Scattering of Solitons for Dirac Equation Coupled to a Particle

A. I. Komech [1]
Fakultät für Mathematik, Universität Wien
and Institute for Information Transmission Problems RAS
e-mail: alexander.komech@univie.ac.at

E. A. Kopylova [2]
Institute for Information Transmission Problems RAS
B.Karetnyi 19, Moscow 101447,GSP-4, Russia
e-mail: elena.kopylova@univie.ac.at

H. Spohn
Faculty of Mathematics, TU Munich, Garching, 85747 Germany
email: spohn@ma.tum.de

Abstract

We establish soliton-like asymptotics for finite energy solutions to the Dirac equation coupled to a relativistic particle. Any solution with initial state close to the solitary manifold, converges in long time limit to a sum of traveling wave and outgoing free wave. The convergence holds in global energy norm. The proof uses spectral theory and symplectic projection onto solitary manifold in the Hilbert phase space.

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1 Introduction

We prove long time convergence to sum of a soliton and dispersive wave for the Dirac equation coupled to a relativistic particle. The convergence holds in global energy norm for finite energy solutions with initial state close to the solitary manifold. Our main motivation is to develop techniques of Buslaev and Perelman [2, 3] in the context of the Dirac equation. The development is not straightforward because of known peculiarities of the Dirac equation: nonpositivity of the energy, algebra of the Dirac matrices, etc. We expect that the result might be extended to relativistic nonlinear Dirac equations relying on an appropriate development of our techniques.

Let $\psi(x) \in \mathbb{C}^4$ be a Dirac spinor field in $\mathbb{R}^3$, coupled to a relativistic particle with position q and momentum p, governed by

$$\left\{\begin{array}{l} i\dot\psi(x,t) = [-i\alpha_1\partial_1 - i\alpha_2\partial_2 - i\alpha_3\partial_3 + \beta m]\psi(x,t) + \rho(x-q(t)) \\ \\ \dot q(t) = p(t)/\sqrt{1+p^2(t)}, \qquad \dot p(t) = \mathrm{Re}\langle\psi(x,t), \nabla\rho(x-q(t))\rangle \end{array}\right| \quad x\in\mathbb{R}^3 \tag{1.1}$$

where $\rho \in C(\mathbb{R}^3, \mathbb{C}^4)$ and $\langle\cdot,\cdot\rangle$ stands for the Hermitian scalar product on $L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$. Here $\partial_j = \partial/\partial x_j$, α_j and β are 4×4 Dirac matrices. The standard representation for the Dirac matrices α_j and β (in 2×2 blocks) is

$$\beta = \alpha_0 = \left(\begin{array}{cc} I_2 & 0 \\ 0 & -I_2 \end{array}\right), \quad \alpha_j = \left(\begin{array}{cc} 0 & \sigma_j \\ \sigma_j & 0 \end{array}\right), \quad j = 1,2,3$$

where I_2 denotes the unit 2×2 matrix, and

$$\sigma_1 = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right), \quad \sigma_2 = \left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right), \quad \sigma_3 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right)$$

The matrices α_j, $j = 0,1,2,3$ are Hermitian, and satisfy the anticommutation relations

$$\alpha_j^* = \alpha_j, \quad \alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk} \tag{1.2}$$

We will use the following *real orthogonality relations*

$$\beta\psi\cdot\alpha_j\psi = 0, \quad j = 1,3, \quad \text{and} \quad \alpha_2\psi\cdot\psi = 0, \qquad \psi\in\mathbb{R}^4 \tag{1.3}$$

Here and below " $\cdot$ " stands for the Hermitian scalar product on $\mathbb{C}^n$ with appropriate n. System (1.1) is translation-invariant, and admits soliton solutions

$$s_{a,v}(t) = (\psi_v(x-vt-a), vt+a, p_v), \quad p_v = v/\sqrt{1-v^2} \tag{1.4}$$

for all $a, v\in\mathbb{R}^3$ with $|v|<1$. The states $S_{a,v} := s_{a,v}(0)$ form the solitary manifold

$$\mathcal{S} := \{S_{a,v} : a, v\in\mathbb{R}^3, |v|<1\}$$

Our main result is soliton-type asymptotics

$$\psi(x,t) \sim \psi_{v_\pm}(x - v_\pm t - a_\pm) + W_0(t)\phi_\pm, \quad t\to\pm\infty \tag{1.5}$$

for solutions to (1.1) with initial data close to solitary manifold $\mathcal{S}$. Here $W_0(t)$ is the dynamical group of the free Dirac equation, $\phi_\pm$ are the corresponding *asymptotic scattering states*, and the asymptotics hold in the *global norm* of the Hilbert space $L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$. For the particle trajectory we prove that

$$\dot q(t) \to v_\pm, \quad q(t) \sim v_\pm t + a_\pm, \quad t\to\pm\infty \tag{1.6}$$

The results are established under the following conditions on the complex valued charge distribution: for some $\nu > 5/2$

$$(1+|x|)^{\nu}|\partial^{\alpha}\rho(x)| \in L^2(\mathbb{R}^3), \quad |\alpha| \le 3 \tag{1.7}$$

We assume that $\rho(x)$ is spherically symmetric for simplicity of calculations. Finally, we assume the Wiener condition for the Fourier transform $\hat{\rho} = (2\pi)^{-3/2}\displaystyle\int e^{ikx}\rho(x)dx$:

$$\mathcal{B}(k) = m\beta\hat{\rho}(k)\cdot\hat{\rho}(k) > 0, \quad k \in \mathbb{R}^3 \tag{1.8}$$

which is nonlinear version of the Fermi Golden Rule in our case (cf. [4, 13, 14, 15]): nonlinear perturbation is not orthogonal to eigenfunctions of continuous spectrum of the linear part. The examples are easily constructed. Namely, let us rewrite (1.8) in the form

$$\mathcal{B}(k) = m[|\hat{\rho}_1(k)|^2 + |\hat{\rho}_2(k)|^2 - |\hat{\rho}_3(k)|^2 - |\hat{\rho}_4(k)|^2] > 0, \quad k \in \mathbb{R}^3 \tag{1.9}$$

Therefore, we can take e.g. ρ_1 constructed in [12], and $\rho_2 = \rho_3 = \rho_4 = 0$.

The system (1.1) describes charged particle interacting with its "own" Dirac field. The asymptotics (1.5)-(1.6) mean asymptotic stability of uniform motion, i.e. "law of inertia". The stability is caused by "radiative damping", i.e. radiation of energy to infinity appearing analytically as a local energy decay for solutions to linearized equation. The radiative damping was suggested first by M.Abraham in 1905 in the context of the Maxwell-Lorentz equations, [1].

One could expect asymptotics (1.5) for small perturbations of solitons for relativistic nonlinear Dirac equations and for coupled nonlinear Maxwell-Dirac equations whose solitons were constructed in [6]. Our result models this situation though the relativistic case is still open problem.

Asymptotics of type (1.5)-(1.6) were obtained previously for the Klein-Gordon and Schrödinger equations coupled to the particle [8, 10]. More weak asymptotics of type (1.5) in the *local energy norms*, and without the dispersive wave, were obtained in [7] and [11] for *all finite energy solutions* to the Maxwell-Lorentz and wave equations.

Let us comment on our approach. For 1D translation invariant Schrödinger equation, the asymptotics of type (1.5) were proved for the first time by Buslaev and Perelman [2, 3, 4], and extended by Cuccagna [5] to higher dimensions. We prove the asymptotics (1.5)– (1.6) applying approach [8] where general Buslaev and Perelman strategy has been developed for the case of 3D Klein-Gordon equation: i) symplectic orthogonal decomposition of the dynamics near the solitary manifold, ii) modulation equations for the symplectic projection onto the manifold, and iii) time decay in transversal directions, etc (see more details in Introduction [8]). Our main novelties are the following.

I. It is well known that the Hamiltonian of the Dirac equation is nonpositive. Respectively, the energy conservation does not provide a uniform a priori estimate for the solution. We obtain an estimate for L^2 norm of the solution using unitarity of free Dirac propagator. This estimate provides existence of global solution.

On the other hand, for linearized equation unitarity of free propagator provides only exponential bounds $\sim e^{\alpha t}$ for L^2 norm of the solution. This guarantees existence of global solution and analyticity of the resolvent for $\operatorname{Re}\lambda > \alpha$. However, we need the analyticity for $\operatorname{Re}\lambda > 0$ which we obtain by bifurcation arguments relying on our analysis of the Fourier-Laplace transform (Lemma 14.4).

II. We establish an appropriate decay of the linearized dynamics:

i) We prove time decay $\sim t^{-3/2}$ in weighted norms for a modified free Dirac equation (Lemma 15.1). The proof relies on a "soft version" of strong Huygens principle for the Dirac equation. Namely, the free Dirac propagator is concentrated mainly near the light cone, while contribution of inner zone is a Hilbert-Schmidt operator.

ii) We compute all needed spectral properties of the linearized equation at the soliton in contrast to majority of works in the field, where the corresponding spectral properties are postulated. Namely, we find that under the Wiener condition (1.8), discrete spectrum consists only of one zero point with algebraic multiplicity 6. The multiplicity is totally due to the translation invariance of the system (1.1).

iii) We exactly calculate the symplectic orthogonality conditions (15.18) for initial data of the linearized equation. These conditions are necessary for the proof of the decay.

All computations differ significantly from the case of the Klein-Gordon equation [8] because of algebra of the Dirac matrices. An important role play the real orthogonality relations (1.3).

Our paper is organized as follows. In Section 2, we formulate the main result. In Section 3, we introduce symplectic projection onto the solitary manifold. The linearized equation is considered in Sections 4 and 5. In Section 6, we split the dynamics in two components: along the solitary manifold, and in transversal directions. The time decay of transversal component is established in sections 7-10. In Section 11 we prove main result. In Sections 12 - 15 we justify the time decay of the linearized dynamics relying on weighted decay for the free Dirac equation in a moving frame. In Appendices A, B and C we collect some technical calculations.

2 Main results

2.1 Existence of dynamics

We consider the Cauchy problem for the system (1.1) which we write as

$$\dot{Y}(t) = F(Y(t)), \quad t \in \mathbb{R}: \quad Y(0) = Y_0 \tag{2.1}$$

Here $Y(t) = (\psi(t), q(t), p(t))$, $Y_0 = (\psi(0), q_0, p_0)$. We introduce a suitable phase space for equation (2.1). Let L^2_α, $\alpha \in \mathbb{R}$ be weighted Agmon spaces with norm $\|\psi\|_\alpha = \|\psi\|_{L^2_\alpha} := \|(1+|x|)^\alpha \psi\|_{L^2}$.

Definition 2.1. *i) The phase space $\mathcal{E}$ is the Hilbert space $L^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ of states $Y = (\psi, q, p)$ with finite norm*

$$\|Y\|_{\mathcal{E}} = \|\psi\|_0 + |q| + |p| \ .$$

ii) $\mathcal{E}_\alpha$ is the space $L^2_\alpha \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with finite norm

$$\|\, Y\|_{\mathcal{E}_\alpha} = \|\psi\|_\alpha + |q| + |p| \ .$$

Proposition 2.2. *Let (1.7) hold. Then*
(i) For every $Y_0 \in \mathcal{E}$ the Cauchy problem (2.1) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$.
(ii) For every $t \in \mathbb{R}$, map $U(t) : Y_0 \mapsto Y(t)$ is continuous on $\mathcal{E}$.

Proof. Step i) First, we fix an arbitrary $b > 0$ and prove *(i)-(ii)* for $Y_0 \in \mathcal{E}$ such that $\|\psi_0\|_0 \le b$ and $|t| \le \varepsilon = \varepsilon(b)$ for some sufficiently small $\varepsilon(b) > 0$. We rewrite (2.1) as

$$\dot{Y}(t) = F_1(Y(t)) + F_2(Y(t)), \quad t \in \mathbb{R}, \quad Y(0) = Y_0 \ , \tag{2.2}$$

$$F_1 : Y \mapsto ((-\alpha\cdot\nabla - i\beta m)\psi, 0, 0), \quad F_2 : Y \mapsto (-i\rho(x-q),\ p/\sqrt{1+p^2},\ \mathrm{Re}\langle\psi, \nabla\rho(x-q)\rangle)$$

The Fourier transform provides existence and uniqueness of solution $Y_1(t) \in C(\mathbb{R}, \mathcal{E})$ to (2.2) with $F_2 = 0$. Let $U_1(t) : Y_0 \mapsto Y_1(t)$ be the corresponding strongly continuous group of bounded linear operators on $\mathcal{E}$. Then (2.2) for $Y(t) \in C(\mathbb{R}, \mathcal{E})$ is equivalent to integral Duhamel equation

$$Y(t) = U_1(t)Y_0 + \int_0^t ds\ U_1(t-s)F_2(Y(s))\ . \tag{2.3}$$

Further, the map $F_2 : Y \mapsto F_2(Y)$ is locally Lipschitz continuous in $\mathcal{E}$: for each $b > 0$ there exist a $\varkappa = \varkappa(b) > 0$ such that for all $Y_1 = (\psi_1, q_1, p_1), Y_2 = (\psi_2, q_2, p_2) \in \mathcal{E}$ with $\|\psi_1\|_0,\ \|\psi_2\|_0 \le b$,

$$\|F_2(Y_1) - F_2(Y_2)\|_{\mathcal{E}} \le \varkappa\|Y_1 - Y_2\|_{\mathcal{E}}\ .$$

Therefore, by the contraction mapping principle, equation (2.3) has a unique local solution $Y(\cdot) \in C([-\varepsilon, \varepsilon], \mathcal{E})$ with $\varepsilon > 0$ depending only on b.

Step ii) Second, we derive *a priori* estimate. Consider $\psi_0 \in C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$. Then

$$\frac{d}{dt}\|\psi\|_0^2 = \int(\psi\cdot\dot\psi + \dot\psi\cdot\psi)dx = 2\int \mathrm{Im}\,\big(\rho(x-q)\cdot\psi(x)\big)dx \le C\|\psi\|_0\ .$$

Hence,

$$\|\psi(t)\|_0 \le \frac{1}{2}Ct + \|\psi(0)\|_0$$

Last two equalities (1.1) imply *a priori* estimates for $|\dot p|$ and $|\dot q|$. The *a priori* estimates for general initial data $\psi_0 \in L^2$ follow by approximating initial data by functions from $C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$.

Step iii) Properties *(i)-(ii)* for arbitrary $t \in \mathbb{R}$ now follow from the same properties for small $|t|$ and from *a priori* estimate. □

2.2 Solitary manifold and main result

Let us compute the solitons (1.4). Substitution to (1.1) gives stationary equations

$$\left.\begin{aligned} -iv\cdot\nabla\psi_v(y) &= [-i\alpha\cdot\nabla + \beta m]\psi_v(y) + \rho(y) \\ v = p_v/\sqrt{1+p_v^2}, &\quad 0 = \mathrm{Re}\langle\psi_v(y), \nabla\rho(y)\rangle \end{aligned}\right| \tag{2.4}$$

Applying Fourier transform to first equation in (2.4) we obtain

$$(-v\cdot k + \alpha\cdot k - \beta m)\hat\psi_v(k) = \hat\rho(k)\ .$$

Then

$$\hat\psi_v(k) = -\frac{(v\cdot k + \alpha\cdot k - \beta m)\hat\rho(k)}{(v\cdot k + \alpha\cdot k - \beta m)(v\cdot k - \alpha\cdot k + \beta m)} = \frac{(v\cdot k + \alpha\cdot k - \beta m)\hat\rho(k)}{k^2 + m^2 - (v\cdot k)^2}\ . \tag{2.5}$$

The soliton is given by the formula

$$\psi_v(x) = \frac{i\gamma}{4\pi}(v\cdot\nabla + \alpha\cdot\nabla + i\beta m)\int \frac{e^{-m|\gamma(y-x)_\parallel + (y-x)_\perp|}\rho(y)d^3y}{|\gamma(y-x)_\parallel + (y-x)_\perp|}, \quad p_v = \gamma v = \frac{v}{\sqrt{1-v^2}}\ . \tag{2.6}$$

Here we set $\gamma = 1/\sqrt{1-v^2}$ and $x = x_\parallel + x_\perp$, where $x_\parallel \| v$ and $x_\perp \perp v$ for $x \in \mathbb{R}^3$. It remains to prove that last equation of (2.4) holds. Indeed, (2.5) and Parseval identity imply

$$\mathrm{Re}\langle\psi_v(y), \partial_j\rho(y)\rangle = \mathrm{Re}\langle ik_j\hat{\psi}_v(k), \hat{\rho}(k)\rangle = \mathrm{Re}\int ik_j \frac{(v\cdot k+\alpha\cdot k-\beta m)\hat{\rho}(k)\cdot\hat{\rho}(k)}{k^2+m^2-(v\cdot k)^2}dk = 0$$

since the integrand is pure imaginary by (1.3). Hence, the soliton solution (1.4) exists and is defined uniquely for $|v|<1$ and $a \in \mathbb{R}^3$. Denote $V := \{v \in \mathbb{R}^3 : |v| < 1\}$.

Definition 2.3. *A soliton state is* $S(\sigma) := (\psi_v(x-b), b, v)$, *where* $\sigma := (b, v)$ *with* $b \in \mathbb{R}^3$ *and* $v \in V$.

Obviously, the soliton solution admits representation $S(\sigma(t))$, where

$$\sigma(t) = (b(t), v(t)) = (vt + a, v)\ . \tag{2.7}$$

Definition 2.4. *A solitary manifold is the set* $\mathcal{S} := \{S(\sigma) : \sigma \in \Sigma := \mathbb{R}^3 \times V\}$.

The main result of our paper is the following theorem.

Theorem 2.5. *Let* (1.7)-*(1.8) hold,* $\nu > 5/2$ *be the number from* (1.7), *and* $Y(t)$ *be the solution to the Cauchy problem (2.1) with initial state* Y_0 *which is sufficiently close to the solitary manifold:*

$$d_0 := \mathrm{dist}_{\mathcal{E}_\nu}(Y_0, \mathcal{S}) \ll 1\ . \tag{2.8}$$

Then the asymptotics hold for $t \to \pm\infty$,

$$\dot{q}(t) = v_\pm + \mathcal{O}(|t|^{-2}), \quad q(t) = v_\pm t + a_\pm + \mathcal{O}(|t|^{-1}) \tag{2.9}$$

$$\psi(x,t) = \psi_{v\pm}(x - v_\pm t - a_\pm) + W_0(t)\phi_\pm + r_\pm(x,t) \tag{2.10}$$

with

$$\|r_\pm(t)\|_0 = \mathcal{O}(|t|^{-1/2})\ .$$

It suffices to prove asymptotics (2.9)-(2.10) for $t \to +\infty$ since system (1.1) is time reversible.

3 Symplectic projection

3.1 Hamiltonian structure

Denote $\psi_1 = \mathrm{Re}\,\psi$, $\psi_2 = \mathrm{Im}\,\psi$, $\rho_1 = \mathrm{Re}\,\rho$, $\rho_2 = \mathrm{Im}\,\rho$, $\tilde{\alpha}_2 = -i\alpha_2$. Then (1.1) reads

$$\left\{\begin{array}{l} \dot{\psi}_1(x,t) = -(\alpha_1\partial_1+\alpha_3\partial_3)\psi_1(x,t) + (\tilde{\alpha}_2\partial_2+\beta m)\psi_2(x,t) + \rho_2(x-q(t)) \\ \dot{\psi}_2(x,t) = -(\tilde{\alpha}_2\partial_2+\beta m)\psi_1(x,t) - (\alpha_1\partial_1+\alpha_3\partial_3)\psi_2(x,t) - \rho_1(x-q(t)) \\ \dot{q}(t) = p(t)/\sqrt{1+p^2(t)} \\ \dot{p}(t) = \displaystyle\int \big(\psi_1(x,t)\cdot\nabla\rho_1(x-q(t)) + \psi_2(x,t)\cdot\nabla\rho_2(x-q(t))\big)dx \end{array}\right| \quad x \in \mathbb{R}^3 \tag{3.1}$$

This is a Hamilton system with the Hamilton functional

$$\mathcal{H}(\psi_1,\psi_2,q,p) = \frac{1}{2}\int(\psi_1\cdot(\tilde{\alpha}_2\partial_2+\beta m)\psi_1 + \psi_2\cdot(\tilde{\alpha}_2\partial_2+\beta m)\psi_2 + 2\psi_1\cdot(\alpha_1\partial_1+\alpha_3\partial_3)\psi_2)dx$$
$$+\int(\psi_1(x)\cdot\rho_1(x-q) + \psi_2(x)\cdot\rho_2(x-q))dx + \sqrt{1+p^2}\ .$$

Equation (3.1) can be written as a Hamilton system

$$\dot{Y} = JD\mathcal{H}(Y), \quad Y = (\psi_1, \psi_2, q, p), \quad J := \begin{pmatrix} 0 & I_4 & 0 & 0 \\ -I_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_3 \\ 0 & 0 & -I_3 & 0 \end{pmatrix} \tag{3.2}$$

where $D\mathcal{H}$ is the Fréchet derivative with respect to ψ_{1k}, ψ_{2k}, $k = 1, 2, 3, 4$, p and q of the Hamilton functional.

3.2 Symplectic projection onto solitary manifold

Let us identify tangent space to $\mathcal{E}$, at every point, with $\mathcal{E}$. Consider symplectic form Ω defined on $\mathcal{E}$ by $\Omega = \displaystyle\int d\psi_1(x) \wedge d\psi_2(x)\, dx + dq \wedge dp$, i.e.

$$\Omega(Y^1, Y^2) = \langle Y^1, JY^2 \rangle, \quad Y^j = (\psi_1^j, \psi_2^j, q^j, p^j) \in \mathcal{E}, \quad j = 1, 2 \tag{3.3}$$

where $\langle Y^1, Y^2 \rangle := \langle \psi_1^1, \psi_1^2 \rangle + \langle \psi_2^1, \psi_2^2 \rangle + q^1 \cdot q^2 + p^1 \cdot p^2$. It is clear that the form Ω is non-degenerate, i.e.

$$\Omega(Y^1, Y^2) = 0 \ \text{ for every } \ Y^2 \in \mathcal{E} \implies Y^1 = 0$$

Definition 3.1. *i)* $Y^1 \nmid Y^2$ *means that* $Y^1 \in \mathcal{E}$, $Y^2 \in \mathcal{E}$, *and* Y^1 *is symplectic orthogonal to* Y^2, *i.e.* $\Omega(Y^1, Y^2) = 0$.

ii) A projection operator $\mathbf{P} : \mathcal{E} \to \mathcal{E}$ *is called symplectic orthogonal if* $Y^1 \nmid Y^2$ *for* $Y^1 \in \operatorname{Ker} \mathbf{P}$ *and* $Y^2 \in \operatorname{Im} \mathbf{P}$.

Let us consider tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ to the manifold $\mathcal{S}$ at a point $S(\sigma)$. Vectors $\tau_j := \partial_{\sigma_j} S(\sigma)$, where $\partial_{\sigma_j} := \partial_{b_j}$ and $\partial_{\sigma_{j+3}} := \partial_{v_j}$ with $j = 1, 2, 3$, form a basis in $\mathcal{T}_\sigma\mathcal{S}$. In detail,

$$\left.\begin{array}{rcl} \tau_j = \tau_j(v) & := & \partial_{b_j} S(\sigma) = (\,-\partial_j \psi_{v1}(y)\,, -\partial_j \psi_{v2}(y)\,,\ e_j\,,\quad 0\ \,) \\ \tau_{j+3} = \tau_{j+3}(v) & := & \partial_{v_j} S(\sigma) = (\ \ \partial_{v_j} \psi_{v1}(y)\,,\ \partial_{v_j} \psi_{v2}(y)\,,\ 0\ ,\ \partial_{v_j} p_v\,) \end{array}\right| \ j = 1, 2, 3 \tag{3.4}$$

where $\psi_{v1} = \operatorname{Re} \psi_v$, $\psi_{v2} = \operatorname{Im} \psi_v$, $y := x - b$ is the "moving frame coordinate", $e_1 = (1, 0, 0)$ etc. Formula (2.6) and condition (1.7) imply that

$$\tau_j(v) \in \mathcal{E}_\alpha, \quad v \in V, \quad j = 1, \ldots, 6, \quad \forall \alpha \in \mathbb{R}\ . \tag{3.5}$$

Lemma 3.2. *The matrix with the elements* $\Omega(\tau_l(v), \tau_j(v))$ *is non-degenerate* $\forall v \in V$.

Proof. The elements are computed in Appendix A. As a result, matrix $\Omega(\tau_l, \tau_j)$ reads

$$\Omega(v) := (\Omega(\tau_l, \tau_j))_{l,j=1,\ldots,6} = \begin{pmatrix} 0 & \Omega^+(v) \\ -\Omega^+(v) & 0 \end{pmatrix} \tag{3.6}$$

where the 3×3-matrix $\Omega^+(v)$ equals

$$\Omega^+(v) = K + (1 - v^2)^{-1/2} E + (1 - v^2)^{-3/2} v \otimes v\ . \tag{3.7}$$

Here K is a symmetric 3×3-matrix with the elements

$$K_{ij} = \int dk k_i k_j \mathcal{B}(k) \frac{k^2 + m^2 + 3(v \cdot k)^2}{(k^2 + m^2 - (v \cdot k)^2)^3} \tag{3.8}$$

where $\mathcal{B}(k) > 0$ is defined in (1.8). The matrix K is the integral of symmetric nonnegative definite matrix $k \otimes k = (k_i k_j)$ with a positive weight. Hence, the matrix K is nonnegative definite. Since unite matrix E is positive definite, the matrix $\Omega^+(v)$ is symmetric and positive definite, hence non-degenerate. Then the matrix $\Omega(\tau_l, \tau_j)$ also is non-degenerate. $\square$

Let us introduce translations $T_a : (\psi(\cdot), q, p) \mapsto (\psi(\cdot - a), q + a, p)$, $a \in \mathbb{R}^3$. Note that the manifold $\mathcal{S}$ is invariant with respect to the translations. Let us denote $v(p) := p/\sqrt{1+p^2}$ for $p \in \mathbb{R}^3$.

Definition 3.3. *i) For any $\alpha \in \mathbb{R}$ and $\overline{v} < 1$ denote by $\mathcal{E}_\alpha(\overline{v}) = \{Y = (\psi, q, p) \in \mathcal{E}_\alpha : |v(p)| \le \overline{v}\}$. We set $\mathcal{E}(\overline{v}) := \mathcal{E}_0(\overline{v})$.*
ii) For any $\tilde{v} < 1$ denote by $\Sigma(\tilde{v}) = \{\sigma = (b, v) : b \in \mathbb{R}^3, |v| \le \tilde{v}\}$.

Next Lemma means that in a small neighborhood of the soliton manifold $\mathcal{S}$ a "symplectic orthogonal projection" onto $\mathcal{S}$ is well-defined.

Lemma 3.4. *(cf.[8, Lemma 3.4]) Let (1.7) hold, $\alpha \in \mathbb{R}$ and $\overline{v} < 1$. Then*
i) there exists a neighborhood $\mathcal{O}_\alpha(\mathcal{S})$ of $\mathcal{S}$ in $\mathcal{E}_\alpha$ and a map $\mathbf{\Pi} : \mathcal{O}_\alpha(\mathcal{S}) \to \mathcal{S}$ such that $\mathbf{\Pi}$ is uniformly continuous on $\mathcal{O}_\alpha(\mathcal{S}) \cap \mathcal{E}_\alpha(\overline{v})$ in the metric of $\mathcal{E}_\alpha$,

$$\mathbf{\Pi} Y = Y \quad \text{for} \quad Y \in \mathcal{S}, \qquad \text{and} \qquad Y - S \nmid \mathcal{T}_S\mathcal{S}, \quad \text{where} \quad S = \mathbf{\Pi} Y \ . \tag{3.9}$$

ii) $\mathcal{O}_\alpha(\mathcal{S})$ is invariant with respect to translations T_a, and

$$\mathbf{\Pi} T_a Y = T_a \mathbf{\Pi} Y, \qquad \text{for} \quad Y \in \mathcal{O}_\alpha(\mathcal{S}) \quad \text{and} \quad a \in \mathbb{R}^3 \ .$$

iii) For any $\overline{v} < 1$ there exists a $\tilde{v} < 1$ s.t. $\mathbf{\Pi} Y = S(\sigma)$ with $\sigma \in \Sigma(\tilde{v})$ for $Y \in \mathcal{O}_\alpha(\mathcal{S}) \cap \mathcal{E}_\alpha(\overline{v})$.

iv) For any $\tilde{v} < 1$ there exists an $r_\alpha(\tilde{v}) > 0$ s.t. $S(\sigma) + Z \in \mathcal{O}_\alpha(\mathcal{S})$ if $\sigma \in \Sigma(\tilde{v})$ and $\|Z\|_\alpha < r_\alpha(\tilde{v})$.

We will call $\mathbf{\Pi}$ a symplectic orthogonal projection onto $\mathcal{S}$.

Corollary 3.5. *Condition (2.8) implies that $Y_0 = S + Z_0$ where $S = S(\sigma_0) = \mathbf{\Pi} Y_0$, and*

$$\|Z_0\|_\nu \ll 1 \ . \tag{3.10}$$

Proof. Lemma 3.4 implies that $\mathbf{\Pi} Y_0 = S$ is well defined for small $d_0 > 0$. Furthermore, condition (2.8) means that there exists a point $S_1 \in \mathcal{S}$ such that $\|Y_0 - S_1\|_\nu = d_0$. Hence, $Y_0, S_1 \in \mathcal{O}_\nu(\mathcal{S}) \cap \mathcal{E}_\nu(\overline{v})$ with a $\overline{v} < 1$ which does not depend on d_0 for sufficiently small d_0. On the other hand, $\mathbf{\Pi} S_1 = S_1$, hence the uniform continuity of the map $\mathbf{\Pi}$ implies that $\|S_1 - S\|_\nu \to 0$ as $d_0 \to 0$. Therefore, finally, $\|Z_0\|_\nu = \|Y_0 - S\|_\nu \le \|Y_0 - S_1\|_\nu + \|S_1 - S\|_\nu \le d_0 + o(1) \ll 1$ for small d_0. □

4 Linearization on solitary manifold

Let us consider a solution to the system (3.1), and split it as the sum

$$Y(t) = S(\sigma(t)) + Z(t) \tag{4.1}$$

where $\sigma(t) = (b(t), v(t)) \in \Sigma$ is an arbitrary smooth function of $t \in \mathbb{R}$. In detail, denote $Y = (\psi, q, p)$ and $Z = (\Psi, Q, P)$. Then (4.1) means that

$$\psi(x,t) = \psi_{v(t)}(x - b(t)) + \Psi(x - b(t), t), \quad q(t) = b(t) + Q(t), \quad p(t) = p_{v(t)} + P(t) \ . \tag{4.2}$$

Let us substitute (4.2) to (1.1), and linearize the equations in Z. Setting $y = x - b(t)$ which is "moving frame coordinate", we obtain that

$$\left.\begin{array}{rcl}\dot{\psi} &=& \dot{v}\cdot\nabla_v\psi_v(y) - \dot{b}\cdot\nabla\psi_v(y) + \dot{\Psi}(y,t) - \dot{b}\cdot\nabla\Psi(y,t)\\[2ex] &=& [-\alpha\cdot\nabla - i\beta m](\psi_v(y)+\Psi(y,t)) - i\rho(y-Q)\\[2ex] \dot{q} &=& \dot{b}+\dot{Q} = \dfrac{p_v+P}{\sqrt{1+(p_v+P)^2}}\\[2ex] \dot{p} &=& \dot{v}\cdot\nabla_v p_v + \dot{P} = \mathrm{Re}\langle\psi_v(y)+\Psi(y,t),\nabla\rho(y-Q)\rangle\end{array}\right| \tag{4.3}$$

Let us extract linear terms in Q. First note that $\rho(y-Q) = \rho(y) - Q\cdot\nabla\rho(y) + N_1(Q)$, $\nabla\rho(y-Q) = \nabla\rho(y) - \nabla(Q\cdot\nabla\rho(y)) + \tilde{N}_1(Q)$.

Condition (1.7) implies that for $N_1(Q)$ and $\tilde{N}_1(Q)$ the bound holds,

$$\|N_1(Q)\|_\nu + \|\tilde{N}_1(Q)\|_\nu \le C_\nu(\overline{Q})Q^2 \tag{4.4}$$

uniformly in $|Q| \le \overline{Q}$ for any fixed $\overline{Q}$. Second, the Taylor expansion gives

$$\frac{p_v+P}{\sqrt{1+(p_v+P)^2}} = v + \frac{1}{\gamma}(P - v(v\cdot P)) + N_2(v,P)$$

where $1/\gamma = \sqrt{1-v^2} = (1+p_v^2)^{-1/2}$, and

$$|N_2(v,P)| \le C(\tilde{v})P^2 \tag{4.5}$$

uniformly with respect to $|v| \le \tilde{v} < 1$. Using (2.4), we obtain from (4.3) the following equations for components of $Z(t)$:

$$\left.\begin{array}{rcl}\dot{\Psi}(y,t) &=& [-\alpha\cdot\nabla - i\beta m]\Psi(y,t) + \dot{b}\cdot\nabla\Psi(y,t) + iQ\cdot\nabla\rho(y)\\[2ex] && +\ (\dot{b}-v)\cdot\nabla\psi_v(y) - \dot{v}\cdot\nabla_v\psi_v(y) - iN_1\\[2ex] \dot{Q}(t) &=& \frac{1}{\gamma}(E - v\otimes v)P + (v-\dot{b}) + N_2\\[2ex] \dot{P}(t) &=& -\dot{v}\cdot\nabla_v p_v + \mathrm{Re}\langle\Psi(y,t),\nabla\rho(y)\rangle + \mathrm{Re}\langle\nabla\psi_v(y), Q\cdot\nabla\rho(y)\rangle + N_3(v,Z)\end{array}\right| \tag{4.6}$$

where $N_3(v,Z) = -\mathrm{Re}\langle\nabla\psi_v, N_1(Q)\rangle - \mathrm{Re}\langle\Psi,\nabla(Q\cdot\nabla\rho)\rangle + \mathrm{Re}\langle\Psi,\tilde{N}_1(Q)\rangle$. Clearly, $N_3(v,Z)$ satisfies the following estimate

$$|N_3(v,Z)| \le C_\nu(\rho,\overline{v},\overline{Q})\Big[Q^2 + \|\Psi\|_{-\nu}|Q|\Big] \tag{4.7}$$

uniformly in $|v| \le \tilde{v}$ and $|Q| \le \overline{Q}$ for any fixed $\tilde{v} < 1$. For the vector version $Z = (\Psi_1,\Psi_2,Q,P)$ with $\Psi_1 = \mathrm{Re}\,\Psi$, $\Psi_2 = \mathrm{Im}\,\Psi$ we rewrite equations (4.6) as

$$\dot{Z}(t) = A(t)Z(t) + T(t) + N(t),\ \ t\in\mathbb{R}\ . \tag{4.8}$$

Here operator $A(t) = A_{v,w}(t)$ depends on two parameters, $v = v(t)$, and $w = \dot{b}(t)$ and can be written as

$$A_{v,w}\begin{pmatrix}\Psi_1\\ \Psi_2\\ Q\\ P\end{pmatrix} = \begin{pmatrix}-\alpha_1\partial_1 - \alpha_3\partial_3 + w\cdot\nabla & \tilde{\alpha}_2\partial_2 + \beta m & -\nabla\rho_2\cdot & 0\\ -(\tilde{\alpha}_2\partial_2 + \beta m) & -\alpha_1\partial_1 - \alpha_3\partial_3 + w\cdot\nabla & \nabla\rho_1\cdot & 0\\ 0 & 0 & 0 & B_v\\ \langle\cdot,\nabla\rho_1\rangle & \langle\cdot,\nabla\rho_2\rangle & \langle\nabla\psi_{vj},\cdot\nabla\rho_j\rangle & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\\ Q\\ P\end{pmatrix} \tag{4.9}$$

where $B_v = \frac{1}{\gamma}(E - v \otimes v)$. Furthermore, $T(t) = T_{v,w}(t)$ and $N(t) = N(t, \sigma, Z)$ in (4.8) stand for

$$T_{v,w} = \begin{pmatrix} (w - v) \cdot \nabla\psi_{v1} - \dot{v} \cdot \nabla_v \psi_{v1} \\ (w - v) \cdot \nabla\psi_{v2} - \dot{v} \cdot \nabla_v \psi_{v2} \\ v - w \\ -\dot{v} \cdot \nabla_v p_v \end{pmatrix}, \quad N(\sigma, Z) = \begin{pmatrix} N_{12}(Z) \\ -N_{11}(Z) \\ N_2(v, Z) \\ N_3(v, Z) \end{pmatrix} \tag{4.10}$$

where $v = v(t)$, $w = w(t)$, $\sigma = \sigma(t) = (b(t), v(t))$, and $Z = Z(t)$. Estimates (4.4), (4.5) and (4.7) imply that

$$\|N(\sigma, Z)\|_\nu \leq C(\tilde{v}, \overline{Q})\|Z\|^2_{-\nu} \tag{4.11}$$

uniformly in $\sigma \in \Sigma(\tilde{v})$ and $\|Z\|_{-\nu} \leq r_{-\nu}(\tilde{v})$ for any fixed $\tilde{v} < 1$.

Remark 4.1. i) Term $A(t)Z(t)$ in right hand side of equation (4.8) is linear in $Z(t)$, and $N(t)$ is a *high order term* in $Z(t)$. On the other hand, $T(t)$ is a zero order term which does not vanish at $Z(t) = 0$ since $S(\sigma(t))$ generally is not a soliton solution if (2.7) does not hold (though $S(\sigma(t))$ belongs to the solitary manifold).
ii) Formulas (3.4) and (4.10) imply:

$$T(t) = -\sum_{l=1}^{3}[(w - v)_l \tau_l + \dot{v}_l \tau_{l+3}] \tag{4.12}$$

and hence $T(t) \in \mathcal{T}_{S(\sigma(t))}\mathcal{S}$, $t \in \mathbb{R}$.

5 Linearized equation

Here we collect some Hamiltonian and spectral properties of generator (4.9) of the linearized equation. First, let us consider linear equation

$$\dot{X}(t) = A_{v,w}X(t), \quad t \in \mathbb{R}, \quad v \in V, \quad w \in \mathbb{R}^3 \tag{5.1}$$

Lemma 5.1. *(cf. Lemma 5.1 [8]) i) For any $v \in V$ and $w \in \mathbb{R}^3$ equation (5.1) can be written as the Hamilton system (cf. (3.2)),*

$$\dot{X}(t) = JD\mathcal{H}_{v,w}(X(t)), \quad t \in \mathbb{R} \tag{5.2}$$

where $D\mathcal{H}_{v,w}$ is the Fréchet derivative with respect to Ψ_{1k}, Ψ_{2k}, $k = 1, 2, 3, 4$, P and Q of the Hamilton functional

$$\mathcal{H}_{v,w}(X) = \frac{1}{2}\int(\Psi_1 \cdot (\tilde{\alpha}_2\partial_2 + \beta m)\Psi_1 + \Psi_2 \cdot (\tilde{\alpha}_2\partial_2 + \beta m)\Psi_2 + 2\Psi_1 \cdot (\alpha_1\partial_1 + \alpha_3\partial_3)\Psi_2)dy$$
$$+\int \rho_j(y)Q \cdot \nabla\Psi_j dy + \frac{1}{2}P \cdot B_v P - \frac{1}{2}\langle Q \cdot \nabla\psi_{vj}(y), Q \cdot \nabla\rho_j(y)\rangle, \quad X = (\Psi_1, \Psi_2, Q, P) \in \mathcal{E}\ .$$

ii) The skew-symmetry relation holds,

$$\Omega(A_{v,w}X_1, X_2) = -\Omega(X_1, A_{v,w}X_2), \quad X_1 \in \mathcal{E}, \quad X_2 \in H^1(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \oplus \mathbb{R}^3 \oplus \mathbb{R}^3\ . \tag{5.3}$$

Lemma 5.2. *Operator $A_{v,w}$ acts on tangent vectors $\tau_j(v)$ to the solitary manifold as follows,*

$$A_{v,w}[\tau_j(v)] = (w - v) \cdot \nabla\tau_j(v), \;\; A_{v,w}[\tau_{j+3}(v)] = (w - v) \cdot \nabla\tau_{j+3}(v) + \tau_j(v), \;\; j = 1, 2, 3\ .$$

Proof. In detail, we have to show that

$$A_{v,w}\begin{pmatrix} -\partial_j\psi_{v1} \\ -\partial_j\psi_{v2} \\ e_j \\ 0 \end{pmatrix} = \begin{pmatrix} (v-w)\cdot\nabla\partial_j\psi_{v1} \\ (v-w)\cdot\nabla\partial_j\psi_{v2} \\ 0 \\ 0 \end{pmatrix} .$$

$$A_{v,w}\begin{pmatrix} \partial_{v_j}\psi_{v1} \\ \partial_{v_j}\psi_{v2} \\ 0 \\ \partial_{v_j}p_v \end{pmatrix} = \begin{pmatrix} (w-v)\cdot\nabla\partial_{v_j}\psi_{v1} \\ (w-v)\cdot\nabla\partial_{v_j}\psi_{v2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\partial_j\psi_{v1} \\ -\partial_j\psi_{v2} \\ e_j \\ 0 \end{pmatrix} . \qquad (5.4)$$

Indeed, differentiate equations (2.4) in b_j and v_j, and obtain that derivatives of soliton state in parameters satisfy the following equations,

$$\left.\begin{aligned} -v\cdot\nabla\partial_j\psi_v &= [-\alpha\cdot\nabla - i\beta m]\partial_j\psi_v - i\partial_j\rho \\ -\partial_j\psi_v - v\cdot\nabla\partial_{v_j}\psi_v &= [-\alpha\cdot\nabla - i\beta m]\partial_{v_j}\psi_v \\ \partial_{v_j}p_v &= e_j(1-v^2)^{-1/2} + v\frac{v_j}{(1-v^2)^{3/2}} \\ 0 &= \langle\partial_{v_j}\psi_{v1},\nabla\rho_1\rangle + \langle\partial_{v_j}\psi_{v2},\nabla\rho_2\rangle \end{aligned}\right| \qquad (5.5)$$

for $j = 1,2,3$. Then (5.4) follows from (5.5) by definition of $A_{v,v}$ in (4.9) □

Corollary 5.3. *Let $w = v \in V$. Then $\tau_j(v)$ are eigenvectors, and $\tau_{j+3}(v)$ are root vectors of operator $A_{v,v}$, corresponding to zero eigenvalue, i.e.*

$$A_{v,v}[\tau_j(v)] = 0, \;\; A_{v,v}[\tau_{j+3}(v)] = \tau_j(v), \;\; j = 1,2,3 \; . \qquad (5.6)$$

6 Symplectic decomposition of dynamics

Here we decompose the dynamics in two components: along manifold $\mathcal{S}$ and in transversal directions. Equation (4.8) is obtained without any assumption on $\sigma(t)$ in (4.1). We are going to choose $S(\sigma(t)) := \mathbf{\Pi}Y(t)$, but then we need to know that

$$Y(t) \in \mathcal{O}_{-\nu}(\mathcal{S}), \qquad t \in \mathbb{R} \; . \qquad (6.1)$$

It is true for $t = 0$ by our main assumption (2.8) with sufficiently small $d_0 > 0$. Then $S(\sigma(0)) = \mathbf{\Pi}Y(0)$ and $Z(0) = Y(0) - S(\sigma(0))$ are well defined. We will prove below that (6.1) holds if d_0 is sufficiently small. Let us choose an arbitrary $\tilde{v}$ such that $|v(0)| < \tilde{v} < 1$ and let $\delta = \tilde{v} - |v(0)|$. Denote by $r_{-\nu}(\tilde{v})$ the positive numbers from Lemma 3.4 iv) which corresponds to $\alpha = -\nu$. Then $S(\sigma) + Z \in \mathcal{O}_{-\nu}(\mathcal{S})$ if $\sigma = (b,v)$ with $|v| < \tilde{v}$ and $\|Z\|_{-\nu} < r_{-\nu}(\tilde{v})$. Note that $\|Z(0)\|_{-\nu} < r_{-\nu}(\tilde{v})$ if d_0 is sufficiently small. Therefore, $S(\sigma(t)) = \mathbf{\Pi}Y(t)$ and $Z(t) = Y(t) - S(\sigma(t))$ are well defined for $t \geq 0$ so small that $|v| < \tilde{v}$ and $\|Z(t)\|_{-\nu} < r_{-\nu}(\tilde{v})$. This is formalized by the following standard definition.

Definition 6.1. *t_* is "exit time",*

$$t_* = \sup\{t > 0 : \|Z(s)\|_{-\nu} < r_{-\nu}(\tilde{v}), \;\; |v(s) - v(0)| < \delta, \;\; 0 \leq s \leq t\} \; . \qquad (6.2)$$

One of our main goals is to prove that $t_* = \infty$ if d_0 is sufficiently small. This would follow if we show that

$$\|Z(t)\|_{-\nu} < r_{-\nu}(\tilde{v})/2, \quad |v(s) - v(0)| < \delta/2, \quad 0 \le t < t_* \ . \tag{6.3}$$

Note that

$$|Q(t)| \le \overline{Q} := r_{-\nu}(\tilde{v}), \qquad 0 \le t < t_* \ . \tag{6.4}$$

Now $N(t)$ in (4.8) satisfies, by (4.11), the following estimate,

$$\|N(t)\|_{\nu} \le C_{\nu}(\tilde{v})\|Z(t)\|^2_{-\nu}, \ \ 0 \le t < t_* \ . \tag{6.5}$$

6.1 Modulation equations

From now on we fix the decomposition $Y(t) = S(\sigma(t)) + Z(t)$ for $0 < t < t_*$ by setting $S(\sigma(t)) = \mathbf{\Pi} Y(t)$ which is equivalent to symplectic orthogonality condition of type (3.9),

$$Z(t) \nmid \mathcal{T}_{S(\sigma(t))}\mathcal{S}, \ \ 0 \le t < t_* \ . \tag{6.6}$$

This allows us to simplify drastically asymptotic analysis of dynamical equations (4.8) for the transversal component $Z(t)$. As a first step, we derive "modulation equations" for the parameters $\sigma(t)$. For this purpose, we write (6.6) in the form

$$\Omega(Z(t), \tau_j(t)) = 0, \ \ j = 1, \ldots, 6, \qquad 0 \le t < t_* \ . \tag{6.7}$$

where vectors $\tau_j(t) = \tau_j(\sigma(t))$ span tangent space $\mathcal{T}_{S(\sigma(t))}\mathcal{S}$. Note that $\sigma(t) = (b(t), v(t))$, where $|v(t)| \le \tilde{v} < 1$ for $0 \le t < t_*$ by Lemma 3.4 iii). It would be convenient for us to use some other parameters (c, v) instead of $\sigma = (b, v)$, where

$$c(t) = b(t) - \int_0^t v(\tau)d\tau, \quad \dot{c}(t) = \dot{b}(t) - v(t) = w(t) - v(t), \quad 0 \le t < t_* \ . \tag{6.8}$$

The following statement can be proved similar to Lemma 6.2 from [8].

Lemma 6.2. *Let $Y(t)$ be a solution to the Cauchy problem (3.1), and (4.1), (6.7) hold. Then*

$$|\dot{c}(t)| + |\dot{v}(t)| \le C(\tilde{v})\|Z\|^2_{-\nu} \ . \tag{6.9}$$

6.2 Decay for transversal dynamics

In Section 11 we will show that our main Theorem 2.5 can be derived from the following time decay of the transversal component $Z(t)$:

Proposition 6.3. *Let all conditions of Theorem 2.5 hold. Then $t_* = \infty$, and*

$$\|Z(t)\|_{-\nu} \le \frac{C(\rho, \overline{v}, d_0)}{(1 + |t|)^{3/2}}, \qquad t \ge 0 \ . \tag{6.10}$$

We will derive (6.10) in Sections 7-10 from equation (4.8) for the transversal component $Z(t)$. This equation can be specified using Lemma 6.2. Indeed, the lemma implies that

$$\|T(t)\|_{\nu} \le C(\tilde{v})\|Z(t)\|^2_{-\nu}, \qquad 0 \le t < t_* \tag{6.11}$$

by (4.10) since $w - v = \dot{c}$. Thus (4.8) becomes the equation

$$\dot{Z}(t) = A(t)Z(t) + \tilde{N}(t), \qquad 0 \le t < t_* \tag{6.12}$$

where $A(t) = A_{v(t),w(t)}$, and $\tilde{N}(t) := T(t) + N(t)$ satisfies the estimate

$$\|\tilde{N}(t)\|_\nu \le C(\tilde{v}, \overline{Q})\|Z(t)\|^2_{-\nu}, \qquad 0 \le t < t_* \ . \tag{6.13}$$

In all remaining part of our paper we will analyze mainly **basic equation** (6.12) to establish decay (6.10). We are going to derive the decay using bound (6.13) and orthogonality condition (6.6).

Similarly [8] we reduce the problem to analysis of *frozen* linear equation,

$$\dot{X}(t) = A_1 X(t), \quad t \in \mathbb{R} \tag{6.14}$$

where $A_1 = A_{v_1,v_1}$ with $v_1 = v(t_1)$ and a fixed $t_1 \in [0, t_*)$. Then we can apply some methods of scattering theory and then estimate the error by the method of majorants.

Note, that even for the frozen equation (6.14), decay of type (6.10) for all solutions does not hold without orthogonality condition of type (6.6). Namely, by (5.6) equation (6.14) admits *secular solutions*

$$X(t) = \sum_1^3 C_j \tau_j(v) + \sum_1^3 D_j[\tau_j(v)t + \tau_{j+3}(v)] \tag{6.15}$$

which arise by differentiation of soliton (1.4) in the parameters a and v.

Remark 6.4. The solution (6.15) lies in tangent space $\mathcal{T}_{S(\sigma_1)}\mathcal{S}$ with $\sigma_1 = (b_1, v_1)$ (for an arbitrary $b_1 \in \mathbb{R}$) that suggests an unstable character of the nonlinear dynamics *along the solitary manifold.*

Further, we will apply the corresponding symplectic orthogonal projection which kills "runaway solutions" (6.15).

Definition 6.5. *i) For $v \in V$, denote by $\mathbf{\Pi}_v$ symplectic orthogonal projection of $\mathcal{E}$ onto tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$, and $\mathbf{P}_v = \mathbf{I} - \mathbf{\Pi}_v$.*
ii) Denote by $\mathcal{Z}_v = \mathbf{P}_v\mathcal{E}$ the space symplectic orthogonal to $\mathcal{T}_{S(\sigma)}\mathcal{S}$ with $\sigma = (b, v)$.

Note that by linearity,

$$\mathbf{\Pi}_v Z = \sum \mathbf{\Pi}_{jl}(v)\tau_j(v)\Omega(\tau_l(v), Z), \qquad Z \in \mathcal{E} \tag{6.16}$$

with some smooth coefficients $\mathbf{\Pi}_{jl}(v)$. Hence, projector $\mathbf{\Pi}_v$, in variable $y = x - b$, does not depend on b. Now we have the symplectic orthogonal decomposition

$$\mathcal{E} = \mathcal{T}_{S(\sigma)}\mathcal{S} + \mathcal{Z}_v, \qquad \sigma = (b, v) \ , \tag{6.17}$$

and symplectic orthogonality (6.6) can be written in the following equivalent forms,

$$\mathbf{\Pi}_{v(t)}Z(t) = 0, \quad \mathbf{P}_{v(t)}Z(t) = Z(t), \qquad 0 \le t < t_* \ . \tag{6.18}$$

Remark 6.6. The tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ is invariant under operator $A_{v,v}$ by Lemma 5.3 i), hence the space $\mathcal{Z}_v$ also is invariant by (5.3): $A_{v,v}Z \in \mathcal{Z}_v$ for *sufficiently smooth* $Z \in \mathcal{Z}_v$.

Below in section 12-16 we will prove the following proposition which will be one of main ingredients for proving (6.10). Consider the Cauchy problem for equation (6.14) with $A = A_{v,v}$ for a fixed $v \in V$. Recall that parameter $\nu > 5/2$ is also fixed.

Proposition 6.7. *Let conditions* (1.7)- (1.8) *hold, $|v| \le \tilde{v} < 1$, and $X_0 \in \mathcal{E}$. Then*
i) Equation (6.14), with $A = A_{v,v}$, admits the unique solution $e^{At}X_0 := X(t) \in C(\mathbb{R}, \mathcal{E})$ with initial condition $X(0) = X_0$.
ii) For $X_0 \in \mathcal{Z}_v \cap \mathcal{E}_\nu$, the decay holds,

$$\|e^{At}X_0\|_{-\nu} \le \frac{C_\nu(\rho, \tilde{v})}{(1 + |t|)^{3/2}}\|X_0\|_\nu, \quad t \in \mathbb{R} \ . \tag{6.19}$$

7 Frozen transversal dynamics

Now let us fix an arbitrary $t_1 \in [0, t_*)$, and rewrite equation (6.12) in a "frozen form"

$$\dot{Z}(t) = A_1 Z(t) + (A(t) - A_1)Z(t) + \tilde{N}(t), \qquad 0 \le t < t_* \tag{7.1}$$

where $A_1 = A_{v(t_1),v(t_1)}$ and

$$A(t)-A_1 = \begin{pmatrix} [w(t)-v(t_1)]\cdot\nabla & 0 & 0 & 0 \\ 0 & [w(t)-v(t_1)]\cdot\nabla & 0 & 0 \\ 0 & 0 & 0 & B_{v(t)}-B_{v_1(t)} \\ 0 & 0 & \langle\nabla(\psi_{v(t)j}-\psi_{v(t_1)j}),\nabla\rho_j\rangle & 0 \end{pmatrix}$$

Next trick allows us to kill the "bad terms" $[w(t)-v(t_1)]\cdot\nabla$ in operator $A(t) - A_1$.

Definition 7.1. *Let us change the variables $(y, t) \mapsto (y_1, t) = (y + d_1(t), t)$, where*

$$d_1(t) := \int_{t_1}^{t} (w(s) - v(t_1))ds, \qquad 0 \le t \le t_1 \ . \tag{7.2}$$

Next define

$$Z_1(t) := (\Psi_1(y_1 - d_1(t), t), \Psi_2(y_1 - d_1(t), t), Q(t), P(t)) \ .$$

Then we obtain final form of the "frozen equation" for the transversal dynamics

$$\dot{Z}_1(t) = A_1 Z_1(t) + B_1(t)Z_1(t) + \tilde{N}_1(t), \ \ 0 \le t \le t_1 \tag{7.3}$$

where $\tilde{N}_1(t) = \tilde{N}(t)$ expressed in terms of $y = y_1 - d_1(t)$, and

$$B_1(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{v(t)} - B_{v_1(t)} \\ 0 & 0 & \langle\nabla(\psi_{v(t)j}-\psi_{v(t_1)j}),\nabla\rho_j\rangle & 0 \end{pmatrix} .$$

Let us estimate the "remainder terms" $B_1(t)Z_1(t)$ and $\tilde{N}_1(t)$.

Lemma 7.2. *The bound holds*

$$\|B_1(t)Z_1(t)\|_\nu \le C(\tilde{v})\|Z(t)\|_{-\nu} \int_t^{t_1} \|Z(s)\|_{-\nu}^2 ds, \quad 0 \le t \le t_1 \ . \tag{7.4}$$

Proof. Lemma 6.2 implies

$$|B_{v(t)} - B_{v_1(t)}| \le |\int_{t_1}^{t} \dot{v}(s)\cdot\nabla_v B_{v(s)} ds| \le C(\tilde{v}) \int_{t_1}^{t} \|Z(s)\|_{-\nu}^2 ds$$

$$|\langle\nabla(\psi_{v(t)j} - \psi_{v(t_1)j}), \nabla\rho_j| \le C(\tilde{v}) \int_{t_1}^{t} \|Z(s)\|_{-\nu}^2 ds \ .$$

Therefore,

$$\begin{aligned} \|B_1(t)Z_1(t)\|_\nu &= |\langle\nabla(\psi_{v(t)j} - \psi_{v(t_1)j}), \nabla\rho_j\rangle Q_1(t)| + |(B_{v(t)} - B_{v_1(t)})P_1(t)| \\ &\le C(\tilde{v})(|Q(t)| + |P(t)|) \int_{t_1}^{t} \|Z(s)\|_{-\nu}^2 ds \le C(\tilde{v})\|Z(t)\|_{-\nu} \int_t^{t_1} \|Z(s)\|_{-\nu}^2 ds \ . \end{aligned}$$

$\square$

Lemma 7.3. *The bounds hold*

$$\|\tilde{N}_1(t)\|_\nu \le C(\tilde{v}, \overline{Q})(1+|d_1(t)|)^\nu \|Z(t)\|^2_{-\nu}, \quad 0 \le t \le t_1 \ . \tag{7.5}$$

Proof. For any $\Phi \in L^2_\alpha$ and $d \in \mathbb{R}^3$ we have

$$\begin{aligned} \|\Phi(y-d)\|^2_\alpha &= \int |\Phi(y-d)|^2(1+|y|)^{2\alpha}dy = \int |\Phi(y)|^2(1+|y+d|)^{2\alpha}dy \\ &\le \int |\Phi(y)|^2(1+|y|)^{2\alpha}(1+|d|)^{2\alpha}dy \le (1+|d|)^{2\alpha}\|\Phi\|^2_\alpha, \quad \alpha \in \mathbb{R} \ . \end{aligned}$$

Hence, bound (7.5) follows. □

8 Integral inequality

Equation (7.3) can be written in integral form:

$$Z_1(t) = e^{A_1 t}Z_1(0) + \int_0^t e^{A_1(t-s)}[B_1 Z_1(s) + \tilde{N}_1(s)]ds, \quad 0 \le t \le t_1 \tag{8.1}$$

Now we apply symplectic orthogonal projection $\mathbf{P}_1 := \mathbf{P}_{v(t_1)}$ to both sides of (8.1):

$$\mathbf{P}_1 Z_1(t) = e^{A_1 t}\mathbf{P}_1 Z_1(0) + \int_0^t e^{A_1(t-s)}\mathbf{P}_1[B_1 Z_1(s) + \tilde{N}_1(s)]ds \ .$$

Projector $\mathbf{P}_1$ commutes with the group $e^{A_1 t}$ since the space $\mathcal{Z}_1 := \mathbf{P}_1\mathcal{E}$ is invariant with respect to $e^{A_1 t}$ by Remark 6.6. Applying (6.19) we obtain that

$$\|\mathbf{P}_1 Z_1(t)\|_{-\nu} \le C\frac{\|\mathbf{P}_1 Z_1(0)\|_\nu}{(1+t)^{3/2}} + C\int_0^t \frac{\|\mathbf{P}_1[B_1 Z_1(s) + \tilde{N}_1(s)]\|_\nu \ ds}{(1+|t-s|)^{3/2}} \ .$$

Operator $\mathbf{P}_1 = \mathbf{I} - \mathbf{\Pi}_1$ is continuous in $\mathcal{E}_\nu$ by (6.16). Hence, (7.4)-(7.5) imply

$$\begin{aligned} &\|\mathbf{P}_1 Z_1(t)\|_{-\nu} \le \frac{C(\overline{d}_1(0))}{(1+t)^{3/2}}\|Z(0)\|_\nu \qquad (8.2) \\ &+C(\overline{d}_1(t))\int_0^t \frac{1}{(1+|t-s|)^{3/2}}\left[\|Z(s)\|_{-\nu}\int_s^{t_1}\|Z(\tau)\|^2_{-\nu}d\tau + \|Z(s)\|^2_{-\nu}\right]ds, \quad 0 \le t \le t_1 \end{aligned}$$

where $\overline{d}_1(t) := \sup_{0\le s\le t}|d_1(s)|$. Let us introduce the "majorant"

$$m(t) := \sup_{s\in[0,t]}(1+s)^{3/2}\|Z(s)\|_{-\nu}, \quad t \in [0, t_*) \ . \tag{8.3}$$

Now we reduce further the exit time. Denote by $\varepsilon < 1$ a fixed positive number which we will specify below.

Definition 8.1. t'_* *is the exit time*

$$t'_* = \sup\{t \in [0, t_*) : m(s) \le \varepsilon, \quad 0 \le s \le t\} \ . \tag{8.4}$$

To estimate $d_1(t)$, note that

$$w(s) - v(t_1) = w(s) - v(s) + v(s) - v(t_1) = \dot{c}(s) + \int_s^{t_1}\dot{v}(\tau)d\tau \tag{8.5}$$

by (6.8). Hence, (7.2), Lemma 6.2 and definition (8.3) imply that for $t_1 < t_*'$

$$|d_1(t)| = |\int_{t_1}^{t} (w(s) - v(t_1))ds| \le \int_t^{t_1} \left(|\dot{c}(s)| + \int_s^{t_1} |\dot{v}(\tau)| d\tau \right) ds \tag{8.6}$$
$$\le C(\tilde{v}) m^2(t_1) \int_t^{t_1} \left(\frac{1}{(1+s)^3} + \int_s^{t_1} \frac{d\tau}{(1+\tau)^3} \right) ds \le C(\tilde{v}) m^2(t_1) \le C(\tilde{v}), \quad 0 \le t \le t_1 \ .$$

Now we can to replace $C(\overline{d}_1)$ with $C(\tilde{v})$ in (8.2): for $t_1 < t_*'$

$$\|\mathbf{P}_1 Z_1(t)\|_{-\nu} \le \frac{C(\tilde{v})}{(1+t)^{3/2}} \|Z(0)\|_\nu$$
$$+ C(\tilde{v}) \int_0^t \frac{1}{(1+|t-s|)^{3/2}} \left[\|Z(s)\|_{-\nu} \int_s^{t_1} \|Z(\tau)\|_{-\nu}^2 d\tau + \|Z(s)\|_{-\nu}^2 \right] ds, \quad 0 \le t \le t_1 \ . \tag{8.7}$$

9 Symplectic orthogonality

Finally, we are going to change $\mathbf{P}_1 Z_1(t)$ by $Z(t)$ in the left hand side of (8.7). We will prove that it is possible using again that $d_0 \ll 1$ in (2.8).

Lemma 9.1. *(cf.[8]) For sufficiently small $\varepsilon > 0$, we have for $t_1 < t_*'$*

$$\|Z(t)\|_{-\nu} \le C \|\mathbf{P}_1 Z_1(t)\|_{-\nu} \ , \qquad 0 \le t \le t_1 \ , \tag{9.1}$$

where C depends only on ρ and $\overline{v}$.

Proof. Since $|d_1(t)| \le C$ for $t \le t_1 < t_*'$ then $\|Z(t)\|_{-\nu} \le C\|Z_1(t)\|_{-\nu}$, and it suffices to prove that

$$\|Z_1(t)\|_{-\nu} \le 2\|\mathbf{P}_1 Z_1(t)\|_{-\nu} \ , \quad 0 \le t \le t_1 \ . \tag{9.2}$$

Recall that $\mathbf{P}_1 Z_1(t) = Z_1(t) - \mathbf{\Pi}_{v(t_1)} Z_1(t)$. Then estimate (9.2) will follow from

$$\|\mathbf{\Pi}_{v(t_1)} Z_1(t)\|_{-\nu} \le \frac{1}{2} \|Z_1(t)\|_{-\nu} \ , \quad 0 \le t \le t_1 \ . \tag{9.3}$$

Symplectic orthogonality (6.18) implies

$$\mathbf{\Pi}_{v(t),1} Z_1(t) = 0 \ , \quad t \in [0, t_1] \ , \tag{9.4}$$

where $\mathbf{\Pi}_{v(t),1} Z_1(t)$ is $\mathbf{\Pi}_{v(t)} Z(t)$ expressed in terms of variable $y_1 = y + d_1(t)$. Hence, (9.3) follows from (9.4) if difference $\mathbf{\Pi}_{v(t_1)} - \mathbf{\Pi}_{v(t),1}$ is small uniformly in t, i.e.

$$\|\mathbf{\Pi}_{v(t_1)} - \mathbf{\Pi}_{v(t),1}\| < 1/2 \ , \qquad 0 \le t \le t_1 \ . \tag{9.5}$$

It remains to justify (9.5) for small enough $\varepsilon > 0$. Formula (6.16) implies

$$\mathbf{\Pi}_{v(t),1} Z_1(t) = \sum \mathbf{\Pi}_{jl}(v(t)) \tau_{j,1}(v(t)) \Omega(\tau_{l,1}(v(t)), Z_1(t)) \ , \tag{9.6}$$

where $\tau_{j,1}(v(t))$ are vectors $\tau_j(v(t))$ expressed in variable y_1. Since $|d_1(t)| \le C$ and $\nabla \tau_j$ are smooth and fast decaying at infinity functions, we have

$$\|\tau_{j,1}(v(t)) - \tau_j(v(t))\|_\nu \le C|d_1(t)|^\nu \le C \ , \quad 0 \le t \le t_1 \tag{9.7}$$

for all $j = 1, 2, \ldots, 6$. Furthermore,

$$\tau_j(v(t)) - \tau_j(v(t_1)) = \int_t^{t_1} \dot{v}(s) \cdot \nabla_v \tau_j(v(s)) ds \ ,$$

and therefore

$$\|\tau_j(v(t)) - \tau_j(v(t_1))\|_\nu \le C \int_t^{t_1} |\dot{v}(s)| ds \ , \quad 0 \le t \le t_1 \ . \tag{9.8}$$

Similarly,

$$|\mathbf{\Pi}_{jl}(v(t)) - \mathbf{\Pi}_{jl}(v(t_1))| = |\int_t^{t_1} \dot{v}(s) \cdot \nabla_v \mathbf{\Pi}_{jl}(v(s)) ds| \le C \int_t^{t_1} |\dot{v}(s)| ds, \quad 0 \le t \le t_1 \ . \tag{9.9}$$

Hence, bounds (9.5) will follow from (6.16), (9.6) and (9.7)-(9.9) if we establish that integral in the right hand side of (9.8) can be made as small as we please by choosing $\varepsilon > 0$ small enough. Indeed,

$$\int_t^{t_1} |\dot{v}(s)| ds \le C m^2(t_1) \int_t^{t_1} \frac{ds}{(1+s)^3} \le C\varepsilon^2 \ , \quad 0 \le t \le t_1 \ . \tag{9.10}$$

□

10 Decay of transversal component

Here we prove Proposition 6.3.
Step i) We fix $0 < \varepsilon < 1$ and $t'_* = t'_*(\varepsilon)$ for which Lemma 9.1 holds. Then bound of type (8.7) holds with $\|\mathbf{P}_1 Z_1(t)\|_{-\nu}$ in the left hand side replaced by $\|Z(t)\|_{-\nu}$:

$$\begin{aligned} \|Z(t)\|_{-\nu} &\le \frac{C}{(1+t)^{3/2}} \|Z(0)\|_\nu \\ &+ C \int_0^t \frac{1}{(1+|t-s|)^{3/2}} \Big[\|Z(s)\|_{-\nu} \int_s^{t_1} \|Z(\tau)\|^2_{-\nu} d\tau + \|Z(s)\|^2_{-\nu} \Big] ds \ , \quad 0 \le t \le t_1 \end{aligned} \tag{10.1}$$

for $t_1 < t'_*$. This implies an integral inequality for majorant $m(t)$ defined in (8.3). Namely, multiplying both sides of (10.1) by $(1+t)^{3/2}$, and taking supremum in $t \in [0, t_1]$, we get

$$m(t_1) \le C\|Z(0)\|_\nu + C \sup_{t \in [0,t_1]} \int_0^t \frac{(1+t)^{3/2}}{(1+|t-s|)^{3/2}} \Big[\frac{m(s)}{(1+s)^{3/2}} \int_s^{t_1} \frac{m^2(\tau) d\tau}{(1+\tau)^3} + \frac{m^2(s)}{(1+s)^3} \Big] ds$$

for $t_1 \le t'_*$. Taking into account that $m(t)$ is a monotone increasing function, we get

$$m(t_1) \le C\|Z(0)\|_\nu + C[m^3(t_1) + m^2(t_1)] I(t_1) \ , \quad t_1 \le t'_* \tag{10.2}$$

where

$$I(t_1) = \sup_{t \in [0,t_1]} \int_0^t \frac{(1+t)^{3/2}}{(1+|t-s|)^{3/2}} \Big[\frac{1}{(1+s)^{3/2}} \int_s^{t_1} \frac{d\tau}{(1+\tau)^3} + \frac{1}{(1+s)^3} \Big] ds \le \overline{I} < \infty \ .$$

Therefore, (10.2) becomes

$$m(t_1) \le C\|Z(0)\|_\nu + C\overline{I}[m^3(t_1) + m^2(t_1)] \ , \quad t_1 < t'_* \ . \tag{10.3}$$

This inequality implies that $m(t_1)$ is bounded for $t_1 < t'_*$, and moreover,

$$m(t_1) \le C_1 \|Z(0)\|_\nu \ , \qquad t_1 < t'_* \tag{10.4}$$

since $m(0) = \|Z(0)\|_\nu$ is sufficiently small by (3.10).
Step ii) The constant C_1 in estimate (10.4) does not depend on t_* and t'_* by Lemma 9.1. We choose d_0 in (2.8) so small that $\|Z(0)\|_\nu < \varepsilon/(2C_1)$. It is possible due to (3.10). Then estimate (10.4) implies that $t'_* = t_*$ and therefore (10.4) holds for all $t_1 < t_*$. Further,

$$|v(t) - v(0)| \le \int_0^t |\dot v(s)| ds \le C m^2(t) \int_0^t \frac{ds}{(1+s)^3} \le C m^2(t) \ .$$

Hence both inequalities (6.3) also holds if $\|Z(0)\|_\nu$ is sufficiently small by (8.3). Finally, this implies that $t_* = \infty$, hence also $t'_* = \infty$ and (10.4) holds for all $t_1 > 0$ if d_0 is small enough. It complete proof of Proposition 6.3.

11 Soliton asymptotics

Here we prove our main Theorem 2.5 under the assumption that decay (6.10) holds. First we will prove asymptotics (2.9) for vector components, and afterwards asymptotics (2.10) for the fields.
Asymptotics for vector components. From (4.3) we have $\dot q = \dot b + \dot Q$, and from (6.12), (6.13), (4.9) it follows that $\dot Q = P + \mathcal{O}(\|Z\|^2_{-\nu})$. Thus,

$$\dot q = \dot b + \dot Q = v(t) + \dot c(t) + P(t) + \mathcal{O}(\|Z\|^2_{-\nu}) \tag{11.1}$$

Bounds (6.9) and (6.10) imply that

$$|\dot c(t)| + |\dot v(t)| \le \frac{C_1(\rho, \overline{v}, d_0)}{(1+t)^3} \ , \qquad t \ge 0 \ . \tag{11.2}$$

Therefore, $c(t) = c_+ + \mathcal{O}(t^{-2})$ and $v(t) = v_+ + \mathcal{O}(t^{-2})$, $t \to \infty$. Since $|P| \le \|Z\|_{-\nu}$, estimate (6.10), and (11.1)-(11.2), imply that

$$\dot q(t) = v_+ + \mathcal{O}(t^{-3/2}) \ , \quad b(t) = c(t) + \int_0^t v(s) ds = v_+ t + a_+ + \mathcal{O}(t^{-1}) \ .$$

Hence second part of (1.6) follows:

$$q(t) = b(t) + Q(t) = v_+ t + a_+ + \mathcal{O}(t^{-1})$$

since $Q(t) = \mathcal{O}(t^{-3/2})$ by (6.10).
Asymptotics for fields. For field part of the solution $\psi(x,t)$ let us define the "accompanying soliton field" as $\psi_{\mathrm{v}(t)}(x - q(t))$, where we define now $\mathrm{v}(t) = \dot q(t)$, cf. (11.1). Then for difference $z(x,t) = \psi(x,t) - \psi_{\mathrm{v}(t)}(x - q(t))$ we obtain the equation

$$\dot z(x,t) = [-\alpha \cdot \nabla - i\beta m] z(x,t) - i\dot{\mathrm{v}} \cdot \nabla_{\mathrm{v}} \psi_{\mathrm{v}(t)}(x - q(t)) \ .$$

Then

$$z(t) = W_0(t) z(0) - \int_0^t W_0(t-s)[i\dot{\mathrm{v}}(s) \cdot \nabla_{\mathrm{v}} \psi_{\mathrm{v}(s)}(\cdot - q(s))] ds \ . \tag{11.3}$$

To obtain asymptotics (2.10) it suffices to prove that $z(t) = W_0(t)\phi_+ + r_+(t)$ with some $\phi_+ \in L^2_0$ and $\|r_+(t)\|_0 = \mathcal{O}(t^{-1/2})$. This is equivalent to

$$W_0(-t) z(t) = \phi_+ + r'_+(t) \tag{11.4}$$

where $\|r'_+(t)\|_0 = \mathcal{O}(t^{-1/2})$ since $W_0(t)$ is a unitary group in L^2_0 by charge conservation for the free Dirac equation. Finally, (11.4) holds since (11.3) implies

$$W_0(-t)z(t) = z(0) - \int_0^t W_0(-s)f(s)ds \ , \quad f(s) = i\dot{\mathrm{v}}(s)\cdot\nabla_{\mathrm{v}}\psi_{\mathrm{v}(s)}(\cdot - q(s))$$

where the integral in the right hand side converges in L^2_0 with rate $\mathcal{O}(t^{-1/2})$. The latter holds since $\|W_0(-s)f(s)\|_0 = \mathcal{O}(s^{-3/2})$ by unitarity of $W_0(-s)$ and the decay rate $\|f(s)\|_0 = \mathcal{O}(s^{-3/2})$. Let us prove this rate of decay. It suffices to prove that $|\dot{\mathrm{v}}(s)| = \mathcal{O}(s^{-3/2})$, or equivalently $|\dot{p}(s)| = \mathcal{O}(s^{-3/2})$. Substitute (4.2) to last equation of (1.1) and obtain

$$\begin{aligned}
\dot{p}(t) &= \mathrm{Re}\int \left[\psi_{v(t)}(x-b(t)) + \Psi(x-b(t),t)\right]\cdot\nabla\rho(x-b(t)-Q(t))dx \\
&= \mathrm{Re}\int \psi_{v(t)}(y)\cdot\nabla\rho(y)dy + \mathrm{Re}\int \psi_{v(t)}(y)\cdot\left[\nabla\rho(y-Q(t)) - \nabla\rho(y)\right]dy \\
&+ \ \mathrm{Re}\int \Psi(y,t)\cdot\nabla\rho(y-Q(t))dy \ .
\end{aligned}$$

First integral in the right hand side is zero by stationary equations (2.4). The second integral is $\mathcal{O}(t^{-3/2})$, since $Q(t) = \mathcal{O}(t^{-3/2})$, and by conditions (1.7) on ρ. Finally, the third integral is $\mathcal{O}(t^{-3/2})$ by estimate (6.10). The proof is complete.

12 Decay for linearized dynamics

In remaining sections we prove Proposition 6.7. Applying the Gronwall inequality to frozen linear equation (6.14) we obtain

$$\|X(t)\|_{\mathcal{E}} \le Ce^{\alpha t}\|X(0)\|_{\mathcal{E}} \tag{12.1}$$

with some $\alpha > 0$. Now we can apply the Fourier-Laplace transform

$$\tilde{X}(\lambda) = \int_0^\infty e^{-\lambda t}X(t)dt \ , \qquad \mathrm{Re}\,\lambda > \alpha \tag{12.2}$$

to (6.14). Integral (12.2) converges and is analytic for $\mathrm{Re}\,\lambda > \alpha$. We will write A and v instead of A_1 and v_1 in remaining part of the paper. After the Fourier-Laplace transform, equation (6.14) reads

$$\lambda\tilde{X}(\lambda) = A\tilde{X}(\lambda) + X_0 \ , \quad \mathrm{Re}\,\lambda > \alpha \ . \tag{12.3}$$

We will construct the resolvent $R(\lambda)) = (A-\lambda)^{-1}$ for $\mathrm{Re}\,\lambda > 0$ and prove that it is a continuous operator in $\mathcal{E}_{-\nu}$. Then $\tilde{X}(\lambda) = -(A-\lambda)^{-1}X_0 \in \mathcal{E}_{-\nu}$ and is an analytic function for $\mathrm{Re}\,\lambda > 0$.

This analyticity and the Paley-Wiener arguments (see [9]) should provide existence of a $\mathcal{E}_{-\nu}$ - valued distribution $X(t)$, $t \in \mathbb{R}$, with a support in $[0,\infty)$. Formally,

$$X(t) = \Lambda^{-1}\tilde{X} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}\tilde{X}(i\omega+0)d\omega \ , \quad t \in \mathbb{R} \tag{12.4}$$

To check continuity of $X(t)$ for $t \ge 0$, we need additionally an asymptotics for $\tilde{X}(i\omega+0)$ at large $|\omega|$. Finally, for time decay of $X(t)$, we need an additional information on smoothness and decay of $\tilde{X}(i\omega+0)$. More precisely, we should prove that $\tilde{X}(i\omega+0)$

i) is smooth outside $\omega = 0$ and $\omega = \pm\mu$, where $\mu = \mu(v) > 0$;

ii) decays in a certain sense as $|\omega| \to \infty$;

iii) admits the Puiseux expansion at $\omega = \pm\mu$;

iv) is analytic at $\omega = 0$ if $X_0 \in \mathcal{Z}_v := \mathbf{P}_v\mathcal{E}$ and $X_0 \in \mathcal{E}_\nu$.

Then decay (6.19) would follow from the Fourier-Laplace representation (12.4).

13 Solving linearized equation

Here we construct the resolvent. By (12.3)

$$(A-\lambda)\begin{pmatrix}\tilde\Psi_1\\ \tilde\Psi_2\\ \tilde Q\\ \tilde P\end{pmatrix}=-\begin{pmatrix}\Psi_{01}\\ \Psi_{02}\\ Q_0\\ P_0\end{pmatrix}.$$

It is system of equations

$$\left.\begin{aligned}(-\alpha_1\partial_1-\alpha_3\partial_3+v\cdot\nabla-\lambda)\tilde\Psi_1+(\beta m+\tilde\alpha_2\partial_2)\tilde\Psi_2-\tilde Q\cdot\nabla\rho_2&=-\Psi_{01}\\ -(\beta m+\tilde\alpha_2\partial_2)\tilde\Psi_1+(-\alpha_1\partial_1-\alpha_3\partial_3+v\cdot\nabla-\lambda)\tilde\Psi_2+\tilde Q\cdot\nabla\rho_1&=-\Psi_{02}\\ B_v\tilde P-\lambda\tilde Q&=-Q_0\\ -\langle\nabla\tilde\Psi_j,\rho_j\rangle+\langle\nabla\psi_{vj},\tilde Q\cdot\nabla\rho_j\rangle-\lambda\tilde P&=-P_0\end{aligned}\right|\qquad(13.1)$$

Step i) Let us study first two equations. First, we compute matrix integral kernel $G_\lambda(y-y')$ of the Green operator

$$G_\lambda=\begin{pmatrix}-\alpha_1\partial_1-\alpha_3\partial_3+v\cdot\nabla-\lambda & \beta m+\tilde\alpha_2\partial_2\\ -\beta m-\tilde\alpha_2\partial_2 & -\alpha_1\partial_1-\alpha_3\partial_3+v\cdot\nabla-\lambda\end{pmatrix}^{-1}.\qquad(13.2)$$

In Fourier space

$$\hat G_\lambda(k)=\begin{pmatrix}i\alpha_1k_1+i\alpha_3k_3-iv\cdot k-\lambda & \beta m-\alpha_2k_2\\ -\beta m+\alpha_2k_2 & i\alpha_1k_1+i\alpha_3k_3-iv\cdot k-\lambda\end{pmatrix}^{-1}.$$

To invert the matrix, we solve the system

$$\left.\begin{aligned}af_1+bf_2&=g_1\\ -bf_1+af_2&=g_2\end{aligned}\right|\qquad(13.3)$$

where $a=i\alpha_1k_1+i\alpha_3k_3-iv\cdot k-\lambda$, $b=\beta m-\alpha_2k_2$. Multiplying first equation of (13.3) by $c=-i\alpha_1k_1-i\alpha_3k_3-iv\cdot k-\lambda$ and the second equation by $-b$, we obtain

$$\left.\begin{aligned}caf_1+cbf_2&=cg_1\\ b^2f_1-cbf_2&=-bg_2\end{aligned}\right|\qquad(13.4)$$

since $ba=cb$ by anticommutations (1.2). Further, $b^2+ac=k^2+m^2+(iv\cdot k+\lambda)^2$. Therefore, summing up equations (13.4), we obtain that

$$f_1=\frac{cg_1-bg_2}{k^2+m^2+(iv\cdot k+\lambda)^2}$$

Similarly, we obtain

$$f_2=\frac{bg_1+cg_2}{k^2+m^2+(iv\cdot k+\lambda)^2}$$

Hence

$$\hat G_\lambda(k)=\frac{1}{k^2+m^2+(iv\cdot k+\lambda)^2}\begin{pmatrix}-i\alpha_1k_1-i\alpha_3k_3-iv\cdot k-\lambda & -\beta m+\alpha_2k_2\\ \beta m-\alpha_2k_2 & -i\alpha_1k_1-i\alpha_3k_3-iv\cdot k-\lambda\end{pmatrix}\qquad(13.5)$$

Taking the inverse Fourier transform we obtain

$$G_\lambda(y) = \begin{pmatrix} \alpha_1\partial_1 + \alpha_3\partial_3 + v\cdot\nabla - \lambda & -\beta m - \tilde\alpha_2\partial_2 \\ \beta m + \tilde\alpha_2\partial_2 & \alpha_1\partial_1 + \alpha_3\partial_3 + v\cdot\nabla - \lambda \end{pmatrix} g_\lambda(y) \tag{13.6}$$

where

$$g_\lambda(y) = F^{-1}_{k\to y}\frac{1}{k^2+m^2+(iv\cdot k+\lambda)^2}, \quad y\in\mathbb{R}^3 \tag{13.7}$$

Note that denominator in RHS (13.7) does not vanish for $\mathrm{Re}\,\lambda > 0$ since $|v| < 1$. This implies

Lemma 13.1. *Operator G_λ with integral kernel $G_\lambda(y-y')$, is continuous operator $L^2_0 \oplus L^2_0 \to L^2_0 \oplus L^2_0$ for* $\mathrm{Re}\,\lambda > 0$.

From now on we use system of coordinates in y-space in which $v = (|v|, 0, 0)$, hence $v\cdot k = |v|k_1$. Let us compute the function $g_\lambda(y)$. One has

$$k^2+m^2+(i|v|k_1+\lambda)^2 = \frac{1}{\gamma^2}k_1^2+k_2^2+k_3^2+2i|v|k_1\lambda+\lambda^2+m^2 = \frac{1}{\gamma^2}(k_1+i\gamma^2|v|\lambda)^2+k_2^2+k_3^2+\varkappa^2$$

where

$$\gamma = 1/\sqrt{1-v^2}\ , \quad \varkappa^2 = \frac{v^2\lambda^2}{1-v^2}+\lambda^2+m^2 = \frac{\lambda^2}{1-v^2}+m^2 = \gamma^2(\lambda^2+\mu^2)\ , \quad \mu := m/\gamma\ . \tag{13.8}$$

Hence,

$$\begin{aligned} g_\lambda(y) &= \frac{1}{(2\pi)^{3/2}}\int\frac{e^{-iky}dk}{\frac{1}{\gamma^2}(k_1+i\gamma^2|v|\lambda)^2+k_2^2+k_3^2+\kappa^2} = \frac{e^{-\gamma^2|v|\lambda y_1}}{(2\pi)^{3/2}}\int\frac{e^{-iky}dk}{\frac{1}{\gamma^2}k_1^2+k_2^2+k_3^2+\kappa^2} \\ &= \frac{\gamma e^{-\gamma|v|\lambda\tilde y_1}}{(2\pi)^{3/2}}\int\frac{e^{-ik\tilde y}dk}{k_1^2+k_2^2+k_3^2+\kappa^2} = \gamma e^{-\gamma|v|\lambda\tilde y_1}R_0(\tilde y, -\kappa^2) \end{aligned} \tag{13.9}$$

Here $\tilde y_1 = \gamma y_1$, $\tilde y = (\gamma y_1, y_2, y_3)$, and $R_0(y-y',\zeta)$ is integral kernel of operator $R_0(\zeta) = (-\Delta-\zeta)^{-1}$. It is well known that $R_0(y,\zeta) = e^{i\sqrt{\zeta}|y|}/4\pi|y|$. Therefore,

$$g_\lambda(y) = \frac{e^{-\varkappa|\tilde y|-\varkappa_1\tilde y_1}}{4\pi|\tilde y|}, \quad \varkappa = \gamma\sqrt{\lambda^2+\mu^2}, \quad \varkappa_1 := \gamma|v|\lambda \tag{13.10}$$

We choose $\mathrm{Re}\,\kappa > 0$ for $\mathrm{Re}\,\lambda > 0$. Note that for $0 < |v| < 1$

$$0 < \mathrm{Re}\,\varkappa_1 < \mathrm{Re}\,\varkappa\ , \qquad \mathrm{Re}\,\lambda > 0\ . \tag{13.11}$$

Let us state the result which we have got above.

Lemma 13.2. *i) The function $g_\lambda(y)$ decays exponentially in y for* $\mathrm{Re}\,\lambda > 0$.
ii) Formulas (13.10) and (13.8) imply that for every fixed y, function $g_\lambda(y)$ admits an analytic continuation in λ to the Riemann surface of algebraic function $\sqrt{\lambda^2+\mu^2}$ with branching points $\lambda = \pm i\mu$.

Thus, from (13.1) and (13.2) we obtain the representation

$$\begin{aligned} \tilde\Psi_1 &= -G^{11}_\lambda\Psi_{01} - G^{12}_\lambda\Psi_{02} - (G^{12}_\lambda\nabla\rho_1)\cdot\tilde Q + (G^{11}_\lambda\nabla\rho_2)\cdot\tilde Q\ , \\ \tilde\Psi_2 &= -G^{21}_\lambda\Psi_{01} - G^{22}_\lambda\Psi_{02} - (G^{22}_\lambda\nabla\rho_1)\cdot\tilde Q + (G^{21}_\lambda\nabla\rho_2)\cdot\tilde Q \end{aligned} \tag{13.12}$$

Step ii) Now we proceed to last two equations (13.1):

$$-\lambda\tilde{Q}+B_v\tilde{P}=-Q_0\ ,\qquad \langle\nabla\psi_{vj}\ ,\tilde{Q}\cdot\nabla\rho_j\rangle-\langle\nabla\tilde{\Psi}_j\ ,\rho_j\rangle-\lambda\tilde{P}=-P_0 \tag{13.13}$$

We rewrite equations (13.12) as $\tilde{\Psi}_j=\tilde{\Psi}_j(\tilde{Q})+\tilde{\Psi}_j(\Psi_0)$, where

$$\tilde{\Psi}_1(\Psi_0)=-G^{11}_\lambda\Psi_{01}-G^{12}_\lambda\Psi_{02}\ ,\quad \tilde{\Psi}_2(\Psi_0)=-G^{21}_\lambda\Psi_{01}-G^{22}_\lambda\Psi_{02} \tag{13.14}$$

$$\tilde{\Psi}_1(\tilde{Q})=(-G^{12}_\lambda\nabla\rho_1+G^{11}_\lambda\nabla\rho_2)\cdot\tilde{Q}\ ,\quad \tilde{\Psi}_2(\tilde{Q})=(-G^{22}_\lambda\nabla\rho_1+G^{21}_\lambda\nabla\rho_2)\cdot\tilde{Q}\ . \tag{13.15}$$

Then $\langle\nabla\tilde{\Psi}_j,\rho_j\rangle=\langle\nabla\tilde{\Psi}_j(\tilde{Q}),\rho_j\rangle+\langle\nabla\tilde{\Psi}_j(\Psi_0),\rho_j\rangle$, and last equation (13.13) becomes

$$\langle\nabla\psi_{vj},\tilde{Q}\cdot\nabla\rho_j\rangle-\langle\nabla\tilde{\Psi}_j(\tilde{Q}),\rho_j\rangle-\lambda\tilde{P}=-P_0+\langle\nabla\tilde{\Psi}_j(\Psi_0),\rho_j\rangle=:-P_0-\Phi(\lambda)$$

where

$$\Phi(\lambda)=\langle\tilde{\Psi}_j(\Psi_0),\nabla\rho_j\rangle\ . \tag{13.16}$$

First we compute the term

$$\langle\nabla\psi_{vj},\tilde{Q}\cdot\nabla\rho_j\rangle=\sum_{lj}\langle\nabla\psi_{vj},\tilde{Q}_l\partial_l\rho_j\rangle=\sum_{lj}\langle\nabla\psi_{vj},\partial_l\rho_j\rangle\tilde{Q}_l\ .$$

Applying the Fourier transform $F_{y\to k}$, we have by the Parseval identity and (A.25) that

$$\begin{aligned}\sum_j\langle\partial_i\psi_{vj},\partial_l\rho_j\rangle\ &=\ \sum_j\langle -ik_i\hat{\psi}_{vj},-ik_l\hat{\rho}_j\rangle=\int k_ik_l(\hat{\psi}_{v1}\cdot\hat{\rho}_1+\hat{\psi}_{v2}\cdot\hat{\rho}_2)dk \\ &=\ -\int k_ik_lm\frac{\beta\hat{\rho}_1\cdot\hat{\rho}_1+\beta\hat{\rho}_2\cdot\hat{\rho}_2}{k^2+m^2-(|v|k_1)^2}dk=-\int\frac{k_ik_l\mathcal{B}(k)dk}{k^2+m^2-(|v|k_1)^2}=:-L_{il}\ .\end{aligned} \tag{13.17}$$

As a result, $\langle\nabla\psi_{vj},\tilde{Q}\cdot\nabla\rho_j\rangle=-L\tilde{Q}$, where L is 3×3 matrix with matrix elements L_{il}. The matrix L is diagonal and positive defined by (1.8).

Now we compute the term $-\langle\nabla\tilde{\Psi}_j(\tilde{Q}),\rho_j\rangle=\langle\tilde{\Psi}_j(\tilde{Q}),\nabla\rho_j\rangle$. One has

$$\langle\tilde{\Psi}_j(\tilde{Q}),\partial_i\rho_j\rangle=\sum_l\Big(\langle -G^{12}_\lambda\partial_l\rho_1+G^{11}_\lambda\partial_l\rho_2,\partial_i\rho_1\rangle-\langle G^{22}_\lambda\partial_l\rho_1-G^{21}_\lambda\partial_l\rho_2,\partial_i\rho_2\rangle\Big)\tilde{Q}_l=\sum_l H_{il}(\lambda)\tilde{Q}_l$$

and by the Parseval identity and (1.2)-(1.3) we have

$$\begin{aligned}H_{il}(\lambda):\ &=\ \langle -G^{12}_\lambda\partial_l\rho_1+G^{11}_\lambda\partial_l\rho_2,\partial_i\rho_1\rangle-\langle G^{22}_\lambda\partial_l\rho_1-G^{21}_\lambda\partial_l\rho_2,\partial_i\rho_2\rangle \\ &=\ \langle[(\beta m-\alpha_2k_2)\hat{\rho}_1-(i\alpha_1k_1+i\alpha_3k_3+i|v|k_1+\lambda)\hat{\rho}_2]\hat{g}_\lambda k_l,k_i\hat{\rho}_1\rangle \\ &+\ \langle[(i\alpha_1k_1+i\alpha_3k_3+i|v|k_1+\lambda)\hat{\rho}_1+(\beta m-\alpha_2k_2)\hat{\rho}_2]\hat{g}_\lambda k_l,k_i\hat{\rho}_2\rangle \\ &=\ \int k_ik_lm\frac{\beta\hat{\rho}_1\cdot\hat{\rho}_1+\beta\hat{\rho}_2\cdot\hat{\rho}_2}{k^2+m^2-(|v|k_1-i\lambda)^2}dk=\int\frac{k_ik_l\mathcal{B}(k)dk}{k^2+m^2-(|v|k_1-i\lambda)^2}\ .\end{aligned} \tag{13.18}$$

The matrix H is well defined for $\operatorname{Re}\lambda>0$ since the denominator does not vanish. The matrix H is diagonal. Indeed, if $i\neq l$, then at least one of these indices is not equal to one, and the integrand in (13.17) is odd with respect to the corresponding variable. Thus, $H_{il}=0$. As a result, $\langle\tilde{\Psi}_j(\tilde{Q}),\nabla\rho_j\rangle=H\tilde{Q}$, where H is the matrix with matrix elements H_{il}. Finally, (13.13) becomes

$$M(\lambda)\left(\begin{array}{c}\tilde{Q}\\ \tilde{P}\end{array}\right)=\left(\begin{array}{c}Q_0\\ P_0+\Phi(\lambda)\end{array}\right),\ \text{where } M(\lambda)=\left(\begin{array}{cc}\lambda E & -B_v\\ L-H(\lambda) & \lambda E\end{array}\right)\ . \tag{13.19}$$

Assume for a moment that the matrix $M(\lambda)$ is invertible (later we will prove this). Then we obtain

$$\left(\begin{array}{c}\tilde{Q}\\ \tilde{P}\end{array}\right)=M^{-1}(\lambda)\left(\begin{array}{c}Q_0\\ P_0+\Phi(\lambda)\end{array}\right)\ ,\qquad\qquad \operatorname{Re}\lambda>0\ . \tag{13.20}$$

Finally, formula (13.20) and formulas (13.12), where $\tilde{Q}$ is expressed from (13.20), give the expression of the resolvent $R(\lambda)=(A-\lambda)^{-1}$, $\operatorname{Re}\lambda>0$.

Lemma 13.3. *i) $M(\lambda)$ admits an analytic continuation from complex half-plane* $\operatorname{Re}\lambda > 0$ *to the Riemann surface Σ of the function $\sqrt{\mu^2+\lambda^2}$;*
ii)$M(\lambda)$ is Hölder continuous on each compact set in Σ;
iii) $M^{-1}(\lambda)$ is meromorphic on Σ.

Proof. i) The analytic continuation of $M(\lambda)$ exists by Lemma 13.1, conditions (1.7), and last line in (13.18):

$$H_{jj}(\lambda) = \langle mg_\lambda\beta * \partial_j\rho_1, \partial_j\rho_1\rangle + \langle mg_\lambda\beta * \partial_j\rho_2, \partial_j\rho_2\rangle\ , \quad j=1,2,3 \tag{13.21}$$

since g_λ is analytic on Σ by (13.10).
ii) The Hölder continuity holds by the same arguments.
iii) The inverse matrix is meromorphic since it exists for large $\operatorname{Re}\lambda$. The latter follows from (13.19) since $H(\lambda)\to 0$, $\operatorname{Re}\lambda\to\infty$, by (13.18). □

14 Inverse matrix

Here we study smoothness of $M^{-1}(\lambda)$ on imaginary axis and in half-plane $\operatorname{Re}\lambda > 0$.

14.1 Regularity on imaginary axis

By Lemma 13.3, the limit matrix $M(i\omega) := M(i\omega+0)$ exists for $\omega\in\mathbb{R}$, and its entries are continuous functions of $\omega\in\mathbb{R}$, smooth for $|\omega|<\mu$ and $|\omega|>\mu$.

Proposition 14.1. *Let ρ satisfy conditions (1.7)- (1.8), and $|v|<1$. Then the matrix $M(i\omega)$ is invertible for $\omega\in\mathbb{R}\setminus 0$.*

This proposition follows by methods from [8, Proposition 15.1].

Now let us obtain asymptotics of $M^{-1}(\lambda)$ near singular points $\lambda=0$ and $\lambda=\pm i\mu$.

I. First we consider the points $\lambda=\pm i\mu$.

Lemma 14.2. *The asymptotics hold*

$$M^{-1}(\lambda)=C^{\pm}+\mathcal{O}((\lambda\mp i\mu)^{\frac12})\ ,\ \partial M^{-1}(\lambda)=\mathcal{O}((\lambda\mp i\mu)^{-\frac12})\ ,\ \partial^2 M^{-1}(\lambda)=\mathcal{O}((\lambda\mp i\mu)^{-\frac32}) \tag{14.1}$$

Proof. It suffices to prove similar asymptotics for $M(\lambda)$. Then (14.1) holds also for $M^{-1}(\lambda)$, since the matrices $M(\pm i\mu)$ are invertible. The asymptotics for $M(\lambda)$ hold by convolution representation (13.21) since g_λ admits the corresponding asymptotics by (13.10). Namely

$$g_\lambda(y) = \frac{1}{4\pi|\tilde y|} + r_{\pm}(\lambda,y)\ , \quad \lambda\to\pm i\mu, \quad \operatorname{Re}\lambda>0$$

where

$$r_{\pm}(\lambda,y)=\mathcal{O}((\lambda\mp i\mu)^{\frac12})\ ,\ \ \partial_\lambda r_{\pm}(\lambda,y)=\mathcal{O}((\lambda\mp i\mu)^{-\frac12})\ ,\ \ \partial_\lambda^2 r_{\pm}(\lambda,y)=\mathcal{O}((1+|y|)(\lambda\mp i\mu)^{-\frac32})$$

Condition (1.7) provides convergence of all integrals arising in $\partial_\lambda^k H_{jj}$. □

II. Second, we consider the point $\omega=0$ which is an isolated pole of a finite degree by Lemma 13.3. In Appendix B we prove that determinant of $M(i\omega)$ can be written as

$$\det M(i\omega) = -\omega^6\big(1+\frac{f_{11}(\omega)}{\gamma^3}\big)\big(1+\frac{f_{22}(\omega)}{\gamma}\big)\big(1+\frac{f_{33}(\omega)}{\gamma}\big) \tag{14.2}$$

where $f_{jj}(\omega)\in C^\infty(-\mu,\mu)$ and $f_{jj}(0)>0$.

14.2 Behavior at infinity

Here we study asymptotic behavior of $M^{-1}(\lambda)$ at infinity.

Lemma 14.3. *There exist a matrix D_0 and a matrix-function $D_1(\lambda)$, such that*

$$M^{-1}(\lambda) = \frac{D_0}{\lambda} + D_1(\lambda) \ , \quad |\lambda| \to \infty \ , \quad \operatorname{Re}\lambda > 0 \tag{14.3}$$

where

$$|\partial_\lambda^k D_1(\lambda)| \le \frac{C(k)}{|\lambda|^2} \ , \qquad |\lambda| \to \infty \ , \quad \operatorname{Re}\lambda > 0 \ , \quad k = 0, 1, 2 \ . \tag{14.4}$$

Proof. The structure of $M(\lambda)$ provides that it suffices to prove the estimate for $H_{jj}(\lambda)$:

$$|\partial_\lambda^k H_{jj}(\lambda)| \le C(k) \ , \quad \lambda \in \mathbb{C} \ , \quad |\lambda| \ge \mu + 1 \ , \quad j = 1, 2, 3 \ , \quad k = 0, 1, 2 \ . \tag{14.5}$$

This estimate follows from representation (13.21) and the bounds

$$|g_\lambda(y)| \le \frac{C_1}{|y|}, \quad |\partial_\lambda g_\lambda(y)| \le \frac{C_2}{|y|} + C_3, \quad |\partial_\lambda^2 g_\lambda(y)| \le \frac{C_4}{|y|} + C_5|y|, \quad \operatorname{Re}\lambda > 0$$

$\square$

14.3 Analyticity in half-plane

Lemma 14.4. $M^{-1}(\lambda)$ *is holomorphic in* $\mathbb{C}^+ := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$.

Proof. We apply a bifurcation argument. Namely, we replace ρ by $\varepsilon\rho$ with $\varepsilon \in [0, 1]$, and write $M_\varepsilon(\lambda)$ for the corresponding matrix $M(\lambda)$ with $\varepsilon^2 L$ and $\varepsilon^2 H(\lambda)$ instead of L and $H(\lambda)$. Then (13.19) in the case $\varepsilon = 0$ yields

$$M_0(\lambda) = \begin{pmatrix} \lambda E & -B_v \\ 0 & \lambda E \end{pmatrix} \tag{14.6}$$

Hence, $M_0^{-1}(\lambda)$ is a holomorphic matrix function for $\lambda \in \mathbb{C}^+$. Let us extend this analyticity to $M_\varepsilon^{-1}(\lambda)$ with $\varepsilon \in (0, 1]$.

Step i) Asymptotics of type (14.3)-(14.4) hold for $M_\varepsilon^{-1}(\lambda)$ uniformly in $\varepsilon \in [0, 1]$. Therefore, there exists an $R > 0$ such that $M_\varepsilon^{-1}(\lambda)$ is a holomorphic matrix function of $\lambda \in \mathbb{C}^+$ with $|\lambda| > R$ for all $\varepsilon \in (0, 1]$.

Step ii) Similarly, formulas (14.2), (14.6) imply asymptotics

$$\det M_\varepsilon(\lambda) \sim \lambda^6 \ , \qquad \lambda \to 0$$

which hold uniformly in $\varepsilon \in [0, 1]$. Therefore, there exists a $\delta_1 > 0$ such that

a) $M_\varepsilon^{-1}(\lambda)$ is holomorphic in the semicircle $\lambda \in \mathbb{C}^+$, $|\lambda| < 2\delta_1$ for all $\varepsilon \in (0, 1]$;

b) $M_\varepsilon^{-1}(\lambda)$ is bounded on the ring $\mathcal{R}(\delta_1) := \{\lambda \in \mathbb{C}^+ : \ \delta_1 < |\lambda| < 2\delta_1\}$ uniformly in $\varepsilon \in (0, 1]$.

Step iii) Proposition 14.1 implies, by a continuity argument, that $M_\varepsilon^{-1}(\lambda)$ is bounded on the set $\{i\omega : \omega \in \mathbb{R}, \ \delta_1 < |\omega| < R + 1\}$ uniformly in $\varepsilon \in (0, 1]$. Furthermore, the Hölder continuity from Lemma 13.3 ii) is obviously uniform in $\varepsilon \in (0, 1]$. Hence, $M_\varepsilon^{-1}(\lambda)$ is holomorphic and bounded on a region

$$\Pi(R, \delta_1, \delta_2) := \{\lambda \in \mathbb{C} : \ \delta_1 < |\operatorname{Im}\lambda| < R + 1, \quad 0 < \operatorname{Re}\lambda \le \delta_2\}$$

uniformly in $\varepsilon \in (0, 1]$, where $\delta_2 = \delta(R, \delta_1) > 0$.

Step iv) Finally, we consider a closed path $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 lies in the union $\Pi(R, \delta_2) \cup \mathcal{R}(\delta_1)$, and Γ_2 lies on the semicircle $|\lambda| = R$, $\operatorname{Re}\lambda > 0$. By the arguments above, $M_\varepsilon^{-1}(\lambda)$ is bounded on Γ uniformly in $\varepsilon \in [0, 1]$. Therefore, $M_\varepsilon^{-1}(\lambda)$ is holomorphic inside Γ for $\varepsilon \in (0, 1]$ as well as for $\varepsilon = 0$. $\square$

15 Transversal decay for linearized equation

Here we prove Proposition 6.7. First, we establish decay in weighted norm for the solution to free Dirac equation.

15.1 Weighted decay for free Dirac equation

Denote by $W_v^{\pm}(t)$ dynamical groups (propagators) of the following "modified" free Dirac equations

$$\partial_t \Psi^{\pm}(x,t) = [\pm\alpha\cdot\nabla \pm i\beta m + v\cdot\nabla]\Psi^{\pm}(x,t)\ . \tag{15.1}$$

Lemma 15.1. *For any $\Phi \in L^2_\nu$ with $\nu > 3/2$ the bound holds*

$$\|W_v^{\pm}(t)\Phi\|_{-\nu} \le \frac{C_\nu(v)\|\Phi\|_\nu}{(1+|t|)^{3/2}}\ , \quad t\ge 0\ . \tag{15.2}$$

Proof. Step i) For concreteness we consider the case "+". Note, that

$$(\partial_t + \alpha\cdot\nabla + i\beta m + v\cdot\nabla)\,(\partial_t - \alpha\cdot\nabla - i\beta m + v\cdot\nabla) = (\partial_t^2 - \Delta + (v\cdot\nabla)^2 + 2v\cdot\nabla\partial_t + m^2)\ .$$

Hence the integral kernel $W_v^+(x-y,t)$ of the operator $W_v^+(t)$ reads

$$W_v^+(z,t) = (\partial_t + \alpha\cdot\nabla + i\beta m + v\cdot\nabla)\,G_v(z,t)\ , \tag{15.3}$$

where $G_v(z,t)$ is a fundamental solution of the "modified" Klein-Gordon operator

$$(\partial_t^2 - \Delta + (v\cdot\nabla)^2 + 2v\cdot\nabla\partial_t + m^2)G_v(z,t) = \delta(z)\delta(t)\ .$$

Let $G_v(t)$, $t\ge 0$ be the operator with the integral kernel $G_v(x-y,t)$. It is easy to see that

$$[G_v(t)\Phi](x) = [G_0(t)\Phi](x-vt)\ , \quad x\in\mathbb{R}^3, \quad t\ge 0\ .$$

Then

$$G_v(z,t) = G_0(z-vt,t) = \frac{\delta(t-|z-vt|)}{4\pi t} - \frac{m}{4\pi}\frac{\theta(t-|z-vt|)J_1(m\sqrt{t^2-|z-vt|^2})}{\sqrt{t^2-|z-vt|^2}}\ , \quad t>0$$

where J_1 is the Bessel function of order 1, and θ is the Heavyside function. Let us fix an arbitrary $\varepsilon\in(|v|,1)$. Well known asymptotics of the Bessel function imply that

$$|\partial_t G_v(z,t)|\ ,\ |\partial_{z_j} G_v(z,t)| \le C(\varepsilon)(1+t)^{-3/2}\ , \quad |z-vt|\le\varepsilon t\ , \ \ t\ge 1\ , \ \ j=1,2,3\ . \tag{15.4}$$

Step ii) Consider an arbitrary $t\ge 1$. Denote $\varepsilon_1 = \varepsilon - |v|$. We split the function Φ in two terms, $\Phi = \Phi_{1,t} + \Phi_{2,t}$ such that

$$\|\Phi_{1,t}\|_{L^2_\nu} + \|\Phi_{2,t}\|_{L^2_\nu} \le C\|\Phi\|_{L^2_\nu}\ , \quad t\ge 1 \tag{15.5}$$

and

$$\Phi_{1,t}(x) = 0 \ \text{ for } |x| > \frac{\varepsilon_1 t}{2}\ , \qquad \text{and} \qquad \Phi_{2,t}(x) = 0 \ \text{ for } |x| < \frac{\varepsilon_1 t}{4}\ . \tag{15.6}$$

Estimate (15.2) for $W_v^+(t)\Phi_{2,t}$ follows by charge conservation for Dirac equation, (15.5) and (15.6):

$$\|W_v^+(t)\Phi_{2,t}\|_{L^2_{-\nu}} \le \|W_v^+(t)\Phi_{2,t}\|_{L^2_0} = \|\Phi_{2,t}\|_{L^2_0} \le \frac{C(\varepsilon)\|\Phi_{2,t}\|_{L^2_\nu}}{(1+t)^\nu} \le \frac{C_1(\varepsilon)\|\Phi\|_{L^2_\nu}}{(1+t)^{3/2}}\ , \quad t\ge 1 \tag{15.7}$$

since $\nu > 3/2$.
Step iii) Next we consider $W_v^+(t)\Phi_{1,t}$. Now we split the operator $W_v^+(t)$ in two terms:

$$W_v^+(t) = (1-\zeta)W_v^+(t) + \zeta W_v^+(t) \ , \qquad t \geq 1$$

where ζ is the operator of multiplication by the function $\zeta(|x|/t)$ such that $\zeta = \zeta(s) \in C_0^\infty(\mathbb{R})$, $\zeta(s) = 1$ for $|s| < \varepsilon_1/4$, $\zeta(s) = 0$ for $|s| > \varepsilon_1/2$. Since $1 - \zeta(|x|/t) = 0$ for $|x| < \varepsilon_1 t/4$, then applying the charge conservation and (15.5), we have for $t \geq 1$

$$\begin{aligned} \|(1-\zeta)W_v^+(t)\Phi_{1,t}\|_{L^2_{-\nu}} &\leq \frac{C(\varepsilon)\|W_v^+(t)\Phi_{1,t}\|_{L_0^2}}{(1+t)^\nu} = \frac{C(\varepsilon)\|\Phi_{1,t}\|_{L_0^2}}{(1+t)^\nu} \\ &\leq \frac{C_1(\varepsilon)\|\Phi_{1,t}\|_{L^2_\nu}}{(1+t)^\nu} \leq \frac{C_2(\varepsilon)\|\Phi\|_{L^2_\nu}}{(1+t)^{3/2}} \ . \end{aligned} \tag{15.8}$$

Step iv) It remains to estimate $\zeta W_v^+(t)\Phi_{1,t}$. Let χ_t be the characteristic function of the ball $|x| \leq \varepsilon_1 t/2$. We will use the same notation for operator of multiplication by this characteristic function. By (15.6), we have

$$\zeta W_v^+(t)\Phi_{1,t} = \zeta W_v^+(t)\chi_t\Phi \tag{15.9}$$

The matrix kernel of the operator $\zeta W_v^+(t)\chi_t$ is equal to

$$\mathcal{W}_v^+(x-y,t) = \zeta(|x|/t)W_v^+(x-y,t)\chi_t(y)$$

Since $\zeta(|x|/t) = 0$ for $|x| > \varepsilon_1 t/2$ and $\chi_t(y) = 0$ for $|y| > \varepsilon_1 t/2$ then $\mathcal{W}_v^+(x-y,t) = 0$ for $|x-y| > \varepsilon_1 t$. On the other hand, $|x-y| \leq \varepsilon_1 t$ implies $|x-y-vt| \leq \varepsilon t$, since $\varepsilon_1 + |v| = \varepsilon$ by definition of ε_1. Hence, (15.3) and (15.4) yield

$$|\mathcal{W}_v^+(x-y,t)| \leq C(1+t)^{-3/2} \ , \qquad t \geq 1 \ . \tag{15.10}$$

The norm of the operator $\zeta W_v^+(t)\chi_t : L^2_\nu \to L^2_{-\nu}$ is equivalent to the norm of the operator

$$\langle x\rangle^{-\nu}\zeta W_v^+(t)\chi_t(y)\langle y\rangle^{-\nu} : L^2 \to L^2 \tag{15.11}$$

Therefore, (15.10) implies that operator (15.11) is Hilbert-Schmidt operator since $\nu > 3/2$, and its Hilbert-Schmidt norm does not exceed $C(1+t)^{-3/2}$. Hence, by (15.9)

$$\|\zeta W_v^+(t)\Phi_{1,t}\|_{L^2_{-\nu}} \leq C(1+t)^{-3/2}\|\Phi\|_{L^2_\nu} \ , \quad t \geq 1 \ . \tag{15.12}$$

Finally, (15.7), (15.8) and (15.12) imply (15.2). $\square$

15.2 Decay of vector components

Here we establish the decay (6.19) for $Q(t)$ and $P(t)$.

Lemma 15.2. *Let $X_0 \in \mathcal{Z}_v \cap \mathcal{E}_\nu$. Then $Q(t)$, $P(t)$ are continuous and*

$$|Q(t)| + |P(t)| \leq C_\nu(\rho,\tilde{v})(1+|t|)^{-3/2} \ , \qquad t \geq 0 \ . \tag{15.13}$$

Proof. The components $Q(t)$ and $P(t)$ are given by the Fourier integral

$$\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \frac{1}{2\pi}\int e^{i\omega t}M^{-1}(i\omega)\begin{pmatrix} Q_0 \\ P_0 + \Phi(i\omega) \end{pmatrix} d\omega \tag{15.14}$$

with $\Phi(i\omega):=\Phi(i\omega+0)$ where $\Phi(\lambda)$ is defined in (13.16). The integral converges in the sense of distributions to a continuous function of $t\geq 0$ by (14.1), (14.3) and (B.31). Note that the condition $X_0\in\mathcal{Z}_v$ implies that the whole trajectory $X(t)$ lies in $\mathcal{Z}_v$. This follows from the invariance of the space $\mathcal{Z}_v$ under the generator $A_{v,v}$ (cf. Remark 6.6). If $X_0\not\in\mathcal{Z}_v$, then $Q(t)$ and $P(t)$ may contain non-decaying terms which correspond to singular point $\omega=0$ since the linearized dynamics admits the secular solutions without decay, see (6.15). We will show that the symplectic orthogonality condition leads to (15.13). We split integral (15.14) into three terms using the partition of unity $\zeta_1(\omega)+\zeta_2(\omega)+\zeta_3(\omega)=1$, $\omega\in\mathbb{R}$:

$$\left(\begin{array}{c}Q(t)\\ P(t)\end{array}\right)=\frac{1}{2\pi}\int e^{i\omega t}(\zeta_1(\omega)+\zeta_2(\omega)+\zeta_3(\omega))M^{-1}(i\omega+0)\left(\begin{array}{c}Q_0\\ P_0+\Phi(i\omega)\end{array}\right)d\omega=\sum_{j=1}^{3}I_j(t)$$

where the functions $\zeta_j(\omega)\in C^\infty(\mathbb{R})$ are supported by

$$\left.\begin{array}{rcl}\operatorname{supp}\zeta_1&\subset&\{\omega\in\mathbb{R}:\varepsilon_0/2<|\omega|<\mu+2\}\\[1ex] \operatorname{supp}\zeta_2&\subset&\{\omega\in\mathbb{R}:|\omega|>\mu+1\}\\[1ex] \operatorname{supp}\zeta_3&\subset&\{\omega\in\mathbb{R}:|\omega|<\varepsilon_0\}\end{array}\right|\qquad(15.15)$$

i) Let us represent $I_j(t)$, $j=1,2$ as

$$\begin{array}{rcl}I_j(t)&=&\displaystyle\frac{1}{2\pi}\int e^{i\omega t}\zeta_j(\omega)\Big[M^{-1}(i\omega)\left(\begin{array}{c}Q_0\\ P_0\end{array}\right)+M^{-1}(i\omega)\left(\begin{array}{c}0\\ \Phi(i\omega)\end{array}\right)\Big]d\omega\\ &=&s_j(t)\left(\begin{array}{c}Q_0\\ P_0\end{array}\right)+s_j(t)*\left(\begin{array}{c}0\\ f(t)\end{array}\right)\end{array}\qquad(15.16)$$

where

$$s_j(t)=\Lambda^{-1}\zeta_j(\omega)M^{-1}(i\omega+0)\ ,\quad f(t)=\Lambda^{-1}\Phi(i\omega)\ .\qquad(15.17)$$

By (13.14) and (15.1)

$$\tilde\Psi_1(\Psi_0)=-\Lambda\operatorname{Re}W_v^+(t)\Psi_0\ ,\quad \tilde\Psi_2(\Psi_0)=-\Lambda\operatorname{Im}W_v^+(t)\Psi_0\ .$$

Hence, (13.16), (15.17) and Lemma 15.1 imply

$$|f(t)|=|\operatorname{Re}\langle W_v^+(t)\Psi_0,\nabla\rho\rangle|\leq C_\nu(\rho,v)(1+t)^{-3/2}\ .$$

Further, the function $s_1(t)$ decays as $(1+|t|)^{-3/2}$ by (14.1), and $s_2(t)$ decays as $(1+|t|)^{-2}$ due to Proposition 14.3. Hence, $I_1(t)$ and $I_2(t)$ decay as $(1+|t|)^{-3/2}$ by (15.16).
iii) Finally, the function $I_3(t)$ decays as $t^{-\infty}$ since

$$\left(\begin{array}{c}\tilde Q(i\omega)\\ \tilde P(i\omega)\end{array}\right)=M^{-1}(i\omega)\left(\begin{array}{c}Q_0\\ P_0+\Phi(i\omega)\end{array}\right)\in C^\infty(-\mu,\mu)\quad\text{if}\quad X_0\in\mathcal{Z}_v\ .$$

Indeed, in Appendix C we prove that the symplectic orthogonality conditions (6.7) at $t=0$ imply

$$P_0+\Phi(0)=0\ ,\quad B_v^{-1}Q_0+\Phi'(0)=0\ .\qquad(15.18)$$

Then

$$P_0+\Phi(i\omega)=\Phi(i\omega)-\Phi(0)=i\omega\Upsilon_1(\omega)$$

$$B_v^{-1}Q_0+\Upsilon_1(\omega)=\frac{\Phi(i\omega)-\Phi(0)}{i\omega}-\Phi'(0)=i\omega\Upsilon_2(\omega)$$

where $\Upsilon_j(\omega)\in C^\infty(-\mu,\mu)$ by (13.6), (13.14) and (13.16). Therefore, (B.31)-(B.32) imply

$$\begin{aligned}\tilde P(i\omega) &= \mathcal{M}_{21}(\omega)Q_0+i\mathcal{M}_{22}(\omega)\Upsilon_1(\omega)\in C^\infty(-\mu,\mu)\\ \tilde Q(i\omega) &= \frac{1}{\omega}\mathcal{M}_{11}(\omega)Q_0+\frac{i}{\omega}\mathcal{M}_{12}(\omega)\Upsilon_1(\omega)=\frac{i}{\omega}\mathcal{M}_{12}(B_v^{-1}Q_0+\Upsilon_1(\omega))\\ &= -\mathcal{M}_{12}\Upsilon_2(\omega)\in C^\infty(-\mu,\mu)\end{aligned}$$

□

15.3 Decay of fields

Here we prove the decay (6.19) for the field components $\Psi_1(y,t),\Psi_2(y,t)$. First two equations of (6.14) may be written as one equation:

$$\dot\Psi(y,t)=[-\alpha\cdot\nabla-i\beta m+v\cdot\nabla]\Psi(y,t)-iQ(t)\cdot\nabla\rho(y)\ ,\quad x\in\mathbb{R}^3\ ,\quad t\in\mathbb{R}\qquad(15.19)$$

where $\Psi(y,t)=\Psi_1(y,t)+i\Psi_2(y,t))$. Applying the Duhamel representation, we obtain

$$\Psi(t)=W_v^-(t)\Psi_0-\int_0^t W_v^-(t-s)Q(s)\cdot\nabla\rho\,ds\ ,\quad t\ge 0$$

where $W_v^-(t)$ is defined in section 15.1. Hence, Lemma 15.1 and the decay of Q from (15.13) yield

$$\|\Psi(t)\|_{-\nu}\le C_\nu(\rho,\tilde v)\|\Psi_0\|_\nu(1+|t|)^{-3/2}\ ,\quad t\ge 0\ .\qquad(15.20)$$

It completes the proof of Proposition 6.7.

A Computing symplectic form

Here we justify formulas (3.6)-(3.8) for the matrix Ω.

1) First, the Parseval identity implies

$$\Omega(\tau_j,\tau_l)=\langle\partial_j\psi_{v1},\partial_l\psi_{v2}\rangle-\langle\partial_j\psi_{v2},\partial_l\psi_{v1}\rangle=\int k_jk_l\,dk(\hat\psi_{v1}\cdot\hat\psi_{v2}-\hat\psi_{v2}\cdot\hat\psi_{v1})=0\ ,\quad j,l=1,2,3\ .$$

since the integrand is odd function.

2) Second, we consider

$$\Omega(\tau_{j+3},\tau_{l+3})=\langle\partial_{v_j}\psi_{v1},\partial_{v_l}\psi_{v2}\rangle-\langle\partial_{v_j}\psi_{v2},\partial_{v_l}\psi_{v1}\rangle\ .\qquad(\text{A.21})$$

Let us derive the formulas for ψ_{v1} and ψ_{v2}. First equation of (2.4) implies

$$[(v\cdot\nabla)^2-\Delta+m^2]\psi_v=[iv\cdot\nabla+i\alpha\cdot\nabla-\beta m]\rho_1\ .$$

Hence

$$[(v\cdot\nabla)^2-\Delta+m^2]\psi_{v1}=-[v\cdot\nabla+\alpha_1\partial_1+\alpha_3\partial_3]\rho_2-[\tilde\alpha_2\partial_2+\beta m]\rho_1\ ,$$

$$[(v\cdot\nabla)^2-\Delta+m^2]\psi_{v2}=[v\cdot\nabla+\alpha_1\partial_1+\alpha_3\partial_3]\rho_1-[\tilde\alpha_2\partial_2+\beta m]\rho_2\ .$$

Applying the Fourier transform, we obtain

$$\left.\begin{aligned}\hat\psi_{v1}&=\frac{[iv\cdot k+i\alpha_1k_1+i\alpha_3k_3]\hat\rho_2+[\alpha_2k_2-\beta m]\hat\rho_1}{-(v\cdot k)^2+k^2+m^2}\\ \hat\psi_{v2}&=\frac{-[iv\cdot k+i\alpha_1k_1+i\alpha_3k_3]\hat\rho_1+[\alpha_2k_2-\beta m]\hat\rho_2}{-(v\cdot k)^2+k^2+m^2}\end{aligned}\right|\qquad(\text{A.22})$$

Differentiating, we get

$$\left.\begin{aligned} \partial_{v_j}\hat\psi_{v1} &= \frac{ik_j\hat\rho_2}{-(v\cdot k)^2+k^2+m^2}+\frac{2k_jv\cdot k\hat\psi_{v1}}{-(v\cdot k)^2+k^2+m^2} \\ \partial_{v_l}\hat\psi_{v2} &= \frac{-ik_l\hat\rho_1}{-(v\cdot k)^2+k^2+m^2}+\frac{2k_lv\cdot k\hat\psi_{v2}}{-(v\cdot k)^2+k^2+m^2} \end{aligned}\right| \tag{A.23}$$

Hence, (A.21) implies

$$\begin{aligned} \Omega(\tau_{j+3},\tau_{l+3}) \ &= \ \int\frac{k_jk_l[\hat\rho_1\cdot\hat\rho_2-\hat\rho_2\cdot\hat\rho_1]dk}{(k^2+m^2-(v\cdot k)^2)^2}+\int\frac{4k_jk_l(v\cdot k)^2[\hat\psi_{v1}\cdot\hat\psi_{v2}-\hat\psi_{v2}\cdot\hat\psi_{v1}]dk}{(k^2+m^2-(v\cdot k)^2)^2} \\ &+ \ \int\frac{2ik_jk_lv\cdot k[\hat\rho_2\cdot\hat\psi_{v2}+\hat\psi_{v2}\cdot\hat\rho_2+\hat\rho_1\cdot\hat\psi_{v1}+\hat\psi_{v1}\cdot\hat\rho_1]}{(k^2+m^2-(v\cdot k)^2)^2}=0 \end{aligned}$$

since all integrands are odd functions.

3) Finally, (A.23) implies

$$\begin{aligned} &\Omega(\tau_j,\tau_{l+3}) = -\langle\partial_j\psi_{v1},\partial_{v_l}\psi_{v2}\rangle+\langle\partial_j\psi_{v2},\partial_{v_l}\psi_{v1}\rangle+e_j\cdot\partial_{v_l}p_v \\ &=\int\frac{ik_j\hat\psi_{v1}\cdot[-ik_l\hat\rho_1+2k_lv\cdot k\hat\psi_{v2})]-ik_j\hat\psi_{v2}\cdot[ik_l\hat\rho_2+2k_lv\cdot k\hat\psi_{v1})]}{k^2+m^2-(v\cdot k)^2}dk+e_j\cdot\partial_{v_l}p_v \\ &=\int k_jk_l\frac{-[\hat\psi_{v1}\cdot\hat\rho_1+\hat\psi_{v2}\cdot\hat\rho_2]+2iv\cdot k[\hat\psi_{v1}\cdot\hat\psi_{v2}-\hat\psi_{v2}\cdot\hat\psi_{v1}]}{k^2+m^2-(v\cdot k)^2}dk+e_j\cdot\partial_{v_l}p_v \end{aligned} \tag{A.24}$$

Recall, that $\rho_j(x)$ are even, then $\hat\rho_j(k)$ are real. Hence (1.2)-(1.3) and (A.22) imply

$$\begin{aligned} &(k^2+m^2-(v\cdot k)^2)(\hat\psi_{v1}\cdot\hat\rho_1+\hat\psi_{v2}\cdot\hat\rho_2)=[\alpha_2k_2-\beta m]\hat\rho_1\cdot\hat\rho_1+[\alpha_2k_2-\beta m]\hat\rho_2\cdot\hat\rho \\ &\quad+[iv\cdot k+i\alpha_1k_1+i\alpha_3k_3]\hat\rho_2\cdot\hat\rho_1-[iv\cdot k+i\alpha_1k_1+i\alpha_3k_3]\hat\rho_1\cdot\hat\rho_2=-\mathcal{B}\hat\rho\cdot\hat\rho \end{aligned} \tag{A.25}$$

$$\begin{aligned} &(k^2+m^2-(v\cdot k)^2)^2(\hat\psi_{v1}\cdot\hat\psi_{v2}-\hat\psi_{v2}\cdot\hat\psi_{v1})=2i(k^2+m^2-(v\cdot k)^2)^2\,\mathrm{Im}(\hat\psi_{v1}\cdot\hat\psi_{v2}) \\ &=-2\beta m\hat\rho_1\cdot[iv\cdot k+i\alpha_1k_1+i\alpha_3k_3]\hat\rho_1-2[iv\cdot k+i\alpha_1k_1+i\alpha_3k_3]\hat\rho_2\cdot\beta m\hat\rho_2 \\ &=-2iv\cdot k\mathcal{B}\hat\rho\cdot\hat\rho \end{aligned} \tag{A.26}$$

Substituting (A.25) and (A.26) into the right hand site of (A.24), we obtain

$$\Omega(\tau_j,\tau_{l+3})=\int k_jk_l\Big(\frac{\mathcal{B}(k)}{(k^2+m^2-(v\cdot k)^2)^2}+\frac{4(v\cdot k)^2\mathcal{B}(k)}{(k^2+m^2-(v\cdot k)^2)^3}\Big)dk+e_j\cdot\partial_{v_l}p_v$$

that correspond to (3.6) - (3.8).

B Computing inverse matrix

Denote $F(\omega):=-L+H(i\omega+0)$ which is diagonal. Then by (13.19) for $\omega\in\mathbb{R}$ we obtain

$$\det M(i\omega)=\det\begin{pmatrix} i\omega E & -B_v \\ -F(\omega) & i\omega E\end{pmatrix}=-\Big(\omega^2+\frac{F_{11}(\omega)}{\gamma^3}\Big)\Big(\omega^2+\frac{F_{22}(\omega)}{\gamma}\Big)\Big(\omega^2+\frac{F_{33}(\omega)}{\gamma}\Big) \tag{B.27}$$

where

$$F_{jj}(\omega)=\int k_j^2\mathcal{B}dk\left(\frac{1}{m^2+k^2-(|v|k_1+\omega)^2}-\frac{1}{m^2+k^2-(|v|k_1)^2}\right)\ ,\ \ j=1,2,3\ . \tag{B.28}$$

Formula (B.27) is obvious since both matrices $F(\omega)$ and B_v are diagonal, hence the matrix $M(i\omega)$ is equivalent to three independent matrices 2×2. Namely, let us transpose the columns and rows of the matrix $M(i\omega)$ in the order (142536). Then we get the matrix with three 2×2 blocks on the main diagonal. Therefore, the determinant of $M(i\omega)$ is product of the determinants of these three matrices. Further,

$$M^{-1}(i\omega)=\begin{pmatrix} \frac{-i\omega\gamma^3}{\omega^2\gamma^3+F_{11}} & 0 & 0 & \frac{-1}{\omega^2\gamma^3+F_{11}} & 0 & 0\\ 0 & \frac{-i\omega\gamma}{\omega^2\gamma+F_{22}} & 0 & 0 & \frac{-1}{\omega^2\gamma+F_{22}} & 0\\ 0 & 0 & \frac{-i\omega\gamma}{\omega^2\gamma+F_{33}} & 0 & 0 & \frac{-1}{\omega^2\gamma+F_{33}}\\ \frac{-\gamma F_{11}}{\omega^2\gamma^3+F_{11}} & 0 & 0 & \frac{-i\omega\gamma^3}{\omega^2\gamma^3+F_{11}} & 0 & 0\\ 0 & \frac{-\gamma F_{22}}{\omega^2\gamma+F_{22}} & 0 & 0 & \frac{-i\omega\gamma}{\omega^2\gamma+F_{22}} & 0\\ 0 & 0 & \frac{-\gamma F_{33}}{\omega^2\gamma+F_{33}} & 0 & 0 & \frac{-i\omega\gamma}{\omega^2\gamma+F_{33}} \end{pmatrix} \tag{B.29}$$

where $F_{jj}=F_{jj}(\omega)$. Let us prove that for $\omega\in(-\mu,\mu)$

$$F_{jj}(\omega)=\omega^2 f_{jj}(\omega),\quad f_{jj}(\omega)\in\mathbb{C}^\infty(-\mu,\mu)\ ,\quad f_{jj}(0)>0\ . \tag{B.30}$$

Indeed, formula (B.28) implies that $F_{jj}(0)=0$. Differentiating (B.28), we obtain

$$F'_{jj}(0)=2\int k_j^2\mathcal{B}(k)dk\frac{|v|k_1}{(k^2+m^2-(|v|k_1)^2)^2}=0$$

since integrand is odd function in respect to k_1, and

$$F''_{jj}(0)=2\int k_j^2\mathcal{B}(k)dk\frac{k^2+m^2+3(|v|k_1)^2}{(k^2+m^2-(|v|k_1)^2)^3}>0\ .$$

By (B.30) we can represent the matrices $M^{-1}(i\omega)$ as

$$M^{-1}(i\omega)=\begin{pmatrix}\frac{1}{\omega}\mathcal{M}_{11}(\omega) & \frac{1}{\omega^2}\mathcal{M}_{12}(\omega)\\[2ex] \mathcal{M}_{21}(\omega) & \frac{1}{\omega}\mathcal{M}_{22}(\omega)\end{pmatrix} \tag{B.31}$$

where

$$\mathcal{M}_{11}(\omega)=\mathcal{M}_{22}(\omega)=\begin{pmatrix}\frac{-i\gamma^3}{\gamma^3+f_{11}} & 0 & 0\\ 0 & \frac{-i\gamma}{\gamma+f_{22}} & 0\\ 0 & 0 & \frac{-i\gamma}{\gamma+f_{33}}\end{pmatrix}$$

$$\mathcal{M}_{12}(\omega)=\begin{pmatrix}\frac{-1}{\gamma^3+f_{11}} & 0 & 0\\ 0 & \frac{-1}{\gamma+f_{22}} & 0\\ 0 & 0 & \frac{-1}{\gamma+f_{33}}\end{pmatrix},\quad \mathcal{M}_{21}(\omega)=\begin{pmatrix}\frac{-\gamma^3 f_{11}}{\gamma^3+f_{11}} & 0 & 0\\ 0 & \frac{-\gamma f_{22}}{\gamma+f_{22}} & 0\\ 0 & 0 & \frac{-\gamma f_{33}}{\gamma+f_{33}}\end{pmatrix}$$

where $f_{jj}:=f_{jj}(\omega)$, $\mathcal{M}_{ij}(\omega)\in C^\infty(-\mu,\mu)$, and

$$\mathcal{M}_{11}=i\mathcal{M}_{12}B_\nu^{-1}\ . \tag{B.32}$$

C Symplectic orthogonality conditions

Here we derive conditions (15.18) from the symplectic orthogonality conditions (6.7). First let us compute $\Phi(0)$. Formulas (13.14) and (13.16) imply

$$(\Phi(0))_j=\langle\hat G_0^{11}\hat\Psi_{01}+\hat G_0^{12}\hat\Psi_{02},ik_j\hat\rho_1\rangle+\langle\hat G_0^{11}\hat\Psi_{02}-\hat G_0^{12}\hat\Psi_{01},ik_j\hat\rho_2\rangle\ ,\quad j=1,2,3\ .$$

On the other hand, by (13.5) formulas (A.22) read

$$\hat\psi_{v1}=-\hat G_0^{11}\hat\rho_2+\hat G_0^{12}\hat\rho_1,\quad \hat\psi_{v2}=\hat G_0^{11}\hat\rho_1+\hat G_0^{12}\hat\rho_2\ .$$

Hence, for $j=1,2,3$

$$\begin{aligned}0&=-\Omega(Z_0,\tau_j)=\langle\Psi_{01},\partial_j\psi_{v2}\rangle-\langle\Psi_{02},\partial_j\psi_{v1}\rangle+P_0\cdot e_j\\&=-\langle\Psi_{01},ik_j(\hat G_0^{11}\hat\rho_1+\hat G_0^{12}\hat\rho_2)\rangle+\langle\Psi_{02},ik_j(\hat G_0^{12}\hat\rho_1-\hat G_0^{11}\hat\rho_2)\rangle+P_0\cdot e_j=(\Phi(0)+P_0)_j\end{aligned}$$

since $(\hat G_0^{11})^*=-\hat G_0^{11}$, $(\hat G_0^{12})^*=\hat G_0^{12}$. Hence the first condition (15.18) follows. Further,

$$\partial_\lambda\hat G_\lambda^{11}\Big|_{\lambda=0}=\frac{-1-2iv\cdot k\hat G_0^{11}}{k^2+m^2-(v\cdot k)^2}\ ,\quad \partial_\lambda\hat G_\lambda^{12}\Big|_{\lambda=0}=\frac{-2iv\cdot k\hat G_0^{12}}{k^2+m^2-(v\cdot k)^2}\ .$$

Then (13.14) and (13.16) imply for $j=1,2,3$

$$\begin{aligned}(\Phi'(0))_j&=-\Big\langle\frac{\hat\Psi_{01}+2iv\cdot k(\hat G_0^{11}\hat\Psi_{01}+\hat G_0^{12}\hat\Psi_{02})}{k^2+m^2-(v\cdot k)^2},ik_j\hat\rho_1\Big\rangle\\&-\Big\langle\frac{\hat\Psi_{02}+2iv\cdot k(\hat G_0^{11}\hat\Psi_{02}-G_0^{12}\hat\Psi_{01})}{k^2+m^2-(v\cdot k)^2},ik_j\hat\rho_2\Big\rangle\end{aligned}$$

On the other hand, from (A.22) and (A.23) it follows that for $j=1,2,3$

$$\partial_{v_j}\hat\psi_{v1}=\frac{ik_j\hat\rho_2+2k_jv\cdot k(-\hat G_0^{11}\hat\rho_2+\hat G_0^{12}\hat\rho_1)}{k^2+m^2-(v\cdot k)^2},\quad \partial_{v_j}\hat\psi_{v2}=\frac{-ik_j\hat\rho_1+2k_jv\cdot k(\hat G_0^{11}\hat\rho_1+\hat G_0^{12}\hat\rho_2)}{k^2+m^2-(v\cdot k)^2}$$

Hence,

$$\begin{aligned}0&=\Omega(Z_0,\tau_{j+3})=\langle\Psi_{01},\partial_{v_j}\psi_{v2}\rangle-\langle\Psi_{02},\partial_{v_j}\psi_{v1}\rangle+Q_0\cdot\partial_{v_j}p_v\\&=\Big\langle\Psi_{01},\frac{-ik_j\hat\rho_1+2k_jv\cdot k(\hat G_0^{11}\hat\rho_1+\hat G_0^{12}\hat\rho_2)}{k^2+m^2-(v\cdot k)^2}\Big\rangle-\Big\langle\Psi_{02},\frac{ik_j\hat\rho_2+2k_jv\cdot k(\hat G_0^{12}\hat\rho_1-\hat G_0^{11}\hat\rho_2)}{k^2+m^2-(v\cdot k)^2}\Big\rangle\\&+Q_0\cdot\partial_{v_j}p_v=(\Phi'(0)+B_v^{-1}Q_0)_j,\quad j=1,2,3\end{aligned}$$

since $Q_0\cdot\partial_{v_j}p_v=Q_0\cdot B_v^{-1}e_j=B_v^{-1}Q_0\cdot e_j$. Hence the second condition (15.18) follows.

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