := \hat{L}_V \cdot V + \hat{L}_\omega \cdot \omega - \hat{L} \tag{4.3}$$

is conserved along the path.

Proof Differentiating formally, we get

$$\frac{d}{dt}\Big(\hat{L}_V \cdot V + \hat{L}_\omega \cdot \omega - \hat{L}\Big) =$$

$$= \Big(\frac{d}{dt}\hat{L}_V(X, V) \cdot V - \hat{L}_X \cdot \dot{X}\Big) + \Big(\frac{d}{dt}\hat{L}_\omega(g, \omega) \cdot \omega - \hat{L}_g \cdot \dot{g}\Big) = 0.$$

Indeed, here the first bracket of the last line vanishes by the first equation of (4.2). The second bracket vanishes by the second equation of (4.2). This follows by the calculations similar to (3.4)–(3.5). □

Further, consider a one-parametric group of diffeomorphisms

$$h^s : (X, g) \mapsto (h^s_1(X), h^s_2(g)). \tag{4.4}$$

Let us suppose that the Lagrangian functional is h^s-invariant, i.e.,

$$L(h^s_1 X, dh^s_1 V, h^s_2 g, dh^s_2 \dot{g}) \equiv L(X, V, g, \dot{g}). \tag{4.5}$$

Theorem 4.2 *Let (4.2), (4.5) hold. Then the sum*

$$\hat{L}_V \cdot \frac{dh^s_1 X}{ds}\Big|_{s=0} + \sum \hat{L}_{\omega_k} w_k(g) \tag{4.6}$$

is conserved along the path.

Proof Let $X(s, t) := h^s_1 X$, $g(s, t) := h^s_2 g$, and let $\omega(s, t)$ be defined as above. Then $\dot{g}(s, t) = \sum_k v_k(g(s, t))\omega_k(s, t)$, and formally,

$$0 = \frac{d}{ds}\hat{L}(X(s, t), \dot{X}(s, t), g(s, t), \omega(s, t)) = \hat{L}_X \cdot X' + \hat{L}_{\dot{X}} \cdot \dot{X}(s, t)' + \hat{L}_g \cdot g' + \sum \hat{L}_{\omega_k}\omega'_k.$$

At $s = 0$, the sum of the first two terms reduces to $\frac{d}{dt}(L_{\dot{X}} \cdot \frac{dh^s_1 X}{ds}\Big|_{s=0})$ as in the proof of the standard theorem on Noether invariants [1]. The sum of the last two terms transforms to $\frac{d}{dt}(\sum \hat{L}_{\omega_k} w_k(g))$ as in the proof of Theorem 3.4 (calculations (3.8)–(3.11)). □

5 Conservation laws for Maxwell-Lorentz equations

We now apply the theory of Noether invariants and Poincare invariants for our system of Maxwell-Lorentz equations with rotating charge. As above, we denote

$$\hat{L}(\mathcal{A}, q, \dot{\mathcal{A}}, \dot{q}, \omega) = L(\mathcal{A}, q, R, \dot{\mathcal{A}}, \dot{q}, \dot{R}) \tag{5.1}$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ is defined by (2.21). In other words, ω_k are coordinates of $\dot{R}$ in the basis $v_1(R), v_2(R)$, $v_3(R)$; recall that $\hat{L}$ does not depend explicitly on R.

5.1 Energy

Let us note that L is independent of $\dot{A}_0$. Hence, Theorem 4.1 formally implies the following corollary.

Corollary 5.1 *Suppose A_0^{ext} end A^{ext} do not depend on time. Then the functional*

$$E(\mathcal{A}, q, \dot{\mathcal{A}}, \dot{q}, R, \omega) := \hat{L}_{\dot{A}} \cdot \dot{A} + \hat{L}_{\dot{q}} \cdot \dot{q} + \hat{L}_{\omega} \cdot \omega - \hat{L} \tag{5.2}$$

is conserved along the regular solutions of the Maxwell-Lorentz system (2.3)–(2.5).

5.2 Momentum

Let the external field

$$\mathcal{A}^{ext}(x) = (A_0^{ext}(x), A^{ext}(x)) \;\text{ be independent of } x_k \text{ for some } k. \tag{5.3}$$

Then the Lagrangian (5.1) is invariant w.r.t to the one-parametric group of spatial translations

$$h_k^s(\mathcal{A}(x), q) = (\mathcal{A}(x - se_k), q + se_k), \tag{5.4}$$

where $e_k \in \mathbb{R}^3$ is the corresponding basis vector. Since the group acts only on the Lagrange coordinates $X := (\mathcal{A}, q)$, $V := (\dot{\mathcal{A}}, \dot{q})$, we may formally apply the Noether theory [1, 7] to obtain

Corollary 5.2 *Under the condition (5.3) the functional*

$$P_k = P_k(X, V, R, \omega) := \hat{L}_V \cdot \frac{dh_k^s X}{ds}\Big|_{s=0} \tag{5.5}$$

is conserved for regular solutions to the Maxwell–Lorentz system (2.3)–(2.5).

Definition 5.3 *P_k is called the k-th component of momentum of a state (X, V, R, ω).*

5.3 Angular momentum

Let the external potential $\mathcal{A}^{ext}$ be axially symmetric,

$$A_0^{ext}(U_k x) = A_0^{ext}(x), \quad A^{ext}(U_k x) = U_k A^{ext}(x), \tag{5.6}$$

where U_k is any rotation around the axis Ox_k.

Lemma 5.4 *Let (5.6) hold. Then the Lagrangian (2.10) is invariant w.r.t. the axial rotations*

$$A_0(x) \mapsto A_0(U_k^{-1}x), \quad A(x) \mapsto U_k A(U_k^{-1}x), \quad \dot{A}(x) \mapsto U_k \dot{A}(U_k^{-1}x), \tag{5.7}$$

$$R \mapsto U_k R, \quad \dot{R} \mapsto U_k \dot{R}, \tag{5.8}$$

$$q \mapsto U_k q, \quad \dot{q} \mapsto U_k \dot{q}. \tag{5.9}$$

Proof By (2.8) the transforms (5.7) of the potentials induce the following transforms of the fields:

$$E(x) \mapsto U_k E(U_k^{-1}x), \quad B(x) \mapsto U_k B(U_k^{-1}x). \tag{5.10}$$

In the operator notation, $\mathcal{J}\omega = \omega\wedge$, where $\omega\wedge$ is the operator of the vector product by ω in $\mathbb{R}^3$. Hence, it is easy to check that $\mathcal{J}(U_k\omega) = U_k\mathcal{J}(\omega)U_k^{-1}$. Thus, for $\omega = \mathcal{J}^{-1}\dot{R}R^{-1}$ we obtain $\mathcal{J}\omega = \dot{R}R^{-1}$ and hence $\mathcal{J}(U_k\omega) = U_k(\dot{R}R^{-1})U_k^{-1} = (U_k\dot{R})(U_kR)^{-1}$. Finally,

$$U_k\omega = \mathcal{J}^{-1}(U_k\dot{R})(U_kR)^{-1}.$$

This means that the transforms (5.8) induce the following transform of ω:

$$\omega \mapsto U_k\omega. \tag{5.11}$$

Hence, the Lagrangian L is invarient w.r.t. the transforms (5.9), (5.10), (5.11) by (5.6), since ρ is spherically symmetric. □

Recall that $\tilde{e}_k$ is the image of the basis vector e_k w.r.t. the isomorphism (2.18). By Lemma 5.4 the Lagrangian $\hat{L}$ (5.1) is invariant w.r.t. the spatial rotations (5.9), (5.10), (5.11). In particular, $\hat{L}$ is invariant under the transform group $h_k^s = e^{s\tilde{e}_k} \in SO(3)$.

In detail, we have the situation of previous section, when $\hat{L}$ depends on the Lagrangian variables $(X; V) = (\mathcal{A}, q; \dot{\mathcal{A}}, \dot{q})$ and on the Poincaré variables (R, ω). The action of this group on the state (X, R) reads

$$h_k^s(X, R) = (\alpha_k^s X, \beta_k^s R): \quad \alpha_k^s X = (A_0(e^{-s\tilde{e}_k}x), e^{s\tilde{e}_k}A(e^{-s\tilde{e}_k}x), e^{s\tilde{e}_k}q); \quad \beta_k^s R = e^{s\tilde{e}_k}R.$$

The currents $w_1^k(R), w_2^k(R), w_3^k(R)$ are defined from

$$\frac{d\beta_k^s R}{ds}\Big|_{s=0} = \sum_{j=1}^{3} w_j^k(R)v_j(R), \quad R \in SO(3). \tag{5.12}$$

Hence, by Theorem 4.2 we come to the following result.

Corollary 5.5 *Under the condition (5.6) the quantity*

$$M_k = M_k(X, V, R, \omega) := \hat{L}_V \cdot \frac{d\alpha_k^s X}{ds}\Big|_{s=0} + \sum_{j=1}^{3} \hat{L}_{\omega_j} w_j^k(R) \tag{5.13}$$

is conserved for regular solutions to the Maxwell-Lorentz system (2.3)–(2.5).

Definition 5.6 M_k *is called k-th component of the angular momentum of a state* (X, V, R, ω).

6 Expressions for energy and momenta

Let us show that the Poincaré invariants from previous section coincide with the classical known expressions considered in [6] (where their conservation was shown by direct differentiation).

Proposition 6.1 *The invariants for the Maxwell-Lorentz system (2.3)–(2.5) read as follows:*

i) The energy:

$$E = \frac{1}{2}\int (|E(x)|^2 + |B(x)|^2)\,dx + \frac{1}{2}\dot q^2 + \frac{1}{2}I\omega^2 + \int A_0^{ext}(x)\rho(x-q)\,dx. \tag{6.1}$$

ii) The momentum:

$$P = \dot q + \int E(x)\wedge B(x)\,dx + \int A^{ext}(x)\rho(x-q)\,dx. \tag{6.2}$$

iii) The angular momentum:

$$M = q\wedge \dot q + I\omega + \int x\wedge E(x)\wedge B(x)\,dx + \int x\wedge A^{ext}(x)\rho(x-q)\,dx. \tag{6.3}$$

Proof i) By (5.1) and (2.10), one has

$$\hat L_{\dot A}\cdot \dot A = -\int E\cdot \dot A\,dx,\ \ \hat L_{\dot q}\cdot \dot q = \dot q^2 + \int \dot q\cdot (A + A^{ext})\rho(x-q)\,dx,$$

and

$$\hat L_\omega\cdot\omega = I\omega^2 + \int (\omega\wedge(x-q))\cdot(A+A^{ext})\rho(x-q)\,dx.$$

Hence,

$$\begin{aligned} E &= \hat L_{\dot A}\cdot \dot A + \hat L_{\dot q}\cdot \dot q + \hat L_\omega\cdot\omega - \hat L = \frac{1}{2}\dot q^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}\int (|B|^2 - |E|^2)\,dx \\ &\quad + \int (-E\cdot \dot A + A_0\rho(x-q)\,dx + \int A_0^{ext}\rho(x-q))\,dx. \end{aligned} \tag{6.4}$$

Since

$$\begin{aligned} \int (-E\cdot\dot A + A_0\rho(x-q))\,dx &= \int(-E\cdot \dot A + A_0\cdot\nabla E)\,dx \\ &= -\int E(\dot A - \nabla A_0)\,dx = \int E^2\,dx, \end{aligned} \tag{6.5}$$

formula (6.4) gives (6.1).

ii) Let us compute P_j. Formula (5.4) implies that

$$\frac{dh_j^s(X)}{ds}|_{s=0} = -(e_j\cdot\nabla A(x),\ \ e_j).$$

As a result, we obtain that

$$\begin{aligned} P_j &= L_V\cdot \frac{dh_j^s(X)}{ds}|_{s=0} = -L_{\dot A}\cdot(e_j\cdot\nabla)A + L_{\dot q}\cdot e_j \\ &= -\int(\nabla A_0 + \dot A)\cdot(e_j\cdot\nabla)A\,dx + \dot q\cdot e_j + \int e_j\cdot A\rho(x-q)\,dx + \int A_j^{ext}\rho(x-q)\,dx \\ &= \dot q_j + \int A_j\rho(x-q)\,dx - \int(\nabla A_0+\dot A)\cdot\partial_j A\,dx + \int A_j^{ext}\rho(x-q)\,dx. \end{aligned} \tag{6.6}$$

By partial integration

$$\begin{aligned}\int A_j(x)\rho(x-q)\,dx &= \int A_j(\nabla\cdot E)\,dx = \int A_j\nabla\cdot(-\nabla A_0 - \dot{A})\,dx \\ &= \int A_j(-\Delta A_0 - \nabla\dot{A})\,dx = \int(\nabla A_0\cdot\nabla A_j + (\dot{A}\cdot\nabla)A_j)\,dx.\end{aligned}$$

Hence,

$$P_j = \dot{q}_j + \int(\nabla A_0\cdot\nabla A_j + (\dot{A}\cdot\nabla)A_j)\,dx - \int(\nabla A_0\cdot\partial_j A + \dot{A}\cdot\partial_j A)\,dx + \int A_j^{ext}\rho(x-q)\,dx. \quad (6.7)$$

On the other hand, the j-th component of the RHS of (6.2) equals

$$\dot{q}_j + \int(E\wedge B)_j\,dx + \int A_j^{ext}\rho(x-q)\,dx.$$

Inserting $E = -\dot{A} - \nabla A_0$ and $B = \nabla\wedge A$, we obtain

$$\dot{q}_j + \int A_j^{ext}\rho(x-q)\,dx + \int\Big((\dot{A}\cdot\nabla)A_j - \dot{A}\cdot\partial_j A + \nabla A_0\cdot\nabla A_j - \nabla A_0\cdot\partial_j A\Big)\,dx$$

which coincides with (6.7).

iii) For example, let us compute M_1. We have

$$\alpha_1^s(X) = (A_0(e^{-s\tilde{e}_1}x), e^{s\tilde{e}_1}A(e^{-s\tilde{e}_1}x), e^{s\tilde{e}_1}q).$$

Hence,

$$\begin{aligned}\frac{d\alpha_1^s X}{ds}\Big|_{s=0} &= (-\tilde{e}_1 e^{-s\tilde{e}_1}x\cdot\nabla)A_0(e^{-s\tilde{e}_1}x), \tilde{e}_1 e^{s\tilde{e}_1}A(e^{-s\tilde{e}_1}x) + e^{s\tilde{e}_1}(-\tilde{e}_1 e^{-s\tilde{e}_1}x\cdot\nabla)A(e^{-s\tilde{e}_1}x), \tilde{e}_1 e^{s\tilde{e}_1}q)|_{s=0} \\ &= (\tilde{e}_1 A_0(x), \tilde{e}_1 A(x) - (\tilde{e}_1 x\cdot\nabla)A(x), \tilde{e}_1 q).\end{aligned}$$

Further,

$$\frac{d\beta_1^s R}{ds}\Big|_{s=0} = \frac{de^{s\tilde{e}_1}R}{ds}\Big|_{s=0} = \tilde{e}_1 R = v_1(R)$$

by definition (2.21) of the fields $v_k(R)$. Hence, for the currents w_j^1 of (5.12) we have $w_1^1 = 1$, $w_2^1 = w_3^1 = 0$. Now using that $\hat{L}$ does not depend on $\dot{A}_0$, we obtain

$$\begin{aligned}M_1 &= \hat{L}_{\dot{A}}\cdot(\tilde{e}_1 A(x) - (\tilde{e}_1 x\cdot\nabla)A(x)) + \hat{L}_{\dot{q}}\cdot(\tilde{e}_1 q) + \hat{L}_{\omega_1} \\ &= \int\Big(\dot{A}\cdot(\tilde{e}_1 A(x) - (\tilde{e}_1 x\cdot\nabla)A(x)) + \nabla A_0\cdot(\tilde{e}_1 A(x) - (\tilde{e}_1 x\cdot\nabla)A(x))\Big)\,dx \\ &\quad + \dot{q}\cdot(\tilde{e}_1 q) + \int(\tilde{e}_1 q)\cdot(A + A^{ext})\rho(x-q)\,dx + I\omega\cdot e_1 + \int(e_1\wedge(x-q))\cdot[A + A^{ext}]\rho(x-q)\,dx \\ &= (q\wedge\dot{q})_1 + I\omega_1 + \int(x_2 A_3^{ext} - x_3 A_2^{ext})\rho(x-q)\,dx \\ &\quad + \int(x_2 A_3 - x_3 A_2)\rho(x-q)\,dx + \int(\dot{A} + \nabla A_0)\cdot((0, -A_3, A_2) + (x_3\partial_2 - x_2\partial_3)A)\,dx. \qquad (6.8)\end{aligned}$$

We have to prove that this expression equals to the first component of the RHS of (6.3). It suffices to prove that the last line (6.8) equals to the first component of $\int x\wedge(E\wedge B)\,dx$. Indeed, $\rho(x-q)=\nabla\cdot E=\nabla\cdot(-\nabla A_0-\dot A)$, and hence,

$$\int(x_2A_3-x_3A_2)\rho(x-q)\,dx=\int(x_2A_3-x_3A_2)(-\nabla\dot A-\nabla^2A_0)\,dx=\int\nabla(x_2A_3-x_3A_2)(\dot A+\nabla A_0)\,dx. \quad (6.9)$$

Now the last line of (6.8) transforms to

$$\int\Big(\partial_1(x_2A_3-x_3A_2)(\dot A_1+\partial_1A_0)+x_2\partial_2A_3(\dot A_2+\partial_2A_0)-x_3\partial_3A_2(\dot A_3+\partial_3A_0)\Big)\,dx$$
$$+\int\Big((x_3\partial_2-x_2\partial_3)A_1(\dot A_1+\partial_1A_0)-x_2\partial_3A_2(\dot A_2+\partial_2A_0)+x_3\partial_2A_3(\dot A_3+\partial_3A_0)\Big)\,dx. \quad (6.10)$$

On the other hand, substituting $E=-\dot A-\nabla A_0$ and $B=\nabla\wedge A$, we obtain that the first component of $\int x\wedge(E\wedge B)\,dx$ equals

$$\int x_2((\partial_1A_3-\partial_3A_1)(\dot A_1+\partial_1A_0)+(\partial_2A_3-\partial_3A_2)(\dot A_2+\partial_2A_0))\,dx$$
$$-\int x_3((\partial_3A_2-\partial_2A_3)(\dot A_3+\partial_3A_0)+(\partial_1A_2-\partial_2A_1)(\dot A_1+\partial_1A_0))\,dx$$

which coincides with (6.10). The proof is complete. □

A The Poincaré equations

Poincaré suggested the form of the Hamilton least action principle for Lagrangian systems on manifolds [8]. We present the derivation of the Poincaré equations [2] since we use some of intermediate calculations.

Let $v_1,\dots,v_n$ be vector fields on a n-dimensional manifold M which are linearly independent at every point $g\in M$. Then the commutation relations hold,

$$[v_i,v_j](g)=\sum c_{ij}^k(g)v_k(g),\qquad g\in M$$

where the commutator $[v_i,v_j]$ is defined by

$$[v_i,v_j](f):=v_i(v_j(f))-v_j(v_i(f)),$$

and $v(f)$ is the derivative of a smooth function f on M w.r.t. the vector field v. Note that by the skew-symmetry property of the commutators one has

$$c_{ij}^k(g)=-c_{ji}^k(g),\ \ \forall\ k=1,...,n. \quad (A.1)$$

If $g(t)$ is a smooth path in M and f is a smooth function on M, one has

$$\frac{d}{dt}f(g(t))=f'(g(t))\cdot\dot g=f'(g(t))\cdot\sum\omega_i(t)v_i(g(t))=\sum v_i(f)\omega_i(t), \quad (A.2)$$

since $\dot g(t)=\sum\omega_i(t)v_i(g(t))$. Now consider an arbitrary variation $g(s,t)$ of the path $g(t)$. Then similarly to (A.2),

$$\partial_sf(g(s,t))=\sum_j v_j(f)w_j(s,t),$$

where $w_j(s,t)$ are the components of $\frac{\partial g}{\partial s}(s,t) \in T_{g(s,t)}M$. Hence

$$\partial_s\partial_t f(g(s,t)) = \sum_i\sum_j v_j(v_i(f))w_j\omega_i + \sum_i v_i(f)\omega_i',$$

$$\partial_t\partial_s f(g(s,t)) = \sum_j\sum_i v_i(v_j(f))w_j\omega_i + \sum_j v_j(f)\dot{w}_j,$$

where the prime (respectively, the dot) stands for the differentiation in s (respectively, in t). However, the differentiations in t and s commute, hence we obtain by subtraction

$$\sum_k v_k(f)\omega_k' = \sum_k\sum_{ij} c_{ij}^k\omega_i w_j v_k(f) + \sum_k v_k(f)\dot{w}_k.$$

Since f is an arbitrary smooth function, we come to the relations

$$\omega_k'(s,t) = \sum_{ij} c_{ij}^k\omega_i w_j + \dot{w}_k. \tag{A.3}$$

Further, let us consider an arbitrary Lagrangian function $L(g,\dot{g})$ on TM. Then $L(g,\dot{g})$ can be expressed in the variables ω: $L(g,\dot{g}) = \hat{L}(g,\omega)$. Let us compute the variation of the corresponding action functional, taking (A.3) into account:

$$\frac{d}{ds}\int_{t_1}^{t_2}\hat{L}(g(s,t),\omega(s,t))dt = \int_{t_1}^{t_2}\left(\sum_k\frac{\partial\hat{L}}{\partial\omega_k}\omega_k' + \nabla_g\hat{L}\cdot g'\right)dt =$$

$$\int_{t_1}^{t_2}\left[\sum_k\frac{\partial\hat{L}}{\partial\omega_k}(\dot{w}_k + \sum_{ij}c_{ij}^k\omega_i w_j) + \nabla_g\hat{L}\cdot\sum_k w_k v_k\right]dt =$$

$$\sum_k\frac{\partial\hat{L}}{\partial\omega_k}w_k\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\sum_k\left[-\frac{d}{dt}\frac{\partial\hat{L}}{\partial\omega_k} + \sum_{ij}c_{ik}^j\omega_i\frac{\partial\hat{L}}{\partial\omega_j} + v_k(\hat{L})\right]w_k\,dt.$$

The variation should be zero by the Hamilton least action principle, under the boundary value conditions

$$g(s,t_1) = g_1,\quad g(s,t_2) = g_2. \tag{A.4}$$

Since $w_k(t_1) = w_k(t_2) = 0$ by (A.4), we obtain the following *Poincaré equations*:

$$\frac{d}{dt}\frac{\partial\hat{L}}{\partial\omega_k} = \sum_{ij}c_{ik}^j\omega_i\frac{\partial\hat{L}}{\partial\omega_j} + v_k(\hat{L}). \tag{A.5}$$

Remarks i) If g is expressed in a local map as $(g_1,...,g_n) \in \mathbb{R}^n$, and $v_k = \partial_{g_k}$, then (A.5) reduce to the standard Euler-Lagrange equations.
ii) If the Lagrangian L does not depend on g, then $\hat{L} = \hat{L}(\omega)$ and so

$$v_k(\hat{L}) = 0. \tag{A.6}$$

Indeed, $v_k(\hat{L}) = \nabla_g\hat{L}\cdot v_k(g) = 0$.
iii) Suppose $M = G$ is a Lie group, and let v_k, $k = 1,...,n$ be left-invariant (or right-invariant) vector fields on G. Then $c_{ij}^k(g)$ are constant:

$$c_{ij}^k(g) \equiv c_{ij}^k,\qquad g \in G. \tag{A.7}$$

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