A NOTE ON CONSTRUCTING AFFINE SYSTEMS FOR L^2

H.-Q. BUI, N. KAIBLINGER AND R. S. LAUGESEN

ABSTRACT. Assume $\psi \in L^2(\mathbb{R}^d)$ has Fourier transform continuous at the origin, with $\widehat{\psi}(0) = 1$, and that $\sum_{l\in\mathbb{Z}^d} |\widehat{\psi}(\xi - l)|^2$ is bounded as a function of $\xi \in \mathbb{R}^d$.

Then every function $f \in L^2(\mathbb{R}^d)$ can be represented by an affine series $f = \sum_{j>0} \sum_{k\in\mathbb{Z}^d} c_{j,k}\psi_{j,k}$ for some coefficients satisfying

$$\|c\|_{\ell^1(\ell^2)} = \sum_{j>0} (\sum_{k\in\mathbb{Z}^d} |c_{j,k}|^2)^{1/2} < \infty.$$

Here $\psi_{j,k}(x) = |\det a_j|^{1/2}\psi(a_j x - k)$ and the dilation matrices a_j expand, for example $a_j = 2^j I$.

The result improves an observation by Daubechies that the linear combinations of the $\psi_{j,k}$ are dense in $L^2(\mathbb{R}^d)$.

1. Introduction and Main Result

Our goal is to prove that the *affine synthesis* operator

$$Sc = \sum_{j>0} \sum_{k\in\mathbb{Z}^d} c_{j,k}\psi_{j,k} \tag{1}$$

represents every function in $L^2(\mathbb{R}^d)$, under suitable conditions on ψ and on the coefficient sequence $c = \{c_{j,k}\}$.

Fix the *dimension* $d \in \mathbb{N}$, and choose invertible $d \times d$ real *dilation matrices* a_j that *expand*, meaning

$$\|a_j^{-1}\| \to 0 \qquad \text{as } j \to \infty. \tag{2}$$

(Here $\|\cdot\|$ denotes the operator norm of a matrix mapping the column vector space $\mathbb{R}^d$ to itself.) For example, one could take $a_j = 2^j I$, or $a_j = M^j$ for any matrix M whose eigenvalues all exceed 1 in magnitude.

Further choose an invertible *translation matrix* b, for example the identity matrix. For $\psi \in L^2 = L^2(\mathbb{R}^d)$ we define a scaled and translated version of ψ by

$$\psi_{j,k}(x) = |\det a_j|^{1/2}\psi(a_j x - bk), \qquad x \in \mathbb{R}^d.$$

Introduce a mixed norm

$$\|c\|_{\ell^1(\ell^2)} = \sum_{j>0} \Big(\sum_{k\in\mathbb{Z}^d} |c_{j,k}|^2\Big)^{1/2}$$

2000 *Mathematics Subject Classification.* Primary 42C40. Secondary 26B40,41A05,41A30.
Key words and phrases. Quasi-interpolation, multiresolution analysis, completeness.

on the doubly indexed sequences $c = \{c_{j,k}\}$, and hence define a Banach space $\ell^1(\ell^2) = \{c : \|c\|_{\ell^1(\ell^2)} < \infty\}$.

We claim every square integrable function can be represented as an infinite series of the $\psi_{j,k}$, provided the periodization of $|\widehat{\psi}|^2$ is bounded and $\widehat{\psi}(0) = 1$.

Theorem 1 (Synthesis onto L^2)**.** *Assume $\psi \in L^2$ has $\widehat{\psi}$ continuous at the origin, with $\widehat{\psi}(0) = 1$, and that $\sum_{l \in \mathbb{Z}^d} |\widehat{\psi}(\xi - lb^{-1})|^2$ is a bounded function of $\xi \in \mathbb{R}^d$.*

Then $S : \ell^1(\ell^2) \to L^2$ is linear, bounded, open, and surjective.

Note that $l \in \mathbb{Z}^d$ and $\xi \in \mathbb{R}^d$ are regarded as row vectors wherever they occur in the paper, and that we take the Fourier transform $\widehat{\psi}(\xi) = \int_{\mathbb{R}^d} \psi(x) e^{-2\pi i \xi x}\, dx$ to have 2π in the exponent.

Section 2 has the proof, which employs weakly convergent quasi-interpolation and the Banach–Saks–Mazur theorem. The double series defining Sc in (1) will be shown to converge unconditionally in L^2. Regarding the surjectivity of synthesis, we show more precisely that if $f \in L^2$ and $\varepsilon > 0$, then a sequence $c \in \ell^1(\ell^2)$ exists with $Sc = f$ and

$$\|c\|_{\ell^1(\ell^2)} \le |\det b|^{1/2} \|f\|_2 + \varepsilon.$$

Motivation. The multiresolution analysis (MRA) generated by a scaling function ψ requires for its completeness that the closed linear span of the $\psi_{j,k}$ should equal all of L^2, or $\overline{\cup_j V_j} = L^2$ in standard MRA notation [8, 15]. Daubechies found conditions that guarantee this MRA completeness [8, Proposition 5.3.2], namely, the conditions assumed in Theorem 1. Her completeness conclusion means the finite linear combinations of the $\psi_{j,k}$ are dense in L^2, so that the synthesis operator S has dense range in L^2.

The point of Theorem 1 is to identify the domain $\ell^1(\ell^2)$ on which synthesis S acts, and to improve the conclusion from dense range to full range. Further, our proof will proceed by explicit construction, in contrast to the duality (Hahn–Banach) methods by which completeness is derived in [8].

Notice neither the Daubechies result nor our Theorem 1 require ψ to satisfy any kind of scaling or refinement relation. Strang–Fix conditions are also not imposed. Thus the assumptions in Theorem 1 are rather weak.

The bounded Gramian hypothesis. We call

$$G_\psi(\xi) = |\det b|^{-1} \sum_{l \in \mathbb{Z}^d} |\widehat{\psi}(\xi - lb^{-1})|^2$$

the *Gramian* of ψ. More precisely, $G_\psi(\xi)$ is the norm of the dual Gramian at ξ, in the terminology of Ron and Shen [17].

The hypothesis that the Gramian is bounded, in Theorem 1, seems natural. As is well known, if $G_\psi \in L^\infty$ then the translates $\psi(\cdot - bk)$ form a *Bessel sequence*, meaning the corresponding analysis operator is bounded from L^2 to ℓ^2:

$$\sum_{k \in \mathbb{Z}^d} |\langle f, \psi(\cdot - bk)\rangle|^2 \le (\text{constant}) \|f\|_2^2, \qquad f \in L^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product. See Lemma 2 below. The converse implication holds also.

If $|\psi|$ has periodization in L^2 (meaning that $\sum_{k\in\mathbb{Z}^d}|\psi(x-bk)| \in L^2_{loc}$) then ψ has bounded Gramian, as follows. Take $b = I$ for simplicity. The Gramian is an integrable periodic function of ξ, with kth Fourier coefficient equal to $g_k = \int_{\mathbb{R}^d} \psi(x-k)\overline{\psi(x)}\,dx$ (see [7, Lemma 7.3.3]). Hence $\sum_{k\in\mathbb{Z}^d}|g_k| \leq \int_{\mathbb{T}^d}(\sum_{k\in\mathbb{Z}^d}|\psi(x-k)|)^2\,dx < \infty$, by periodizing the integral. Thus the Fourier coefficients of the Gramian are absolutely summable, so that the Gramian is essentially bounded.

Hence if $\psi \in L^2$ has compact support or decays fast enough near infinity, then the Gramian is bounded.

Note boundedness of the Gramian does not imply continuity of $\widehat{\psi}$. If, however, one assumes $|\psi|$ has periodization in L^2, and hence in L^1, then $\psi \in L^1$ and so $\widehat{\psi}$ is continuous.

In the reverse direction, if $0 \leq \psi \in L^1 \cap L^2$ and the Gramian of ψ is bounded and is Dini continuous at the origin, then the periodization of ψ belongs to L^2. We omit the proof.

Non-injectivity of synthesis. The synthesis operator is not injective. For example, we could discard the dilation a_1 (in other words, discard all terms with $j = 1$ in the sum defining Sc) and still S would map onto L^2, by applying Theorem 1 with the remaining dilations $a_2, a_3, a_4, \ldots$.

Literature on surjective synthesis and completeness. Surjectivity of affine synthesis from $\ell^1(\ell^p)$ onto L^p was proved already by Terekhin [19, 20] for each $1 \leq p < \infty$, using duality methods based on the complete representation systems of Filippov and Oswald [10, 11]. Bui and Laugesen [6] gave a direct, explicit proof of surjectivity.

These authors assume $|\psi|$ has periodization in L^p, which is stronger for $p = 2$ than the bounded Gramian hypothesis in Theorem 1 (as remarked above). This stronger assumption is needed because the authors work in the spatial domain. In contrast, our results for L^2 will be proved in the frequency domain.

Synthesis onto L^p for $p < 1$ has been treated by Laugesen [14].

Prior work on L^2-completeness of linear combinations of the $\psi_{j,k}$ includes the result of Daubechies described above, and a result of de Boor *et al.* [2, Theorem 4.5] that weakens the continuity assumption to just $\widehat{\psi} \neq 0$ on some neighborhood of the origin but assumes in addition that ψ satisfies a refinement equation. Bruna [4, p. 81] takes a different approach for L^p, applying duality methods to rather general systems of translates and dilates.

Frames. The $\{\psi_{j,k}\}$ in Theorem 1 do not form a wavelet *frame* (which is a more general notion than an orthonormal basis), because $\widehat{\psi}(0) \neq 0$: it is known that a wavelet frame generator must have a vanishing moment. Another obstacle is that the $\psi_{j,k}$ form a frame if and only if the synthesis operator is bounded on the space $\ell^2(\ell^2)$ and maps it onto L^2 (see [7, Theorem 5.5.1]), whereas Theorem 1 guarantees bounded synthesis only on the smaller space $\ell^1(\ell^2)$. This smaller allowable domain reflects the lack of any cancellation between the $\psi_{j,k}$ at different scales.

Conclusion. This paper presents what we believe to be the "right" surjective synthesis result for affine systems in L^2, when the synthesizer ψ satisfies $\widehat{\psi}(0) \neq 0$. In particular, the bounded Gramian assumption in Theorem 1 seems the natural hypothesis in L^2, because it is equivalent to the b-translates of ψ forming a Bessel system.

2. Synthesis $\ell^2 \to L^2$, Analysis $L^2 \to \ell^2$, and Quasi-interpolation

Here we establish preliminary results needed in the next section for the proof of Theorem 1.

Initially we synthesize at a fixed scale j by writing

$$S_j s = \sum_{k \in \mathbb{Z}^d} s_k \psi_{j,k}$$

for sequences $s = \{s_k\}_{k \in \mathbb{Z}^d}$. We also need the analysis operator at scale j, which maps a function f to its sequence of sampled ϕ-averages at scale j:

$$T_j f = \{|\det b| \langle f, \phi_{j,k} \rangle\}_{k \in \mathbb{Z}^d}$$

where the *analyzer* ϕ belongs to L^2.

Lemma 2 (Synthesis into L^2, and Analysis into ℓ^2)**.** *Assume $\psi, \phi \in L^2$ with Gramians G_ψ, G_ϕ that are bounded.*

Then $S_j : \ell^2 \to L^2$ is linear and bounded with norm $\|G_\psi\|_\infty^{1/2}$, and $T_j : L^2 \to \ell^2$ is linear and bounded with norm $|\det b| \, \|G_\phi\|_\infty^{1/2}$.

The lemma is well known (see the proof of [7, Theorem 7.2.3]). In brief, one writes $\mathbb{T}^d = [-1/2, 1/2]^d$ for the unit cube or torus, so that

$$\|S_j s\|_2^2 = \int_{\mathbb{T}^d} \Big| \sum_{k \in \mathbb{Z}^d} s_k e^{-2\pi i \xi k} \Big|^2 G_\psi(\xi b^{-1}) \, d\xi \leq \|s\|_{\ell^2}^2 \, \|G_\psi\|_\infty$$

by Plancherel's identity. Hence the series for $S_j s$ converges unconditionally in L^2, with $\|S_j\| \leq \|G_\psi\|_\infty^{1/2}$. Equality holds here, as one finds by choosing a suitable s. The bound on T_j follows immediately, since analysis is adjoint to synthesis (apart from the constant factor of $|\det b|$).

In the next result, we say the dilations *expand super-linearly* if they are expanding ($\|a_j^{-1}\| \to 0$ as $j \to \infty$) and also satisfy

$$\lim_{j \to \infty} \inf_{n > m > j} |\gamma a_n - \gamma a_m| = \infty \qquad \text{for all } \gamma \in \mathbb{R}^d \setminus \{0\}. \tag{3}$$

This condition says, roughly, that the points in each orbit become farther and farther apart. Easy examples of super-linearly expanding dilations are $a_j = j^p I$ for a fixed $p > 1$, and $a_j = \alpha^j I$ for fixed $\alpha > 1$. A harder example is $a_j = M^j$ when all the eigenvalues of M have magnitude greater than 1, as one justifies using the Jordan form of M.

Proposition 3. *Assume $\psi, \phi \in L^2$ have Gramians G_ψ, G_ϕ that are bounded, and suppose $\widehat{\psi}, \widehat{\phi}$ are continuous at the origin with $\widehat{\psi}(0) = \widehat{\phi}(0) = 1$. Let $f \in L^2$. Then:*
(a) [Weakly convergent quasi-interpolation] $S_j T_j f \rightharpoonup f$ *weakly in L^2, as $j \to \infty$.*
(b) [Norm convergent quasi-interpolation with scale-averaging] *If the dilations a_j expand super-linearly, and the Gramian series $\sum_{\ell \in \mathbb{Z}^d} |\widehat{\psi}(\cdot - \ell b^{-1})|^2$ converges in the norm of $L^\infty(\mathbb{T}^d b^{-1})$, then*

$$\frac{1}{J} \sum_{j=1}^{J} S_j T_j f \to f \qquad \text{in } L^2, \ \text{ as } J \to \infty. \tag{4}$$

The weak quasi-interpolation in part (a) is the key to proving surjectivity of the synthesis operator, in Theorem 1. Part (b), which shows how to improve weak convergence to norm convergence by averaging over dilation scales in a manner that is independent of the signal f, will not be needed later but is interesting also.

Explicitly, part (a) says $|\det b| \sum_{k \in \mathbb{Z}^d} \langle f, \phi_{j,k} \rangle \psi_{j,k} \rightharpoonup f$ weakly in L^2, as $j \to \infty$. This result is known in all L^p spaces under the stronger assumption that $|\psi|$ has periodization in L^p (see [5, Theorem 1]).

Norm convergence holds in part (a) if in addition ψ satisfies the Strang–Fix condition $\widehat{\psi}(lb^{-1}) = 0, l \in \mathbb{Z}^d \setminus \{0\}$; see [5, §3.1] and [12] for discussion of this literature, especially noting the complete characterization of L^2 approximation rates achieved by de Boor, DeVore and Ron [3], and the treatment of the quasi-interpolation case by Jetter and Zhou [13].

Part (b) of the proposition says that regardless of Strang–Fix conditions, the weak convergence can be improved to norm convergence by averaging over dilation scales $j = 1, \ldots, J$. This improvement by scale-averaging was established previously in [5, Theorem 1] for all L^p, under the stronger assumption that $|\psi|$ has periodization in L^p.

The hypothesis in part (b) that the Gramian series should converge in L^∞ is satisfied whenever $\widehat{\psi}$ is bounded and decays fast enough near infinity.

Proof of Proposition 3. Part (a). For the weak convergence $S_j T_j f \rightharpoonup f$ we want to show for each $g \in L^2$ that

$$\langle S_j T_j f, g \rangle \to \langle f, g \rangle \qquad \text{as } j \to \infty. \tag{5}$$

We can assume the translation matrix b equals the identity, by suitably rescaling ψ, ϕ, f, g and the dilations a_j, as follows. Rescale ψ to $\Psi(x) = |\det b| \psi(bx)$ and f to $F(x) = |\det b|^{1/2} f(bx)$, and similarly rescale ϕ and g. Then prove (5) for Ψ, Φ, F, G using dilation matrices $A_j = b^{-1} a_j b$ and translation matrix $B = I$. Undoing the rescaling then yields (5) in its original form. Thus we can suppose $b = I$.

Next, by boundedness of S_j and T_j (in Lemma 2) it suffices to show the desired convergence (5) for a dense subset of $f, g \in L^2$. Thus we can suppose from now on that f and g are bandlimited, meaning $\widehat{f}$ and $\widehat{g}$ are compactly supported.

We may further suppose $\widehat{\psi}$ and $\widehat{\phi}$ are identically equal to 1 on some neighborhood of the origin, as we now explain. The continuity of $\widehat{\psi}$ at the origin and the normalization

$\widehat{\psi}(0) = 1$ ensure that for each $\varepsilon > 0$ there exists a ball around the origin of radius at most $1/2$ on which $|\widehat{\psi} - 1| \leq \varepsilon$. Define $\widehat{\eta}$ to equal 1 on that neighborhood and to equal $\widehat{\psi}$ everywhere else. Then $\widehat{\eta} \in L^2$ and $G_{\psi-\eta} \leq \varepsilon^2$. Notice the difference between S_j using synthesizer ψ and S_j using synthesizer η is precisely S_j using synthesizer $\psi - \eta$, and this last operator has norm $\|G_{\psi-\eta}\|_\infty^{1/2} \leq \varepsilon$. We deduce it suffices to prove the convergence (5) for η instead of ψ. That is, we can suppose $\widehat{\psi} = 1$ on some neighborhood U of the origin, with $U \subset \mathbb{T}^d = [-1/2, 1/2]^d$. Argue similarly for $\widehat{\phi}$.

Suppose for the remainder of the proof that j is so large we have $\operatorname{supp}(\widehat{f})a_j^{-1} \subset U$, which can be achieved because $\|a_j^{-1}\| \to 0$ by hypothesis (2). Then $\widehat{\phi}(\xi a_j^{-1}) = 1$ for all $\xi \in \operatorname{supp}(\widehat{f})$, so that

$$\widehat{f}(\xi)\overline{\widehat{\phi_{j,k}}(\xi)} = \widehat{f}(\xi)|\det a_j|^{-1/2} e^{2\pi i \xi a_j^{-1} k}, \qquad \xi \in \mathbb{R}^d.$$

After integration we deduce

$$\langle f, \phi_{j,k} \rangle = \langle \widehat{f}, \widehat{\phi_{j,k}} \rangle = \widehat{F_j}(-k),$$

where the function $F_j(\xi) = \widehat{f}(\xi a_j)|\det a_j|^{1/2}$ is supported in U and hence in the unit cube $\mathbb{T}^d$ centered at the origin, and $\widehat{F_j}(\cdot)$ denotes the Fourier coefficients of F_j. Similarly $\langle g, \psi_{j,k} \rangle = \widehat{G_j}(-k)$, where $G_j(\xi) = \widehat{g}(\xi a_j)|\det a_j|^{1/2}$.

Now we compute that

$$\begin{aligned}
\langle S_j T_j f, g \rangle &= \sum_{k \in \mathbb{Z}^d} \langle f, \phi_{j,k} \rangle \langle \psi_{j,k}, g \rangle \\
&= \sum_{k \in \mathbb{Z}^d} \widehat{F_j}(k) \overline{\widehat{G_j}(k)} \\
&= \int_{\mathbb{T}^d} F_j(\xi) \overline{G_j(\xi)} \, d\xi \\
&= \langle \widehat{f}, \widehat{g} \rangle = \langle f, g \rangle,
\end{aligned}$$

which proves the weak convergence (5).

Aside. We have essentially invoked the Whittaker–Shannon sampling theorem (for which see Unser's survey [21]) after first reducing to $\widehat{\psi}$ and $\widehat{\phi}$ equalling 1 near the origin. The same idea has been used previously for norm convergent quasi-interpolation, when Strang–Fix conditions are satisfied [9, p. 113].

Part (b). By rescaling like in part (a) we can assume that $b = I$, $\widehat{f}$ is compactly supported, and $\widehat{\psi} = \widehat{\phi} = 1$ on some neighborhood of the origin. Note the dilations still expand super-linearly after rescaling, because $\gamma A_n - \gamma A_m = [(\gamma b^{-1})a_n - (\gamma b^{-1})a_m]b$, recalling here that $A_j = b^{-1} a_j b$.

Consider $l \in \mathbb{Z}^d \setminus \{0\}$. Then the super-linear expansion property (3) implies that the functions $\widehat{f}(\xi - la_j), j > 0$, have disjoint supports for all large j. Hence

$$\frac{1}{J}\sum_{j=1}^{J}|\widehat{f}(\xi - la_j)| \to 0 \qquad \text{in } L^2\text{, as } J \to \infty, \tag{6}$$

because the L^2 norm of the left hand side decays like $J^{-1/2}$ as $J \to \infty$.

The next step is to reduce to $\widehat{\psi}$ being compactly supported, as follows. The series for the Gramian of ψ converges in $L^\infty(\mathbb{T}^d)$, by hypothesis, and so given $\varepsilon > 0$ there exists $L > 0$ such that

$$\operatorname*{ess\,sup}_{\xi\in\mathbb{T}^d} \sum_{|l|>L} |\widehat{\psi}(\xi - l)|^2 \leq \varepsilon^2. \tag{7}$$

Define

$$\widehat{\mu}(\xi) = \begin{cases} \widehat{\psi}(\xi) & \text{if } \xi \in \bigcup_{|l|\leq L}(\mathbb{T}^d - l), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\widehat{\mu} \in L^2$ with $\widehat{\mu}$ continuous near the origin and $\widehat{\mu}(0) = 1$, and $G_{\psi-\mu} \leq \varepsilon^2$ by (7). The synthesis operator S_j using synthesizer $\psi - \mu$ has norm $\|G_{\psi-\mu}\|_\infty^{1/2} \leq \varepsilon$, by Lemma 2. We deduce it suffices to prove (4) for μ instead of ψ. That is, we can suppose $\widehat{\psi}$ has compact support.

We find

$$(S_jT_jf)\widehat{\ }(\xi) = \widehat{\psi}(\xi a_j^{-1}) \sum_{l\in\mathbb{Z}^d} \overline{\widehat{\phi}(\xi a_j^{-1} - l)}\widehat{f}(\xi - la_j) \tag{8}$$

by taking the Fourier transform of $S_jT_jf = \sum_{k\in\mathbb{Z}^d}\langle f, \phi_{j,k}\rangle\psi_{j,k}$ and invoking Lemma 4 below. Only finitely many l-values contribute to the sum in (8), as we now explain. Take $R > 0$ large enough that $\widehat{\psi}$ and $\widehat{f}$ are supported in the closed ball $B_0(R)$ of radius R centered at the origin. We claim that for all sufficiently large j, if $|l| > R$ then

$$\widehat{\psi}(\xi a_j^{-1})\widehat{f}(\xi - la_j) = 0, \qquad \xi \in \mathbb{R}^d.$$

Indeed, for ξ to lie in the support of this product one must have

$$\begin{aligned} \xi &\in \left[\operatorname{supp}(\widehat{\psi}) \cap (l + \operatorname{supp}(\widehat{f})a_j^{-1})\right]a_j \\ &\subset \left[B_0(R) \cap B_l(R\|a_j^{-1}\|)\right]a_j, \end{aligned}$$

which equals the empty set for all large j, since $\|a_j^{-1}\| \to 0$ and $|l| > R$.

After summing (8) from $j = 1$ to $j = J$ and then dividing by J, we see the desired limit (4) will follow once we prove

$$\frac{1}{J}\sum_{j=1}^{J} \widehat{\psi}(\xi a_j^{-1})\overline{\widehat{\phi}(\xi a_j^{-1} - l)}\widehat{f}(\xi - la_j) \to \begin{cases} \widehat{f}(\xi) & \text{if } l = 0, \\ 0 & \text{if } l \neq 0, \end{cases} \tag{9}$$

with convergence in L^2, as $J \to \infty$. (Here we use that the sum over l in (8) has only finitely many terms, $|l| \leq R$, so that we may pass the limit with respect to J inside the sum over l.)

The $l = 0$ case of (9) is immediate, because $\widehat{\psi}(\xi a_j^{-1}) = \widehat{\phi}(\xi a_j^{-1}) = 1$ for all $\xi \in \mathrm{supp}(\widehat{f})$, for all large j, by our assumption that $\widehat{\psi}$ and $\widehat{\phi}$ equal 1 near the origin. The $l \neq 0$ case of (9) follows from (6), since $\widehat{\psi}$ and $\widehat{\phi}$ are bounded (recalling the Gramians are bounded by hypothesis). □

It remains to prove the well known lemma used above.

Lemma 4. *Assume $\phi \in L^2$, $f \in L^2$ and $b = I$. Suppose the Gramian $\sum_{l\in\mathbb{Z}^d} |\widehat{\phi}(\xi - l)|^2$ is bounded for $\xi \in \mathbb{R}^d$. Then for each $j > 0$,*

$$|\det a_j|^{-1/2} \sum_{k\in\mathbb{Z}^d} \langle f, \phi_{j,k}\rangle e^{-2\pi i\xi k} = \sum_{l\in\mathbb{Z}^d} \widehat{f}((\xi - l)a_j)\overline{\widehat{\phi}(\xi - l)} \tag{10}$$

where the series on the left converges in $L^2(\mathbb{T}^d)$, and the series on the right converges absolutely almost everywhere to a function in $L^2(\mathbb{T}^d)$.

Proof of Lemma 4. The series on the right of (10) converges absolutely almost everywhere to a function in $L^2(\mathbb{T}^d)$, by applying Cauchy–Schwarz and the bounded Gramian assumption. The $(-k)$th Fourier coefficient of this function is

$$\begin{aligned}\int_{\mathbb{T}^d} \sum_{l\in\mathbb{Z}^d} \widehat{f}((\xi - l)a_j)\overline{\widehat{\phi}(\xi - l)}\, e^{2\pi i\xi k}\, d\xi &= \int_{\mathbb{R}^d} \widehat{f}(\xi a_j)\overline{\widehat{\phi}(\xi)e^{-2\pi i\xi k}}\, d\xi \\ &= |\det a_j|^{-1/2}\langle f, \phi_{j,k}\rangle\end{aligned}$$

by Parseval, completing the proof. □

3. Proof of Theorem 1 — Synthesis onto L^2

For boundedness of the full synthesis operator $S : \ell^1(\ell^2) \to L^2$, write $Sc = \sum_{j>0} S_j c_j$ where $c_j = \{c_{j,k}\}_{k\in\mathbb{Z}^d}$ gives the jth "level" of c. This sum $Sc = \sum_{j>0} S_j c_j$ converges absolutely in L^2, because $\sum_{j>0} \|c_j\|_{\ell^2} = \|c\|_{\ell^1(\ell^2)} < \infty$ and S_j has norm $\|G_\psi\|_\infty^{1/2}$ independently of j, by Lemma 2. Therefore $\|S\| = \|G_\psi\|_\infty^{1/2}$, and the sum over j and k that defines Sc converges unconditionally.

Linearity of S is obvious.

We must still prove openness and surjectivity. Take ϕ to be any nice function satisfying the hypotheses of Proposition 3 and with $\|G_\phi\|_\infty = |\det b|^{-1}$; for example, one could take $\widehat{\phi}$ to be smooth, nonnegative and compactly supported in a small neighborhood of the origin, with $\widehat{\phi}(0) = 1$ being its maximum value.

Consider $f \in L^2$. For each $m \in \mathbb{N}$, define a sequence c_m by analyzing f at level m:

$$c_{m;j,k} = \begin{cases} (T_j f)_k & \text{if } j = m, k \in \mathbb{Z}^d, \\ 0 & \text{if } j \neq m, k \in \mathbb{Z}^d. \end{cases}$$

Notice $c_m \in \ell^1(\ell^2)$, because $T_j f \in \ell^2$ by Lemma 2. We have

$$Sc_m = S_m T_m f \rightharpoonup f \qquad \text{weakly in } L^2\text{, as } m \to \infty,$$

by Proposition 3(a). This weak convergence can be improved to norm convergence by taking suitable convex means with respect to m: indeed the Banach–Saks theorem [16, p. 80] provides a subsequence $m_1 < m_2 < m_3 < \dots$ for which the nth Cesàro mean $d_n = (c_{m_1} + \cdots + c_{m_n})/n$ converges in norm, meaning

$$Sd_n \to f \qquad \text{in } L^2, \text{ as } n \to \infty.$$

Note that

$$\|d_n\|_{\ell^1(\ell^2)} \le \frac{1}{n}\big(\|T_{m_1}f\|_{\ell^2} + \cdots + \|T_{m_n}f\|_{\ell^2}\big) \le |\det b|^{1/2}\|f\|_2$$

by Lemma 2, since $\|G_\phi\|_\infty = |\det b|^{-1}$.

The above construction shows that if $\|f\|_2 < |\det b|^{-1/2}$ then $\{d_n\}$ is a sequence in the open unit ball of $\ell^1(\ell^2)$ satisfying $Sd_n \to f$. Thus the open ball in L^2 of radius $\delta \overset{\text{def}}{=} |\det b|^{-1/2}$ is contained in the closure of the S-image of the open unit ball in $\ell^1(\ell^2)$.

The open mapping theorem (for example [18, Theorem 4.13(b)⇒(c)]) now yields the stronger conclusion that the open ball in L^2 of radius δ is contained in the S-image of the open unit ball in $\ell^1(\ell^2)$. Hence $S : \ell^1(\ell^2) \to L^2$ is open, and therefore surjective. It follows also that for each $f \in L^2$ and $\varepsilon > 0$ there exists $c \in \ell^1(\ell^2)$ with $Sc = f$ and $\|c\|_{\ell^1(\ell^2)} \le \delta^{-1}\|f\|_2 + \varepsilon$.

Remarks. 1. The Banach–Saks result for L^p spaces [1] predates the more general theorem of Mazur [18, Theorem 3.13] for normed spaces. Note that above we refer only to a proof of the L^2 case of the Banach–Saks theorem.

2. If ψ and the dilations a_j satisfy the stronger hypotheses of part (b) in Proposition 3, then the use of a signal-dependent subsequence is unnecessary, in the above proof, and one may simply average over the full sequence of dilations.

Acknowledgments

This research began at meetings in Strobl, Austria, organized in May 2005 and June 2007 by the Numerical Harmonic Analysis Group at the University of Vienna. Laugesen gratefully acknowledges support from the Scholars' Travel Fund, University of Illinois. H.-Q. Bui acknowledges the hospitality of the Erwin Schrödinger International Institute, Vienna, during the Special Semester on "Modern Methods of Time-Frequency Analysis" in 2005.

We thank M. B. Erdogan for enlightening discussions on the bounded Gramian condition G_ψ and its relation to the periodization of $|\psi|$ belonging to L^2.

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Department of Mathematics, University of Canterbury, Christchurch 8020, New Zealand

E-mail address: Q.Bui@math.canterbury.ac.nz

Faculty of Mathematics, University of Vienna, Nordbergstrasse 15, 1090 Vienna, Austria

E-mail address: Norbert.Kaiblinger@univie.ac.at

Department of Mathematics, University of Illinois, Urbana, IL 61801, U.S.A.

E-mail address: Laugesen@uiuc.edu