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On the Lagrangian theory for rotating charge in the Maxwell field

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ABSTRACT

We justify the Hamilton least action principle for the Maxwell–Lorentz equations coupled with the equations of motion of Abraham's rotating extended electron. The main novelty in the proof is the application of the variational Poincaré equations on the Lie group $SO(3)$.

1. Introduction

We justify Hamilton's least action principle for the system of Maxwell–Lorentz equations coupled with the equations of motion of a rotating charged particle. Our main contribution is the variational derivation of the *Lorentz torque equation*, see Eq. (1.3) below.

First recall the case of a finite system of material points (q_i, m_i). The *angular momentum* is defined by

$$M := \sum q_i \wedge p_i := \sum q_i \wedge m_i \dot{q}_i. \tag{1.1}$$

By the second and the third Newton's laws this implies

$$\dot{M} = \sum q_i \wedge \dot{p}_i = \sum q_i \wedge F_i = \sum q_i \wedge F_i^{\text{ext}} = T, \tag{1.2}$$

where T is called the *external force torque*. Our aim is to derive the similar Abraham *non-relativistic torque equation* for a charged rigid body in the Maxwell field:

$$I\dot{\omega} = e \int (x-q) \wedge \left[E + E^{\text{ext}} + \left(\dot{q} + \omega \wedge (x-q)\right) \wedge \left(B + B^{\text{ext}}\right)\right] \times \rho(x-q)dx. \tag{1.3}$$

Here I is the moment of inertia, ω is the vector angular velocity, $\rho(x)$ is a distribution of the charge, and the right hand side is the torque of the Lorentz force. Note that in our case the fields E and B are generated by the motion of the charged body, and E^{ext}, B^{ext} are external fields.

Formally, the rigid body can be considered as an infinite system of material points. However Eq. (1.3) cannot be obtained directly from (1.2) since we cannot correctly take into account all the forces of mutual interaction between the different pieces of the rigid body. That is why we look for a different approach to the derivation of (1.3). We show that (1.3) follows from the Hamilton variational least action principle with the standard interaction term $-A_0\rho + \vec{A}\cdot\vec{j}$ in the Lagrangian density, $(A_0, \vec{A})$ being the potential of the fields, see (3.1), (3.2) below.

Let us comment on previous works. For the free rigid body (when $E, E^{\text{ext}}, B, B^{\text{ext}} = 0$) Eq. (1.3) reduces to the Euler's equations which have been obtained from the variational principle first by Poincaré [12]. In [1], this result has been extended to an external force field with an axial symmetry.

Eq. (1.3) is well recognized since Abraham's works [2,3]. In [2, Section 11] Abraham computed the Lagrangian of the Maxwell–Lorentz equations as integral of $-A_0\rho + \vec{A}\cdot\vec{j}$ for standing rotating spherically symmetric electron subject to external fields obeying very special symmetry conditions. In this case the Lagrangian depends only on one variable ω, the angular velocity. However, derivation of the torque equation (1.3) from the variational *Hamilton least action principle* remained an open question.

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The main goal of the Nodvik's paper [11] is a variational derivation of the Lorentz-covariant dynamics for the relativistic rotating charged particle in the Maxwell field, and the proof of the corresponding conservation laws. The system of Nodvick's equations is overdetermined, since they do not include rotational bare inertia. The situation was improved by Appel and Kiessling in [4], where they develop the theory for the relativistic rotating particle introducing a renormalization limit.

We propose an invariant derivation of the non-relativistic Abraham equation (1.3) from the Hamilton least action principle relying on the Poincaré equations [1,12] on the Lie group $SO(3)$.

The new interest for the rather old Abraham model is caused by the fact that a broad class of models of this type display *soliton-type asymptotics* and *scattering behavior* as it was discovered in recent years, see e.g. [5–7]. Lagrangian and Hamiltonian structure of the models play a significant role in these methods, so it is of a considerable interest and importance, to include the Abraham model with rotating charge into the class of Lagrangian systems.

The plan of our article is as follows. In Section 2 we state the Maxwell–Lorentz equations for the rotating charge, and in Section 3 we introduce the corresponding Lagrangian functional and formulate the Hamilton principle. In Section 4 we formulate our main result. In Section 5 we deduce the Lorentz force and the Maxwell equations. In Section 6 we deduce the Lorentz torque equation, and in Appendices A and B we collect some auxiliary calculations.

2. Maxwell–Lorentz equations

The Maxwell field consists of the electric field $E(x,t)$ and the magnetic field $B(x,t)$ generated by a motion of a rotating charge. The external fields E^{ext} and B^{ext} are generated by the corresponding external charges and currents. Let the rotating charge be centered at the position q with the velocity $\dot q$. For simplicity we assume that the mass distribution, $m\rho(x)$, and the charge distribution, $e\rho(x)$, are proportional to each other. Here m is the total mass, e is the total charge, and we use a system of units such that $m=1$, $e=1$.

For the coupling function ρ we apply everywhere below the conditions of regularity and spherical symmetry

$$\rho(x)=\rho_r\big(|x|\big),\quad \rho\in C_0^\infty\big(\mathbb{R}^3\big). \tag{2.1}$$

Then the system of Maxwell–Lorentz equations with the rotating charge reads, see [13]

$$\dot E=\nabla\wedge B-\big(\dot q+\omega\wedge(x-q)\big)\rho(x-q), \tag{2.2a}$$

$$\dot B=-\nabla\wedge E, \tag{2.2b}$$

$$\nabla\cdot E(x,t)=\rho\big(x-q(t)\big), \tag{2.3a}$$

$$\nabla\cdot B(x,t)=0, \tag{2.3b}$$

$$\ddot q=\int\big[E+E^{\text{ext}}+\big(\dot q+\omega\wedge(x-q)\big)\wedge\big(B+B^{\text{ext}}\big)\big]\times\rho(x-q)dx, \tag{2.4}$$

$$I\dot\omega=\int(x-q)\wedge\big[E+E^{\text{ext}}+\big(\dot q+\omega\wedge(x-q)\big)\wedge\big(B+B^{\text{ext}}\big)\big]\times\rho(x-q)dx, \tag{2.5}$$

where $\omega(t)$ is the vector angular velocity of the particle rotation (see Appendix A), and I is the moment of inertia defined by

$$I=\frac{2}{3}\int x^2\rho(x)dx. \tag{2.6}$$

Eqs. (2.2) are Maxwell equations with the corresponding charge density and current, and Eqs. (2.3) are the constraints. The back reaction of the field onto the particle is given through the Lorentz force equation (2.4), and the Lorentz torque equation (2.5) deals with rotational degrees of freedom.

Note that in [10], the direct proofs of the corresponding conservation laws are presented.

3. Lagrangian functional and the Hamilton principle

Our main goal is to deduce equations (2.2)–(2.5) from the Hamilton least action principle. First let us introduce *electromagnetic potentials* $\mathcal{A}=(A_0,A)$, $\mathcal{A}^{\text{ext}}=(A_0^{\text{ext}},A^{\text{ext}})$:

$$B=\nabla\wedge A,\qquad E=-\nabla A_0-\dot A. \tag{3.1}$$

$$B^{\text{ext}}=\nabla\wedge A^{\text{ext}},\qquad E^{\text{ext}}=-\nabla A_0^{\text{ext}}-\dot A^{\text{ext}}. \tag{3.2}$$

Next we define the Lagrangian

$$\begin{aligned}L(\mathcal{A},q,R,\dot{\mathcal{A}},\dot q,\dot R)&=\frac12\int\big(E^2-B^2\big)dx+\frac12\dot q^2+\frac12 I\omega^2\\&\quad-\int\big[A_0+A_0^{\text{ext}}\big]\rho(x-q)dx\\&\quad+\int\big(\dot q+\omega\wedge(x-q)\big)\cdot\big[A+A^{\text{ext}}\big]\rho(x-q)dx,\end{aligned} \tag{3.3}$$

where E, B are expressed in terms of $\mathcal{A}$, $\dot{\mathcal{A}}$ by (3.1), and $\omega=\mathcal{J}^{-1}\dot RR^{-1}$ by (A.2). The last two integrals represent the standard interaction term

$$\int\big[\big(A_0+A_0^{\text{ext}}\big)\rho-j\cdot\big(A+A^{\text{ext}}\big)\big]dx$$

in view of (A.1).

Remark 3.1. This Lagrangian functional does not depend on R due to the spherical symmetry of the charge and mass distributions. Respectively, the dynamical equations (2.2)–(2.5) do not involve R as well. In the case of nonradial densities the Lagrangian and the equations should involve R. Moreover, the moment of inertia (2.6) becomes a matrix in this case.

The corresponding action functional has the form

$$S=S(\mathcal{A},q,R):=\int_{t_1}^{t_2}L\big(\mathcal{A}(t),q(t),R(t),\dot{\mathcal{A}}(t),\dot q(t),\dot R(t)\big)dt. \tag{3.4}$$

Then the Hamilton least action principle reads

$$\delta S(\mathcal{A},q,R)=0, \tag{3.5}$$

where the variation is taken over $\mathcal{A}(t),q(t),R(t)$ with the boundary conditions

$$(\delta\mathcal{A},\delta q,\delta R)|_{t=t_1}=(\delta\mathcal{A},\delta q,\delta R)|_{t=t_2}=0. \tag{3.6}$$

4. Regular solutions and main result

Below we state our results for *regular solutions* of the system (2.2)–(2.5). This means that $q\in C^2(\mathbb{R},\mathbb{R}^3)$, $\omega\in C^1(\mathbb{R},\mathbb{R}^3)$, and all the involved functions and fields/potentials are sufficiently smooth and have (with all the necessary derivatives) a sufficient decay as $|x|\to\infty$ so that the partial integrations below are allowed.

Our main result is the following theorem.

Theorem 4.1. *For regular solutions the Maxwell–Lorentz system (2.2)–(2.5) is equivalent to the least action principle (3.5)–(3.6).*

We will analyze the variations in $\mathcal{A}$, q, R separately, namely, we prove that

$$\frac{\delta S}{\delta \mathcal{A}} = 0, \tag{4.1a}$$

$$\frac{\delta S}{\delta q} = 0, \tag{4.1b}$$

$$\frac{\delta S}{\delta R} = 0 \tag{4.1c}$$

is equivalent to (2.2)–(2.5).

5. The Maxwell equations and the Lorentz force

Maxwell equations

Eq. (4.1a) is equivalent to the Euler–Lagrange equations $\frac{d}{dt}\frac{\delta L}{\delta \dot{\mathcal{A}}} = L_{\mathcal{A}}$. Finally, these equations are equivalent to the Maxwell equations (2.2) with the constraints (2.3), see details in [9,8].

Lorentz force equation

For the simplicity of exposition, we omit $A_0^{\rm ext}$, $A^{\rm ext}$, $E^{\rm ext}$, and $B^{\rm ext}$ in the further computations within this section.

Eq. (4.1b) is equivalent to $\frac{d}{dt}L_{\dot q} = L_q$. We check that these Euler–Lagrange equations are equivalent to the Lorentz force equation (2.4). First, we rewrite the sum of last two integrals of (3.3) as

$$-\int A_0(x+q,t)\rho(x) + \int (\dot q + \omega \wedge x)\cdot A(x+q,t)\rho(x)dx.$$

For simplicity of notations we omit the arguments $x+q$ and t and write simply $A, A_j, \dot A, \dot A_j$ instead of $A(x+q,t)$, $A_j(x+q,t)$, $\dot A(x+q,t)$, $\dot A_j(x+q,t)$. We only have to remember that $\frac{d}{dt}A = \dot A + \dot q \cdot \nabla A$. We also write ρ instead of $\rho(x)$. Differentiating, we obtain

$$L_{\dot q} = \dot q + \int A\rho\, dx,$$

$$L_q = -\int \nabla A_0 \rho\, dx + \int (\dot q + \omega\wedge x)\cdot(\nabla A)\rho\, dx.$$

Here, for a vector field $b(x,t)$ we denote $b\cdot(\nabla A) = \sum b_j \nabla A_j$ (which differs from $(b\cdot\nabla)A$). In particular, if b does not depend on x the following identity holds:

$$b\cdot(\nabla A) - (b\cdot\nabla)A = b\wedge(\nabla\wedge A). \tag{5.1}$$

Hence, taking $b = \dot q$, we obtain that the Euler–Lagrange equations

$$\ddot q = \int \big[-\big(\dot A + (\dot q\cdot\nabla)A\big) - \nabla A_0 + (\dot q + \omega\wedge x)\cdot(\nabla A)\big]\rho dx$$
$$= \int \big[-\dot A - \nabla A_0 + \dot q\wedge(\nabla\wedge A) + (\omega\wedge x)\cdot(\nabla A)\big]\rho dx.$$

Now let us proceed to the Lorentz force equation (2.4). Using (3.1) and another change of variables, (2.4) turns into

$$\ddot q = \int \big[-\dot A - \nabla A_0 + \dot q\wedge(\nabla\wedge A) + (\omega\wedge x)\wedge(\nabla\wedge A)\big]\rho\, dx. \tag{5.2}$$

Hence, it remains to check that

$$\int (\omega\wedge x)\cdot(\nabla A)\rho\, dx = \int (\omega\wedge x)\wedge(\nabla\wedge A)\rho\, dx. \tag{5.3}$$

It suffices to show that

$$\int \big[(\omega\wedge x)\cdot(\nabla A) - (\omega\wedge x)\wedge(\nabla\wedge A)\big]\rho\, dx$$
$$= -\int A(\omega\cdot\nabla_\theta)\rho\, dx, \tag{5.4}$$

where $\nabla_\theta = (\nabla_{\theta_1}, \nabla_{\theta_2}, \nabla_{\theta_3})$, and ∇_{θ_j} is the differentiation in the angular coordinate θ_j around the coordinate axis x_j: $\nabla_{\theta_1} = x_2\partial_3 - x_3\partial_2$ etc. Then (5.3) follows since the last integral equals to zero by the spherical symmetry (2.1).

Let us check (5.4) for the first component, for the rest ones the computation is similar. The first component of the LHS of (5.3) equals

$$\int \big[(\omega_2x_3 - \omega_3x_2)\partial_1A_1 + (\omega_3x_1 - \omega_1x_3)\partial_1A_2$$
$$+ (\omega_1x_2 - \omega_2x_1)\partial_1A_3\big]\rho dx.$$

The first component of the RHS of (5.3) equals

$$\int \big[(\omega_3x_1 - \omega_1x_3)(\partial_1A_2 - \partial_2A_1)$$
$$- (\omega_1x_2 - \omega_2x_1)(\partial_3A_1 - \partial_1A_3)\big]\rho dx.$$

For the difference of the LHS and the RHS we apply partial integration, and obtain

$$\int \big[(\omega_2x_3 - \omega_3x_2)\partial_1A_1 + (\omega_3x_1 - \omega_1x_3)\partial_2A_1$$
$$+ (\omega_1x_2 - \omega_2x_1)\partial_3A_1\big]\rho dx$$
$$= -\int A_1\big[(\omega_2x_3 - \omega_3x_2)\partial_1 + (\omega_3x_1 - \omega_1x_3)\partial_2$$
$$+ (\omega_1x_2 - \omega_2x_1)\partial_3\big]\rho dx$$
$$= -\int A_1\big[\omega_1(x_2\partial_3 - x_3\partial_2) + \omega_2(x_3\partial_1 - x_1\partial_3)$$
$$+ \omega_3(x_1\partial_2 - x_2\partial_1)\big]\rho dx$$
$$= -\int A_1(\omega\cdot\nabla_\theta)\rho dx.$$

6. The Poincaré equations

It remains to check that (4.1c) is equivalent to (2.5). We are going to prove the equivalence applying the Poincaré method [1,12] to the variation of the Lagrangian functional

$$\hat L(R,\dot R,t) := L\big(\mathcal{A}(t), q(t), R, \dot{\mathcal{A}}(t), \dot q(t), \dot R\big). \tag{6.1}$$

Poincaré has obtained differential equations in local coordinates on M which are equivalent to the Hamilton least action principle, and replace the Euler–Lagrange equations in the case when the trajectories lye on a manifold M which is not a linear space [1,12].

In our case the trajectories $R(t)$ lye on the Lie group $SO(3) = M$, and respectively, the velocity $\dot R(t) \in T_{R(t)}SO(3)$, and also the variational derivative of the trajectory

$$\delta R(t) := \frac{d}{d\varepsilon}R_\varepsilon(t)\bigg|_{\varepsilon=0} \in T_{R(t)}SO(3).$$

The main problem is to express the variational derivative of the velocity $\delta\dot R(t)$ through the time derivative of $\delta R(t)$. In the case of linear space we would have $\delta\dot R(t) = \frac{d}{dt}\delta R(t)$ which results in the Euler–Lagrange equations after the partial integration.

Applying the Poincaré method, we will fix a suitable basis $v_k(\cdot)$, $k = 1, 2, 3$ of vector fields on $SO(3)$, and expand the velocity and the variation of the trajectory as

$$\dot{R}(t) = \sum_1^3 \omega_k(t) v_k\big(R(t)\big),$$
$$\delta R(t) = \sum_1^3 w_k(t) v_k\big(R(t)\big), \quad t \in \mathbb{R}. \tag{6.2}$$

The Poincaré calculations [1,12] give similar representation for $\delta\dot{R}(t)$ with the components

$$\delta\omega_k(t) = \dot{w}_k(t) + \sum_{i,j} c_{ij}^k\big(R(t)\big)\omega_i\big(R(t)\big)w_j\big(R(t)\big), \tag{6.3}$$

where the functions c_{ij}^k arise from commutation relations

$$[v_i, v_k](R) = \sum c_{ik}^j(R) v_j(R). \tag{6.4}$$

Here the vector fields v_k are identified with the corresponding differential operators (Lie derivatives along the vector fields). Finally, the formula (6.3) reduces the variational principle (4.1c) to the Poincaré equations

$$\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\omega_k} = \sum_{ij} c_{ik}^j \omega_i \frac{\partial\tilde{L}}{\partial\omega_j} + v_k\tilde{L}, \quad k = 1, 2, 3 \tag{6.5}$$

by the partial integration as in the case of Euler–Lagrange. Here $\tilde{L}(R, \omega, t)$ is the Lagrangian functional (6.1) with $\dot{R}$ expressed in ω: according to (6.2) we have $\dot{R} = \sum_1^3 \omega_k v_k(R)$, and hence

$$\tilde{L}(R, \omega, t) := L\left(\mathcal{A}(t), q(t), R, \dot{\mathcal{A}}(t), \dot{q}(t), \sum_1^3 \omega_k v_k(R)\right). \tag{6.6}$$

6.1. Right invariant vector fields

Let us construct suitable vector fields v_k on $SO(3)$ which provide the first expansion of (6.2), and calculate the corresponding functions $c_{ij}^k(\cdot)$. We will construct v_k as the right-invariant vector fields on $SO(3)$. Hence, it suffices to define $v_k(E) \in so(3) := T_E SO(3)$.

The Lie algebra $so(3)$ consists of skew-symmetric 3×3 matrices with the matrix commutation. It is isomorphic to the algebra $\mathbb{R}^3$ with vector product, through the isomorphism $\mathcal{J}$ of (A.2):

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \mathcal{J}(\omega_1, \omega_2, \omega_3). \tag{6.7}$$

In detail, if $A, B \in so(3)$, $a, b \in \mathbb{R}^3$, and $A = \mathcal{J}a$, $B = \mathcal{J}b$ by the isomorphism (6.7), then

$$AB - BA = \mathcal{J}(a \wedge b). \tag{6.8}$$

Finally, denote by $e_1 := (1, 0, 0)$, $e_2 := (0, 1, 0)$, $e_3 := (0, 0, 1)$ the orthonormal basis in $\mathbb{R}^3$. Then

$$e_1 \wedge e_2 = e_3, \quad e_2 \wedge e_3 = e_1, \quad e_3 \wedge e_1 = e_2. \tag{6.9}$$

Hence, setting $\tilde{e}_k := \mathcal{J}e_k$, we obtain from (A.2)

$$\dot{R}(t)R^{-1}(t) = \mathcal{J}\sum \omega_k(t)e_k = \sum \omega_k(t)\tilde{e}_k. \tag{6.10}$$

Therefore, we obtain the first expansion of (6.2):

$$\dot{R}(t) = \dot{R}(t)R^{-1}(t)R(t) = \sum \omega_k(t) v_k\big(R(t)\big),$$
$$v_k(R) := \tilde{e}_k R. \tag{6.11}$$

In other words, $\dot{R}(t)$ has the same coordinates w.r.t. the vector fields v_k at the point $R(t)$ as $\omega(t)$ in the basis $\{e_k\}$. The fields $v_k(R)$ are right translations of $\tilde{e}_k$, and hence they are right-invariant.

The next lemma is proved in Appendix B using the relations (6.9) and the isomorphism (6.7).

Lemma 6.1. *For the above constructed vector fields v_k on $SO(3)$ the following commutation relations hold:*

$$[v_1, v_2] = -v_3, \quad [v_2, v_3] = -v_1,$$
$$[v_3, v_1] = -v_2. \tag{6.12}$$

This lemma gives the corresponding coefficients c_{ij}^k:

$$c_{21}^3 = c_{32}^1 = c_{13}^2 = 1,$$
$$c_{31}^2 = c_{12}^3 = c_{23}^1 = -1, \quad \text{all the other } c_{ik}^j = 0. \tag{6.13}$$

Lemma 6.2. *The Lagrangian functional (6.6) does not depend on R.*

Proof. The Lagrangian (3.3) does not depend on R. Hence, (6.6) depends on R only through the term

$$\frac{1}{2}I\left|\mathcal{J}^{-1}\left[\sum_1^3 \omega_k v_k(R)\right]R^{-1}\right|^2 = \frac{1}{2}I\sum_1^3 |\omega_k e_k|^2 = \frac{1}{2}I\omega^2, \tag{6.14}$$

which does not depend on R. □

6.2. The Lorentz torque equation

Since the Lagrangian $\tilde{L}$ does not depend on R, we have $v_k(\tilde{L}) = 0$ for $k = 1, 2, 3$, and hence the Poincaré equations (6.5) become

$$\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\omega_k} = \sum_{ij} c_{ik}^j \omega_i \frac{\partial\tilde{L}}{\partial\omega_j}, \quad k = 1, 2, 3. \tag{6.15}$$

It remains to check that these equations are equivalent to the Lorentz torque equation (2.5). First,

$$\frac{\partial\tilde{L}}{\partial\omega} = I\omega + \int (x \wedge A)\rho\, dx.$$

Therefore,

$$\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\omega} = I\dot{\omega} + \int x \wedge \big(\dot{A} + (\dot{q} \cdot \nabla)A\big)\rho\, dx.$$

On the other hand, (6.13) implies that the RHS of (6.15), in the vector form, equals

$$\omega \wedge \left(I\omega + \int (x \wedge A)\rho\, dx\right) = \int \omega \wedge (x \wedge A)\rho\, dx$$
$$= \int \big(x(\omega \cdot A) - (\omega \cdot x)A\big)\rho\, dx.$$

Finally, the Poincaré equations read

$$I\dot{\omega} = \int \big[-x \wedge \big(\dot{A} + (\dot{q} \cdot \nabla)A\big) + x(\omega \cdot A) - (\omega \cdot x)A\big]\rho\, dx. \tag{6.16}$$

Now let us proceed to the Lorentz torque equation (2.5). Introduce $E = -\nabla \cdot A_0 - \dot{A}$, $B = \nabla \wedge A$, make the mentioned change of the variables and obtain that (2.5) reads

$$I\dot{\omega} = \int x \wedge (-\nabla \cdot A_0 - \dot{A})\rho\, dx + \int x \wedge \big(\dot{q} \wedge (\nabla \wedge A)\big)\rho\, dx$$
$$+ \int x \wedge \big((\omega \wedge x) \wedge (\nabla \wedge A)\big)\rho\, dx. \tag{6.17}$$

After partial integration the first integral becomes

$$-\int (x\wedge \dot{A})\rho\,dx + \int A_0\nabla_\theta\rho\,dx = -\int (x\wedge \dot{A})\rho\,dx,$$

since $\nabla_\theta\rho = 0$ in view of spherical symmetry of ρ. Similarly, the second integral of (6.17) transforms to

$$-\int \big[x\wedge(\dot{q}\cdot\nabla)A + (\dot{q}\cdot\nabla)\nabla_\theta\big]\rho\,dx = -\int \big(x\wedge(\dot{q}\cdot\nabla)A\big)\rho\,dx.$$

For the third integral of (6.17) we apply the identity

$$x\wedge\big[(\omega\wedge x)\wedge(\nabla\wedge A)\big] = (\omega\wedge x)\big(x\cdot(\nabla\wedge A)\big)$$

and after partial integration obtain

$$\int\big[x(\omega\cdot A) - (\omega\cdot x)A\big]\rho\,dx - \int(\omega\wedge x)(A\cdot\nabla_\theta)\rho\,dx$$
$$= \int\big[x(\omega\cdot A) - (\omega\cdot x)A\big]\rho\,dx.$$

Finally, (2.5) reads

$$I\dot{\omega} = -\int (x\wedge\dot{A})\rho\,dx - \int\big(x\wedge(\dot{q}\cdot\nabla)A\big)\rho\,dx + \int\big[x(\omega\cdot A)-(\omega\cdot x)A\big]\rho\,dx \quad (6.18)$$

which coincides with (6.16). Theorem 4.1 is proved. □

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Appendix A. Angular velocity

We denote by $\omega(t)\in\mathbb{R}^3$ the vector angular velocity "in space" (in the terminology of [1]) of the charge rotation. Namely, the trajectory $x(\cdot)$ of each fixed point of the body is given by

$$x(t) = q(t) + R(t)\big(x(0) - q(0)\big),$$

where $q(t)$ is the "trajectory of the body", and $R(t)\in SO(3)$. Respectively, the velocity of this fixed point reads

$$\dot{x}(t) = \dot{q}(t) + \dot{R}(t)\big(x(0)-q(0)\big) = \dot{q}(t) + \dot{R}(t)R^{-1}(t)\big(x(t)-q(t)\big)$$
$$= \dot{q}(t) + \omega(t)\wedge\big(x(t)-q(t)\big), \quad (A.1)$$

where $\omega(t)\in\mathbb{R}^3$ corresponds to the skew-symmetric matrix $\dot{R}(t)R^{-1}(t)$ by the rule

$$\dot{R}(t)R^{-1}(t) = \mathcal{J}\omega(t) := \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix}. \quad (A.2)$$

Appendix B. Commutators of right-invariant vector fields

We prove Lemma 6.1

Step 1. By (6.8) the isomorphism (6.7) translates relations (6.9) to

$$[\tilde{e}_1,\tilde{e}_2] = \tilde{e}_3, \qquad [\tilde{e}_3,\tilde{e}_1] = \tilde{e}_2, \qquad [\tilde{e}_2,\tilde{e}_3] = \tilde{e}_1 \quad (B.1)$$

in the sense of matrix commutator.

Step 2. Recall that the right-invariant vector fields v_k on $SO(3)$ are defined by right translations $v_k(R) = \tilde{e}_k R$, where $R\in SO(3)$. We should prove (6.12) in the sense of the commutators of vector fields on the Lie group $SO(3)$.

Since the fields v_k are right-invariant, it suffices to check the relations (6.12) at the group unit E. Let us compute the derivative $v_A f$ of a smooth function f on $SO(3)$ along a right-invariant field v_A such that $v_A(E) = A\in SO(3)$. In this case $v_A(R) = AR$ for $R\in SO(3)$. Consider a smooth path $R_1(\varepsilon)\in SO(3)$ such that $R_1(0) = R$, $\dot{R}_1(0) = AR$. Then

$$v_A f(R) := \frac{d}{d\varepsilon} f\big(R_1(\varepsilon)\big)\Big|_{\varepsilon=0}$$
$$= \big[f'\big(R_1(\varepsilon)\big)\cdot\dot{R}_1(\varepsilon)\big]\big|_{\varepsilon=0} = f'(R)\cdot AR.$$

In particular,

$$v_{[A,B]}f(E) = f'(E)\cdot[A,B], \quad (B.2)$$

where $[A,B] = AB - BA$ is the matrix commutator.

Now let us compute $v_A v_B f(E)$ for a right-invariant field v_B such that $v_B(E) = B\in so(3)$, $v_B(R) = BR$. Consider a smooth path $R_2(\varepsilon)\in SO(3)$ such that $R_2(0) = E$, $\dot{R}_2(0) = A$. Then

$$v_A v_B f(E) = \frac{d}{d\varepsilon}\big[f'\big(R_2(\varepsilon)\big)\cdot BR_2(\varepsilon)\big]\Big|_{\varepsilon=0}$$
$$= \frac{d}{d\varepsilon} f'\big(R_2(\varepsilon)\big)\Big|_{\varepsilon=0}\cdot BR_2(\varepsilon)|_{\varepsilon=0} + f'\big(R_2(\varepsilon)\big)\big|_{\varepsilon=0}\cdot\frac{d}{d\varepsilon}BR_2(\varepsilon)\Big|_{\varepsilon=0}$$
$$= \big[f''\big(R_2(\varepsilon)\big)\cdot\dot{R}_2(\varepsilon)\big]\big|_{\varepsilon=0}\cdot BR_2(\varepsilon)|_{\varepsilon=0} + \big[f'\big(R(\varepsilon)\big)\cdot B\dot{R}_2(\varepsilon)\big]\big|_{\varepsilon=0}$$
$$= \big(f''(E)\cdot A\big)\cdot B + f'(E)\cdot BA.$$

Then, since the form $(f''(E)\cdot A)\cdot B$ is symmetric w.r.t. the matrices A, B one has

$$[v_A, v_B]f(E) = v_A v_B f(E) - v_B v_A f(E) = f'(E)\cdot(BA - AB)$$
$$= -v_{[A,B]}f(E)$$

by (B.2). Together with (B.1) this completes the proof.

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