The Quarterly Journal of Mathematics Advance Access published on December 6, 2007
Quart. J. Math. 59 (2008), 15–28; doi:10.1093/qmath/ham023

METAPLECTIC OPERATORS ON $\mathbb{C}^n$

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[Received 6 February 2006; revised 30 April 2007]

Abstract

The metaplectic representation describes a class of automorphisms of the Heisenberg group $H = H(G)$, defined for a locally compact abelian group G. For $G = \mathbb{R}^d$, H is the usual Heisenberg group. For the case when G is the finite cyclic group $\mathbb{Z}_n$, only partial constructions are known. Here we present new results for this case and we obtain an explicit construction of the metaplectic operators on $\mathbb{C}^n$. We also include applications to Gabor frames.

1. Introduction

The metaplectic representation, also called oscillator representation or Segal–Shale–Weil representation, is concerned with a class of automorphisms of the Heisenberg group $H = H(G)$, where G is a locally compact abelian group [23, 27]. The important case $G = \mathbb{R}^d$, where H is the usual Heisenberg group, has been investigated by numerous researchers; we refer to [11, 19, 22]. Here we describe the metaplectic representation for the case when $G = \mathbb{Z}_n$, the finite cyclic group of order n. In this case the metaplectic transformations are operators on vectors in $\mathbb{C}^n$. For developments in this and related directions, see [1, 2, 4, 14, 20, 21, 25], and [26] with its extensive list of references. The important new features of our approach are as follows.

- Our approach works for general n, not limited to the case when $\mathbb{Z}_n$ is a field.
- The results found in the literature provide explicit formulae only for special cases; we present a general and explicit construction.
- We use chirp functions defined for general n (even or odd); see Section 2.

We consider vectors in $\mathbb{C}^n$ as functions on the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$. In particular, all computations of the indices are modulo n. The (cyclic) translation and modulation

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operators on $\mathbb{C}^n$ are defined by

$$T_k v(m) = v(m-k), \quad k \in \mathbb{Z}_n,$$
$$M_l v(m) = e^{2\pi \mathrm{i} lm/n} v(m), \quad l \in \mathbb{Z}_n, \qquad v \in \mathbb{C}^n.$$

Their combination, called time-frequency shift operator, is denoted by

$$\pi(\lambda) = T_k M_l, \qquad \lambda = (k,l) \in \mathbb{Z}_n \times \mathbb{Z}_n.$$

Let $\mathbb{T} = \{\tau \in \mathbb{C} : abs \tau = 1\}$ denote the circle group. The commutation relation

$$M_l T_k = \mathrm{e}^{2\pi \mathrm{i} kl/n} T_k M_l, \qquad k,l \in \mathbb{Z}_n, \tag{1}$$

entails that the composition of time-frequency shift operators is given by

$$\pi(\lambda)\pi(\lambda') = \mathrm{e}^{2\pi \mathrm{i}\langle \lambda, \kappa\lambda'\rangle/n} \pi(\lambda+\lambda'), \qquad \text{where } \kappa = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and the inverse is $\pi(\lambda)^{-1} = \mathrm{e}^{2\pi \mathrm{i}\langle \lambda, \kappa\lambda\rangle/n} \pi(-\lambda)$. The family of operators

$$H_n := \{\tau\pi(\lambda) : la \in \mathbb{Z}_n \times \mathbb{Z}_n, \tau \in \mathbb{T}\} = \{\tau T_k M_l : k,l \in \mathbb{Z}_n, \tau \in \mathbb{T}\}$$

is thus a group under composition, the (reduced) Heisenberg group $H_n = H(\mathbb{Z}_n)$ over $\mathbb{Z}_n$. The Heisenberg group on $\mathbb{Z}_n$ is used, for example, for finite approximations of continuous-time systems in quantum theory and time-frequency analysis [9, 17]. The metaplectic representation is concerned with a class of automorphisms of H_n.

By $\mathrm{M}_{2,2}(\mathbb{Z}_n)$ we denote the set of 2×2 matrices with elements in $\mathbb{Z}_n$. The special linear group $\mathrm{SL}_2(\mathbb{Z}_n)$ consists of all matrices in $\mathrm{M}_{2,2}(\mathbb{Z}_n)$ with determinant 1 (computed modulo n). We call A symmetric if $A = A^{\mathrm{T}}$, where A^{T} is the transpose of A.

DEFINITION 1.1 (i) Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_{2,2}(\mathbb{Z}_n)$, we define $\sigma = \sigma_A \in \mathrm{M}_{2,2}(\mathbb{Z}_n)$ by

$$\sigma_A := A^{\mathrm{T}} \kappa A - \kappa = \begin{pmatrix} ac & bc \\ ad-1 & bd \end{pmatrix} \tag{2}$$

with κ given above. Observe that σ_A is symmetric if and only if $A \in \mathrm{SL}_2(\mathbb{Z}_n)$; cf. Section 6.

(ii) Let $\sigma \in \mathrm{M}_{2,2}(\mathbb{Z}_n)$ be symmetric. A function $\psi : ZZnt \to \mathbb{T}$ that satisfies

$$\psi(\lambda+\lambda') = \psi(\lambda)\psi(\lambda')\mathrm{e}^{2\pi \mathrm{i}\langle \lambda, \sigma\lambda'\rangle/n}, \qquad \lambda, \lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n, \tag{3}$$

is called a second-degree character associated to σ; see Section 2.

We are concerned with automorphisms of H_n that are based on the transformation

$$\pi(\lambda) \mapsto \pi(A\lambda), \quad \lambda \in \mathbb{Z}_n \times \mathbb{Z}_n,$$

for some $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_{2,2}(\mathbb{Z}_n)$, that is, on the modification of time-frequency shift parameters $(k, l) \mapsto (ak + bl, ck + dl)$. The next result is the concrete version for H_n of what is known in a more general framework [27], see also [23, Section 4.1].

Proposition 1.2 *Given* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_{2,2}(\mathbb{Z}_n)$*, suppose that the mapping*

$$\tau\pi(\lambda) \mapsto \tau\psi(\lambda)\pi(A\lambda), \quad \tau \in \mathbb{T}, \ \lambda \in \mathbb{Z}_n \times \mathbb{Z}_n,$$

that is,

$$\tau T_k M_l \mapsto \tau\psi(k, l) T_{ak+bl} M_{ck+dl}, \quad \tau \in \mathbb{T}, \ k, l \in \mathbb{Z}_n,$$

is an automorphism of H_n*, for some assignment* $\psi : ZZnt \to \mathbb{T}$*. Then* A *belongs to* $\mathrm{SL}_2(\mathbb{Z}_n)$ *and* ψ *is a second-degree character associated to* σ_A*.*

Proof. Since the given mapping is a homomorphism, the image of the operator product

$$\pi(\lambda)\pi(\lambda') = \mathrm{e}^{2\pi \mathrm{i}\langle\lambda,\kappa\lambda'\rangle/n}\pi(\lambda + \lambda') \mapsto \mathrm{e}^{2\pi \mathrm{i}\langle\lambda,\kappa\lambda'\rangle/n}\psi(\lambda + \lambda')\pi(A\lambda + A\lambda')$$

must coincide with the composition of the images

$$\psi(\lambda)\pi(A\lambda)\psi(\lambda')\pi(A\lambda') = \mathrm{e}^{2\pi \mathrm{i}\langle A\lambda,\kappa A\lambda'\rangle/n}\psi(\lambda)\psi(\lambda')\pi(A\lambda + A\lambda').$$

Therefore ψ satisfies

$$\frac{\psi(\lambda + \lambda')}{\psi(\lambda)\psi(\lambda')} = \mathrm{e}^{2\pi \mathrm{i}\langle A\lambda,\kappa A\lambda'\rangle/n}\mathrm{e}^{-2\pi \mathrm{i}\langle\lambda,\kappa\lambda'\rangle/n}, \quad \lambda, \lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n, \tag{4}$$

that is, (3) holds with $\sigma = \sigma_A$ given in (2). In particular, σ is necessarily symmetric, since the left-hand side of (4) is invariant when λ and λ' are interchanged. As observed above, if σ is symmetric it implies $A \in \mathrm{SL}_2(\mathbb{Z}_n)$.

Remark 1.3 The proposition describes a class of automorphisms of H_n that is important for many applications, such as the modification of Gabor frames described in Section 5. It is readily shown that this class precisely consists of those automorphisms that leave multiples of the identity operator invariant, that is, that keep the centre $\{\tau \,\mathrm{id} : tau \in \mathbb{T}\}$ of H_n pointwise fixed.

Metaplectic operators on $\mathbb{C}^n$ are unitary operators U that intertwine the automorphisms of the Heisenberg group H_n described in Proposition 1.2. That is, they satisfy

$$U\pi(\lambda)U^{-1} = \psi(\lambda)\pi(A\lambda), \quad \lambda \in \mathbb{Z}_n \times \mathbb{Z}_n,$$

that is,

$$U T_k M_l U^{-1} = \psi(k, l) T_{ak+bl} M_{ck+dl}, \quad k, l \in \mathbb{Z}_n, \tag{5}$$

for given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_n)$ and a second-degree character ψ associated to $\sigma = \sigma_A$. In this paper, we provide an explicit construction of the metaplectic operators on $\mathbb{C}^n$.

The paper is arranged as follows. In Section 2, we discuss and construct second-degree characters. The main result is presented in Section 3, with preliminary lemmas given in Section 4. In Section 5, we mention applications to Gabor frames and Section 6 contains final remarks.

2. Second-degree characters

The notion of second-degree characters on a locally compact abelian (l.c.a.) group was introduced in [27]; see also [23, Section 3.2]. Given an l.c.a. group G, a bicharacter is a continuous mapping $B\colon G \times G \to \mathbb{T}$ such that B is a character when one of the two arguments is fixed. A second-degree character associated to B is a continuous mapping $\psi\colon G \to \mathbb{T}$ such that

$$\psi(x + x') = \psi(x)\psi(x')B(x, x'), \quad x, x' \in G. \tag{6}$$

Second-degree characters are used in cohomology theory as transfer functions for symmetric multipliers (two-cocycles); the corresponding bicharacter is the multiplier. A general existence result for second-degree characters is obtained by Mackey's technique of induced representation; see [3, p. 308] or [23, p. 37] for a concise proof. We derive a more explicit description suitable for our approach. We note that if one second-degree character for a given bicharacter is constructed, then all others associated to the same bicharacter are obtained easily, as described in the next remark.

REMARK 2.1 (i) If two second-degree characters ψ_1, ψ_2 are associated to the same bicharacter, then $\psi_2 = \chi\psi_1$ for some character χ. Indeed it follows from (6) that the quotient $\chi := \psi_2/\psi_1$ is a homomorphism and thus is a character.

(ii) The second-degree characters associated to the trivial bicharacter identical one are just the usual characters. Indeed, in this case (3) reduces to the homomorphism property.

We have briefly discussed the notion of second-degree characters for general l.c.a. groups. Now we mention some concrete cases.

- On $G = \mathbb{R}$ a bicharacter is of the form $B(t, t') = \mathrm{e}^{2\pi \mathrm{i} stt'}$, $t, t' \in \mathbb{R}$, for some $s \in \mathbb{R}$. The function $\psi(t) = \mathrm{e}^{\pi \mathrm{i} st^2}$, $t \in \mathbb{R}$, is an associated second-degree character.
- More generally, let G be a self-dual group where the doubling of elements $x \mapsto x + x$ is invertible. A bicharacter is of the form $B(x, x') = \langle x, cx'\rangle$, with the usual pairing of an element of the group and an element of the dual group, where c is a symmetric homomorphism from G into the dual group. Then one associated second-degree character is constructed easily by $\psi(x) = \langle x, 2^{-1}cx\rangle$, where 2^{-1} denotes the inverse of doubling; see [15, 23, 27]. For example, this construction can be used for $G = \mathbb{R}^d$, $d \geq 1$, and for $G = \mathbb{Z}_n$ when n is odd (the multiplicative inverse of 2 is $(n + 1)/2 \bmod n$ in this case). For $\mathbb{Z}_n$ with n even this construction does not apply since doubling is not injective.

- In Section 1, we use second-degree characters for the product group $G = \mathbb{Z}_n \times \mathbb{Z}_n$. In this case, the bicharacters are of the form

$$B(\lambda, \lambda') = \mathrm{e}^{2\pi \mathrm{i}\langle \lambda, \sigma\lambda'\rangle/n}, \quad \lambda, \lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n,$$

for some symmetric matrix $\sigma = \begin{pmatrix} p & q \\ q & r \end{pmatrix} \in \mathrm{M}_{2,2}(\mathbb{Z}_n)$. Second-degree characters for a product group can be constructed from those for the factors. For example, an associated second-degree character associated to B as above is given by

$$\psi(\lambda) = \psi(k,l) = \psi_1(k)\psi_2(l)\mathrm{e}^{2\pi \mathrm{i} qkl/n}, \quad \lambda = (k,l) \in \mathbb{Z}_n \times \mathbb{Z}_n,$$

where ψ_1, ψ_2 are second-degree characters on $\mathbb{Z}_n$ associated to p and r, respectively.
- For $G = \mathbb{Z}_n$, a bicharacter is of the form

$$B(m, m') = \mathrm{e}^{2\pi \mathrm{i} cmm'/n}, \quad m, m' \in \mathbb{Z}_n,$$

for some $c \in \mathbb{Z}_n$. We will describe the associated second-degree characters.

For $\mathbb{Z}_n$ with n even, the second-degree characters can only be constructed by the induced representation approach mentioned above or by guessing. We present an explicit description, in fact, a unified formula for n even and n odd; cf. [5, 6].

LEMMA 2.2 *Given $c \in \mathbb{Z}_n$, fix a representative $[c] \in \mathbb{Z}$ for c and define*

$$\psi_{[c]}(m) = \mathrm{e}^{\pi \mathrm{i}[c]m^2(n+1)/n}, \quad m \in \mathbb{Z}_n.$$

Then $\psi = \psi_{[c]}$ is a second-degree character associated to c, that is,

$$\psi(m + m') = \psi(m)\psi(m')\mathrm{e}^{2\pi \mathrm{i} cmm'/n}, \quad m, m' \in \mathbb{Z}_n.$$

Proof. First we point out that the choice of representatives for m in the definition of $\psi_{[c]}$ is irrelevant. Indeed

$$\begin{aligned}\psi(m+n) &= \mathrm{e}^{\pi \mathrm{i}[c](m+n)^2(n+1)/n} \\ &= \mathrm{e}^{\pi \mathrm{i}[c]m^2(n+1)/n}\underbrace{\mathrm{e}^{\pi \mathrm{i}[c]n(n+1)}}_{=1}\underbrace{\mathrm{e}^{2\pi \mathrm{i}[c]m(n+1)}}_{=1} = \psi(m), \quad m \in \mathbb{Z}_n.\end{aligned}$$

Observe that this computation holds for any given n, be it even or odd. Thus ψ is well defined and we calculate

$$\begin{aligned}\psi(m+m') &= \mathrm{e}^{\pi \mathrm{i}[c](m+m')^2(n+1)/n} \\ &= \mathrm{e}^{\pi \mathrm{i}[c]m^2(n+1)/n}\mathrm{e}^{\pi \mathrm{i}[c]m'^2(n+1)/n}\mathrm{e}^{2\pi \mathrm{i}[c]mm'(n+1)/n} \\ &= \psi(m)\psi(m')\mathrm{e}^{2\pi \mathrm{i} cmm'/n}, \quad m, m' \in \mathbb{Z}_n.\end{aligned}$$

REMARK 2.3 For n odd, $\psi_{[c]}$ is uniquely defined, independent on the choice of $[c]$. For n even, there are two possible vectors $\psi_{[c]_1}$, $\psi_{[c]_2}$ depending on the choice of $[c]$ and they differ by the modulation

$$\psi_{[c]_2}(m) = (-1)^m \psi_{[c]_1}(m), \quad m \in \mathbb{Z}_n;$$

cf. Remark 2.1

3. The main result

We use the following operators on $\mathbb{C}^n$.

(i) The Fourier transform $\mathcal{F}$, normalized so that it is a unitary operator, is given by

$$\mathcal{F}v(m) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} v(k) \mathrm{e}^{-2\pi \mathrm{i} km/n}, \quad v \in \mathbb{C}^n.$$

(ii) Given an invertible element a in $\mathbb{Z}_n$, we write D_a for the index permutation

$$D_a v(m) = v(a^{-1}m), \quad v \in \mathbb{C}^n.$$

(iii) Given $c \in \mathbb{Z}_n$, the chirp multiplication is denoted by R_c,

$$R_c v(m) = \psi_{[c]}(m) v(m), \quad v \in \mathbb{C}^n,$$

where $\psi_{[c]} \in \mathbb{C}^n$ is the second-degree character constructed in Lemma 2.2, with $[c] \in \mathbb{Z}$ denoting some representative for $c \in \mathbb{Z}_n$.

LEMMA 3.1 (i) *$U = \mathcal{F}$ satisfies (5) for* $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

$$\mathcal{F} T_k M_l \mathcal{F}^{-1} = \mathrm{e}^{-2\pi \mathrm{i} kl/n} T_l M_{-k}.$$

(ii) *$U = D_a$ satisfies (5) for* $A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$,

$$D_a T_k M_l D_{a^{-1}} = T_{ak} M_{a^{-1}l}.$$

(iii) *$U = R_c$ satisfies (5) for* $A = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$

$$R_c T_k M_l R_c^{-1} = \tau_k T_k M_{ck+l}$$

with $\tau_k = \mathrm{e}^{\pi \mathrm{i}[c]k^2(n+1)/n}$.

Proof. (i) By (1), since $\mathcal{F} M_l = T_l \mathcal{F}$ and $\mathcal{F}\mathcal{F}v(k) = v(-k)$.

(ii) $D_a T_k = T_{ak} D_a$ and $M_l D_a = D_a M_{al}$.
(iii) $R_c M_l = M_l R_c$ and $R_c T_k = \tau_k T_k M_{ck} R_c$, cf. the derivations in the proof of Lemma 2.2.

The Weil decomposition used in the proof of Theorem 3.3 below cannot be applied to general matrices $A \in \mathrm{SL}_2(\mathbb{Z}_n)$. The next lemma is an important preparatory step. We note that the methods to prove the analogous result for $G = \mathbb{R}^d$ in [16, Section I.6] cannot be adapted to the present case $G = \mathbb{Z}_n$. A related result for $\mathbb{Z}_n$ is given in [1, Lemma 3]. Our result is valid for general $A \in \mathrm{SL}_2(\mathbb{Z}_n)$ and it is constructive.

LEMMA 3.2 *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_n)$. *Factorize* $n = \prod_i p_i^{s_i}$, *its prime decomposition, and define* $\vartheta := \prod_{i:\, p_i \nmid a} p_i$. *Then* $a_0 := a + \vartheta b$ *is invertible in* $\mathbb{Z}_n$.

Proof. We show that $a_0 \not\equiv 0 \pmod{p_i}$, for any i. There are two cases.

First, if $p_i \nmid a$, then $p_i \mid \vartheta$ and thus $a + \vartheta b \equiv a \not\equiv 0 \pmod{p_i}$.

Secondly, if $p_i \mid a$, then $p_i \nmid \vartheta$. Since $\det A = 1$, we cannot have $p_i \mid a$ and $p_i \mid b$ at the same time. Hence, $p_i \nmid b$ and thus $p_i \nmid \vartheta b$. Consequently, $a + \vartheta b \equiv \vartheta b \not\equiv 0 \pmod{p_i}$.

The next theorem is our main result, the explicit construction of metaplectic operators on $\mathbb{C}^n$ for an arbitrary matrix $A \in \mathrm{SL}_2(\mathbb{Z}_n)$. Given a symmetric matrix σ, we denote by $\Psi(\sigma)$ the set of all second-degree characters associated to σ. For the structure of $\Psi(\sigma)$ recall Remark 2.1.

THEOREM 3.3 *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_n)$. *Let* ϑ *be given by Lemma* 3.2 *and let* $a_0 = a + \vartheta b$ *and* $c_0 = c + \vartheta d$. *Define the operator* U_A *by*

$$U := R_{c_0 a_0^{-1}} \cdot D_{a_0} \cdot \mathcal{F}^{-1} \cdot R_{-a_0^{-1} b} \cdot \mathcal{F} \cdot R_{-\vartheta}.$$

Then U *is unitary and satisfies* (5) *for some* $\psi \in \Psi(\sigma_A)$.

Proof. Since $a = a_0 - \vartheta b$ and $c = c_0 - \vartheta d$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_0 & b \\ c_0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\vartheta & 1 \end{pmatrix}. \tag{7}$$

Observe that the first and third matrix in (7) belong to $\mathrm{SL}_2(\mathbb{Z}_n)$ and hence so does the second. Since a_0 is invertible by Lemma 3.2, we obtain $d = a_0^{-1} + a_0^{-1} b c_0$ and thus we can make use of the Weil decomposition ([27], see [23, Section 4.3])

$$\begin{aligned} \begin{pmatrix} a_0 & b \\ c_0 & d \end{pmatrix} &= \begin{pmatrix} a_0 & b \\ c_0 & a_0^{-1} + a_0^{-1} b c_0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ c_0 a_0^{-1} & 1 \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix} \begin{pmatrix} 1 & a_0^{-1} b \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ c_0 a_0^{-1} & 1 \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a_0^{-1} b & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{8}$$

Combining (7) and (8),

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c_0 a_0^{-1} & 1 \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a_0^{-1} b & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\vartheta & 1 \end{pmatrix}.$$

Thus (5) follows from Lemma 3.1 and the repeated use of Lemma 4.6, stated and proved in Section 4.

REMARK 3.4 We briefly mention one of the constructions of metaplectic operators in the setting of arbitrary locally compact abelian groups in [27]; see also [23]. Applied to our setting it reads as follows. Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_n)$ and $\psi \in \Psi(\sigma_A)$, define the operator V by

$$Vv(k) = \sum_{l \in \mathbb{Z}_n} v(ak + bl)\overline{\psi(k, l)}, \quad v \in \mathbb{C}^n.$$

Then a straightforward computation shows that

$$T_k M_l V = \psi(k, l) V T_{ak+bl} M_{ck+dl}, \quad k, l \in \mathbb{Z}_n. \tag{9}$$

However, it is important to note that without additional assumptions V need not be invertible and thus (9) does not imply (5) (with $U = V^{-1}$). For example, $A = \mathrm{id} \in \mathrm{SL}_2(\mathbb{Z}_n)$ yields $\sigma_A = 0$, and the character $\psi(k, l) = \mathrm{e}^{2\pi \mathrm{i} l/n}$ belongs to $\Psi(\sigma_A) = \Psi(0)$ by Remark 2.1(ii); it turns out that $V = 0$.

4. Preliminary results

We defined the Heisenberg group $H_n = H(\mathbb{Z}_n)$ in the form of the Schrödinger representation, that is, by time-frequency shift operators. It is known that the Schrödinger representation is irreducible, see, for example, [12], and by Schur's lemma this fact is equivalent to the following result. For convenience we include a short direct proof.

LEMMA 4.1 *Suppose that an operator Q satisfies*

$$Q\pi(\lambda) = \pi(\lambda)Q \quad \text{for all } \lambda \in \mathbb{Z}_n \times \mathbb{Z}_n,$$

that is,

$$QT_k M_l = T_k M_l Q \quad \text{for all } k, l \in \mathbb{Z}_n.$$

Then Q is a scalar multiple of the identity operator, $Q = c \cdot \mathrm{id}$ for some $c \in \mathbb{C}$.

Proof. Since Q commutes with the modulation M_1 the matrix elements of $Q = (Q_{k,l})$ satisfy $Q_{k,l}\mathrm{e}^{2\pi \mathrm{i} l/n} = \mathrm{e}^{2\pi \mathrm{i} k/n} Q_{k,l}$. Hence $Q_{k,l} = 0$ when $k \neq l$, that is, Q is given by a diagonal matrix. In other words, Q is a multiplication operator, $Qv(m) = q(m)v(m)$, for some $q \in \mathbb{C}^n$. Since Q also commutes with the translation T_1, it follows that $q(m) = c$ must be constant. Thus $Qv = c \cdot v$ for some $c \in \mathbb{C}$.

We use the following consequence of Lemma 4.1.

LEMMA 4.2 *Given* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_n)$*, the following holds. Two unitary operators* $U = U_1, U_2$ *satisfy* (5) *for the same* $\psi \in \Psi(\sigma_A)$ *if and only if* $U_2 = \tau U_1$ *for some* $\tau \in \mathbb{T}$.

Proof. By assumption $U_1\pi(\lambda)U_1^{-1} = U_2\pi(\lambda)U_2^{-1}$ for all $\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n$. In other words the operator $Q := U_2^{-1}U_1$ satisfies $Q\pi(\lambda)Q^{-1} = \pi(\lambda)$. Hence Lemma 4.1 implies $Q = c\,\mathrm{id}$ for some $c \in \mathbb{C}$. Since Q is unitary $c \in \mathbb{T}$.

The next lemma describes the inner automorphisms of H_n. Note that for $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we have $\sigma_A = 0$ and by Remark 2.1(ii) the associated second-degree characters are just the usual characters. That is, if $\chi \in \Psi(\sigma_A) = \Psi(0)$, then $\chi(\lambda) = \mathrm{e}^{2\pi\mathrm{i}\langle\mu,\lambda\rangle/n}$, for some $\mu \in \mathbb{Z}_n \times \mathbb{Z}_n$.

LEMMA 4.3 *Given* $\lambda' = (k', l') \in \mathbb{Z}_n \times \mathbb{Z}_n$*, the operator* $W = \pi(\lambda')$ *satisfies* (5) *for* $A = \mathrm{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$$W\pi(\lambda)W^{-1} = \mathrm{e}^{2\pi\mathrm{i}\langle\mu,\lambda\rangle/n}\pi(\lambda)$$

with $\mu = J\lambda' = (-l', k')$*, where* $J := \kappa - \kappa^{\mathrm{T}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Proof. By using (1), $(T_{k'}M_{l'})^{-1}T_kM_lT_{k'}M_{l'} = M_{-l'}T_{-k'}T_kM_lT_{k'}M_{l'} = \mathrm{e}^{2\pi\mathrm{i}(k'l-kl')/n}T_kM_l$.

The next result describes the relation between different intertwiners that correspond to the same $A \in \mathrm{SL}_2(\mathbb{Z}_n)$.

LEMMA 4.4 *Given* $A \in \mathrm{SL}_2(\mathbb{Z}_n)$*, suppose that a unitary operator* $U = U_1$ *satisfies* (5)*, for some* $\psi = \psi_1 \in \Psi(\sigma_A)$*. Then a unitary operator* $U = U_2$ *satisfies* (5) *for some, possibly different* $\psi = \psi_2 \in \Psi(\sigma_A)$*, if and only if* $U_2 = U_1W$ *for some* $W \in H_n$.

Proof. ($\Rightarrow$) Since, by assumption,

$$U_1\pi(\lambda)U_1^{-1} = \psi_1(\lambda)\pi(A\lambda) \quad \text{and} \quad U_2\pi(\lambda)U_2^{-1} = \psi_2(\lambda)\pi(A\lambda),$$

we conclude that the operator $W := U_1^{-1}U_2$ satisfies

$$W\pi(\lambda)W^{-1} = \frac{\psi_2(\lambda)}{\psi_1(\lambda)}\pi(\lambda). \tag{10}$$

By Remark 2.1 the quotient $\chi(\lambda) := \psi_2(\lambda)/\psi_1(\lambda)$ is a character. By Lemma 4.3 there is an element $W = W_0 \in H_n$ such that (10) holds. By Lemma 4.2 the choice of W is unique up to a scalar $\tau \in \mathbb{T}$, that is, $W = \tau W_0$. Therefore also W belongs to H_n.
($\Leftarrow$) By Lemma 4.3 the operator $U_2 = U_1W$ satisfies

$$U_1W\pi(\lambda)W^{-1}U_1^{-1} = \chi(\lambda)U_1\pi(\lambda)U_1^{-1} = \chi(\lambda)\psi_1(\lambda)\pi(A\lambda),$$

and we note that by Remark 2.1 the function $\psi_2 := \chi\psi_1$ indeed belongs to $\Psi(\sigma_A)$.

The following lemma formulates useful modifications of second-degree characters.

LEMMA 4.5 *Given* $A, B \in \mathrm{SL}_2(\mathbb{Z}_n)$, *let* $\psi_A \in \Psi(\sigma_A)$ *and* $\psi_B \in \Psi(\sigma_B)$.

(i) *The complex conjugate or reciprocal* $\overline{\psi_A} = \psi_A^{-1}$ *belongs to* $\Psi(-\sigma_A)$.
(ii) $\psi(\lambda) := \overline{\psi_A(A^{-1}\lambda)}$ *belongs to* $\Psi(\sigma_{A^{-1}})$.
(iii) $\psi(\lambda) := \psi_A(B\lambda)\psi_B(\lambda)$ *belongs to* $\Psi(\sigma_{AB})$.

Proof. (i) By direct computation, $\overline{\psi_A} = \psi_A^{-1}$ satisfies (3) with σ replaced by $-\sigma$;

(ii) ψ satisfies (3) with σ replaced by

$$-A^{-\mathrm{T}}\sigma_A A^{-1} = -(\kappa - A^{-\mathrm{T}}\kappa A^{-1}) = \sigma_{A^{-1}};$$

(iii) ψ satisfies (3) with σ replaced by

$$B^{\mathrm{T}}\sigma_A B + \sigma_B = (B^{\mathrm{T}}A^{\mathrm{T}}\kappa AB - B^{\mathrm{T}}\kappa B) + (B^{\mathrm{T}}\kappa B - \kappa) = B^{\mathrm{T}}A^{\mathrm{T}}\kappa AB - \kappa = \sigma_{AB}.$$

Next, we show how to obtain intertwiners that correspond to inverses or products of matrices $A, B \in \mathrm{SL}_2(\mathbb{Z}_n)$.

LEMMA 4.6 *Given* $A, B \in \mathrm{SL}_2(\mathbb{Z}_n)$, *let* $U = U_A, U_B$ *satisfy* (5) *for* $\psi_A \in \Psi(\sigma_A)$, $\psi_B \in \Psi(\sigma_B)$, *respectively.*

(i) $U := (U_A)^{-1}$ *satisfies* (5) *for* $A^{-1} \in \mathrm{SL}_2(\mathbb{Z}_n)$ *and some* $\psi \in \Psi(\sigma_{A^{-1}})$.
(ii) $U := U_A U_B$ *satisfies* (5) *for* $AB \in \mathrm{SL}_2(\mathbb{Z}_n)$ *and some* $\psi \in \Psi(\sigma_{AB})$.

Proof. (i) By assumption we have

$$U_A\pi(\lambda)U_A^{-1} = \psi_A(\lambda)\pi(A\lambda)$$

and hence the substitution $\lambda \mapsto A^{-1}\lambda$ allows us to write

$$U_A^{-1}\pi(\lambda)U_A = \overline{\psi_A(A^{-1}\lambda)}\pi(A^{-1}\lambda).$$

Since by Lemma 4.5(ii) the function $\lambda \mapsto \overline{\psi_A(A^{-1}\lambda)}$ belongs to $\Psi(\sigma_{A^{-1}})$ we thus conclude that $U = U_A^{-1}$ satisfies the intertwining identity (5) with A replaced by A^{-1}.

(ii) Here we have

$$U_A U_B \pi(\lambda) U_B^{-1} U_A^{-1} = \psi_B(\lambda) U_A \pi(B\lambda) U_A^{-1} = \psi_A(B\lambda)\psi_B(\lambda)\pi(AB\lambda).$$

By Lemma 4.5(iii) the function $\lambda \mapsto \psi_A(B\lambda)\psi_B(\lambda)$ belongs to $\Psi(\sigma_{AB})$ and hence $U = U_A U_B$ satisfies the intertwining identity (5) with A replaced by AB.

REMARK 4.7 By Proposition 1.2 the set

$$\{(A, \psi) \colon A \in \mathrm{SL}_2(\mathbb{Z}_n), \psi \in \Psi(\sigma_A)\}$$

can be identified with a class of automorphisms of the Heisenberg group $H_n = H(\mathbb{Z}_n)$. More precisely it constitutes the group of those automorphisms that leave the centre pointwise fixed; see Remark 1.3.

It is interesting to observe that Lemma 4.5 shows that the group law is given by $(A, \psi_A) \circ (B, \psi_B) = (AB, \psi)$, where

$$\psi(\lambda) = \psi_A(B\lambda)\psi_B(\lambda), \quad \lambda \in \mathbb{Z}_n.$$

Beyond our scope one can show that for each $A \in \mathrm{SL}_2(\mathbb{Z}_n)$ two elements $\psi_A^1, \psi_A^2 \in \Psi(\sigma_A)$ can be chosen in such a way that $\{(A, \psi_A^1), (A, \psi_A^2) \colon A \in \mathrm{SL}_2(\mathbb{Z}_n)\}$ is a subgroup, a double cover of $\mathrm{SL}_2(\mathbb{Z}_n)$. Correspondingly for every $A \in \mathrm{SL}_2(\mathbb{Z}_n)$ two U can be selected in such a way that they also form a group and there exists a homomorphism $(A, \psi) \mapsto U$ which is a unitary representation of the double cover. This is the concrete picture for $\mathbb{Z}_n$ of the general paradigm of the metaplectic representation on locally compact abelian groups in [27].

We mention that a simple extension of the material in [21] of the case n an odd prime shows that for n odd, in fact, one element of $\Psi(\sigma_A)$ for each A can be chosen in such a way that indeed a representation of $\mathrm{SL}_2(\mathbb{Z}_n)$ is obtained; see also [8].

5. Application to Gabor frames

The metaplectic representation is an important technique in time–frequency analysis. It is developed and applied as a general framework, for example, in [10; 11; 13, Section 9.4], and it is often used as a tool both in continuous-time and finite settings, such as in [18]. We describe the use of our results for Gabor frames in $\mathbb{C}^n$, for the relevant notation, see [24] and also [7, Chapter 10].

Given a Gabor window $v \in \mathbb{C}^n$ and a time–frequency lattice $\Lambda \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$, the family

$$G(v, \Lambda) := \{\pi(\lambda)v : la \in \Lambda\} = \{T_k M_l v \colon (k, l) \in \Lambda\}$$

is called a Gabor system. The interesting systems are those that span $\mathbb{C}^n$, they are exactly the regular Gabor frames for $\mathbb{C}^n$. The system $G(v, \Lambda)$ is a Gabor frame if and only if the frame operator

$$S_{v,\Lambda} u = \sum_{\lambda \in \Lambda} \langle u, \pi(\lambda)v\rangle \pi(\lambda)v, \quad u \in \mathbb{C}^n,$$

is invertible. The focus of many questions and results in Gabor analysis is the dual window $\widetilde{v}$, defined for a Gabor frame $G(v, \Lambda)$ by

$$\widetilde{v} = S_{v,\Lambda}^{-1} v.$$

Most importantly, the dual window $\widetilde{v}$ allows us to reconstruct any vector $u \in \mathbb{C}^n$ from its Gabor coefficients $\langle u, \pi(\lambda)v\rangle$ with respect to v. Indeed one has the frame expansions

$$u = \sum_{\lambda \in \Lambda} \langle u, \pi(\lambda)v\rangle \pi(\lambda)\widetilde{v} = \sum_{\lambda \in \Lambda} \langle u, \pi(\lambda)\widetilde{v}\rangle \pi(\lambda)v \quad \text{for all } u \in \mathbb{C}^n.$$

The standard time–frequency lattices are rectangular lattices $\Lambda_0 = r\mathbb{Z}_n \times s\mathbb{Z}_n$, with r, s divisors of n. For this case, there are fast algorithms for computing $\widetilde{v}$ by making use of the fast Fourier transform (FFT); see [24] and the list of references therein. An important application of the metaplectic operators is that they allow us to apply the techniques designed for the case of a rectangular lattice also to more general time–frequency lattices. The method is formulated in the corollary below, it is analogous to the continuous-time case in [13, Proposition 9.4.4]. We use the following lemma.

LEMMA 5.1 *A general lattice $\Lambda \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ can always be written $\Lambda = A\Lambda_0$, where Λ_0 is a rectangular lattice and $A \in \mathrm{SL}_2(\mathbb{Z}_n)$.*

Proof. We identify the finite lattice Λ in $\mathbb{Z}_n \times \mathbb{Z}_n$ with an infinite lattice $\bar{\Lambda}$ of $\mathbb{Z} \times \mathbb{Z}$, its preimage $\bar{\Lambda}$ under the canonical epimorphism $\mathbb{Z} \to \mathbb{Z}_n$. Denote by $M_{2,2}(\mathbb{Z})$ the set of 2×2 matrices with integer entries, and by $\mathrm{SL}_2(\mathbb{Z})$ denote the corresponding special linear group of matrices with determinant 1. Let $\bar{L} \in M_{2,2}(\mathbb{Z})$ be a generator matrix for $\bar{\Lambda}$, that is, the columns of $\bar{L}$ are generators of the lattice $\bar{\Lambda}$,

$$\bar{\Lambda} = \bar{L}(\mathbb{Z} \times \mathbb{Z}).$$

The Smith normal form of integer matrices allows us to write

$$\bar{L} = \bar{A}\bar{L}_0\bar{B},$$

where $\bar{A}$, $\bar{B}$ belong to $\mathrm{SL}_2(\mathbb{Z})$ and $\bar{L}_0 \in M_{2,2}(\mathbb{Z})$ is diagonal. Hence,

$$\bar{\Lambda} = \bar{A}\bar{\Lambda}_0, \tag{11}$$

with $\bar{\Lambda}_0$ generated by $\bar{L}_0$. Note that $\bar{B}$ leaves $\mathbb{Z} \times \mathbb{Z}$ invariant, that is, $\bar{B}(\mathbb{Z} \times \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$, as well as $n(\mathbb{Z} \times \mathbb{Z})$. Hence we can consider all integers modulo n and thus the splitting (11) of $\bar{\Lambda}$ in $\mathbb{Z} \times \mathbb{Z}$ yields the desired splitting $\Lambda = A\Lambda_0$ in $\mathbb{Z}_n \times \mathbb{Z}_n$.

Theorem 3.3 and Lemma 5.1 allow us to use the metaplectic operators for time-frequency analysis in $\mathbb{C}^n$ in a similar way as they are used in the continuous-time setting [**13**, Section 9.4]. The next result illustrates this fact.

COROLLARY 5.2 *Let $\Lambda \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ be a general lattice and $v \in \mathbb{C}^n$. Write $\Lambda = A\Lambda_0$ as described in Lemma* 5.1; *let $U = U_A$ be given by Theorem* 3.3; *and let $v_0 := U^{-1}v \in \mathbb{C}^n$. Then the following are equivalent:*

(i) *$G(v, \Lambda)$ is a Gabor frame with dual window $\widetilde{v}$,*
(ii) *$G(v_0, \Lambda_0)$ is a Gabor frame with dual window $\widetilde{v_0}$, and the dual windows are related to each other by $\widetilde{v} = U\widetilde{v_0}$.*

Proof. By Theorem 3.3 we have for $u \in \mathbb{C}^n$

$$\begin{aligned} US_{v_0,\Lambda_0}U^{-1}u &= \sum_{\lambda\in\Lambda_0} \langle U^{-1}u, \pi(\lambda)v_0\rangle U\pi(\lambda)v_0 \\ &= \sum_{\lambda\in\Lambda_0} \langle u, U\pi(\lambda)U^{-1}v\rangle U\pi(\lambda)U^{-1}v \\ &= \underbrace{\overline{\psi_A(\lambda)}\psi_A(\lambda)}_{=1} \sum_{\lambda\in\Lambda_0} \langle u, \pi(A\lambda)v\rangle \pi(A\lambda)v \\ &= \sum_{\lambda\in\Lambda} \langle u, \pi(\lambda)v\rangle \pi(\lambda)v = S_{v,\Lambda}u \end{aligned}$$

and hence $\widetilde{v} = S_{v,\Lambda}^{-1}v = (US_{v_0,\Lambda_0}U^{-1})^{-1}v = US_{v_0,\Lambda_0}^{-1}v_0 = U\widetilde{v_0}$.

The corollary allows us to obtain the dual window for $G(v, \Lambda)$ by computing the dual window for $G(v_0, \Lambda_0)$. Since in this computation the general lattice Λ is replaced by the rectangular lattice Λ_0, the fast algorithms mentioned above can be used.

6. Final remarks

While our paper is focused on a complete and explicit construction for the 'one-dimensional' case $\mathbb{Z}_n$, several parts of our presentation are also valid for the 'multi-dimensional' case $\mathbb{Z}_n^d$, $d \geq 1$, where A consists of four $d \times d$ blocks. In this general case the condition $\det A = 1$ which ensures that $\sigma = \sigma_A$ is symmetric must be replaced by the conditions $a^{\mathrm{T}}c = c^{\mathrm{T}}a$, $b^{\mathrm{T}}d = d^{\mathrm{T}}b$ and $a^{\mathrm{T}}d - c^{\mathrm{T}}b = 1$. Such A that satisfy these conditions are called symplectic. A usual way to express that A is symplectic is the equivalent condition $A^{\mathrm{T}}JA = J$, with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as in Lemma 4.3.

For the special case $\mathbb{Z}_n$ used in our paper we have that A is symplectic if and only if $\det A = 1$. In other words, the symplectic group over $\mathbb{Z}_n$ is the special linear group $\mathrm{SL}_2(\mathbb{Z}_n)$.

REMARK 6.1 The limitations of the transition from $\mathbb{Z}_n$ to $\mathbb{Z}_n^d$ are analogous to the continuous-time case $\mathbb{R}$ to $\mathbb{R}^d$ discussed in [**11**, **13**]. Indeed, for $d = 1$ (i) the symplectic group reduces to the special linear group and (ii) an arbitrary lattice can be reduced to a rectangular lattice by a symplectic transformation (see Lemma 5.1). For $d \geq 2$ these simplifications do not hold.

Acknowledgements

The third and fifth authors were supported by the Austrian Science Fund FWF grants P-15605 and P-15577, respectively.

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