PRODUCT OF TWO HYPERGEOMETRIC FUNCTIONS WITH POWER ARGUMENTS

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Abstract. The hypergeometric product formula describes the product of two hypergeometric functions with arguments xt and yt, respectively. The result is a power series in t whose coefficients are hypergeometric polynomials. Brychkov has extended the product formula such that in one argument the variable t can be squared. We obtain a further extension such that in both arguments the variable t can be raised to positive integer powers.

1. Introduction

We consider hypergeometric functions as generalized hypergeometric series, that is, formal power series of the form [33, p. 404, Definition 16.2.1]

$$ {}_pF_q\left[\begin{matrix} a \\ b \end{matrix} \,\middle|\, t\right] = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{t^k}{k!}, \quad a \in \mathbb{C}^p, \quad b \in \mathbb{C}^q, \quad b_i \notin \{0, -1, -2, \dots\}, \tag{1}$$

where we use the multi-parameter notation $a = (a_1, \dots, a_p)$, which for the Pochhammer symbol or rising factorials $(x)_0 = 1$ and $(x)_k = x(x+1)\cdots(x+k-1)$, for $k = 1, 2, \dots$, means

$$ (a)_k = (a_1)_k \cdots (a_p)_k. \tag{2}$$

The product of two hypergeometric functions with arguments xt and yt, respectively, is a power series in t. The coefficients are hypergeometric polynomials, described by the hypergeometric product formula, see [5, equation (1)]. We remark that letting either $x = 1$ or $y = 1$ is no loss of generality. For one factor the exponential function, the product formulas by Brafman [4, equations (26)] and Srivastava [35, equation (2.5)], [36, equation (3.9)] are such that in one argument the variable t can be replaced by its square t^2 and by an arbitrary positive integer power t^λ, respectively. For the general case of arbitrary hypergeometric factors, the product formula by Brychkov [5, equation (3)] is such that in one argument the variable t can be replaced by its square t^2. The product of hypergeometric functions occurs in various fields, also of generalized hypergeometric functions [10, 20]. Our goal is an extension of Brychkov's formula for general exponents.

By our main result we extend the product formula such that in both arguments the variable t can be replaced by positive integer powers t^λ and t^μ, respectively. We assume that λ and μ are coprime, which is no loss of generality.

In §2 by Theorem 2.3 we state our main result. In §3 we obtain the preliminary results. In §4 the main result is proved. In §5 we give examples. In §6 by Theorem 6.1 we notice a duality that occurs in the main result and in the examples. In §7 by Theorem 7.3 we complement the main result by a polynomial special case that was initially excluded. In §8 by Theorem 8.4 we complement the main result by making use of the regularized version of a hypergeometric function. In §9 we state final remarks.

2010 *Mathematics Subject Classification.* Primary: 33C20. Secondary: 33C99, 33E20, 30C10.

Key words and phrases. Generalized hypergeometric function. Regularized hypergeometric function. Hypergeometric polynomial. Generating function. Appell sequence.

2. Main result

We write the Delta symbol [32, p. 6 and p. 70], for $\lambda = 1, 2, \dots,$ as

$$\Delta_\lambda(x) = \Big(\frac{x}{\lambda}, \frac{x+1}{\lambda}, \dots, \frac{x+\lambda-1}{\lambda}\Big), \tag{3}$$

and for short, $\Delta_\lambda(a) = \big(\Delta_\lambda(a_1), \dots, \Delta_\lambda(a_p)\big)$. For multi-parameters, the order and nesting is irrelevant, for example, $((3,2,1),(1,2)) = (1,1,2,2,3)$. We use the set difference symbol for removing parameters, for example, $(1,1,2,2) \setminus (1) = (1,2,2)$. For any integer k and positive integer n, the symbol $\mathrm{mod}(k,n)$ is defined to be the unique integer r that satisfies $r \equiv k (\mathrm{mod}\ n)$ and $0 \le r < n$.

Definition 2.1. Let $\lambda, \mu = 1, 2, \dots$ such that $\gcd(\lambda,\mu) = 1$. Let $\lambda' \in \mathbb{Z}$ be an inverse of λ modulo μ,

$$\lambda'\lambda \equiv 1 \ (\mathrm{mod}\ \mu),$$

and for $n = 0, 1, \dots,$ define $\mathcal{N}_{\lambda,\mu}(n) = (R, K)$ by

$$R = \mathrm{mod}(\lambda' \cdot n, \mu) \quad \text{ and } \quad K = (n - R\lambda)/\mu.$$

Remark 2.2. (i) The numbers (R,K) are the unique pair of integers that satisfy

$$R \in \{0, 1, \dots, \mu - 1\} \quad \text{ and } \quad R\lambda + K\mu = n.$$

(ii) For (R,K) as a sequence in n, note R is periodic and $K \to \infty$, for $n \to \infty$.
(iii) $\mathcal{N}_{\lambda,\mu}$ is uniquely determined by (λ,μ), since λ' is unique modulo μ.
(iv) Two special cases are

$$\mathcal{N}_{\lambda,1}(n) = (0, n), \qquad \mathcal{N}_{1,\mu}(n) = (\mathrm{mod}(n,\mu), \lfloor n/\mu \rfloor),$$

(v) An example, $(\lambda,\mu) = (7,5)$, is shown in Figure 1.

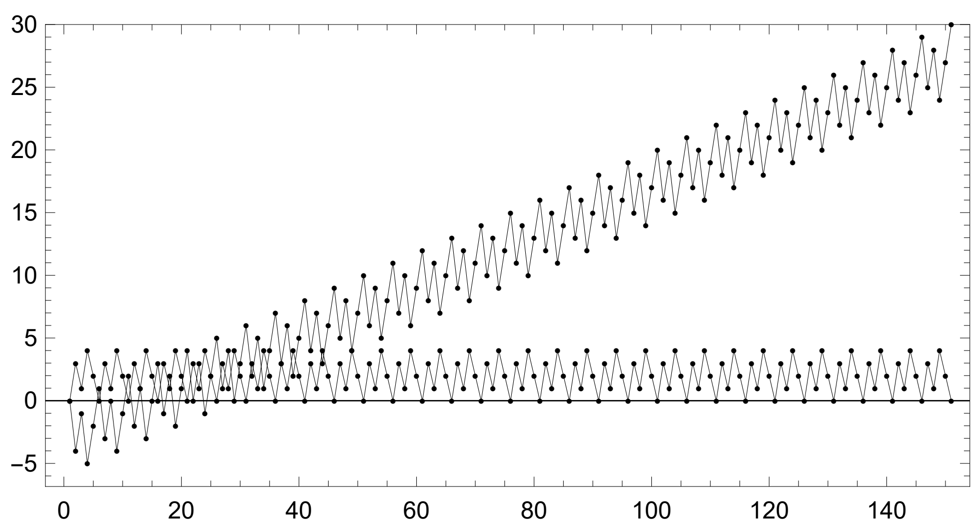

Figure 1. The graphs show $\mathcal{N}_{7,5}(n) = (R,K)$, for $n = 0, 1, 2, \dots, 150$. That is, $R = 0, 3, 1, 4, 2, \dots$ with period 5, and $K = (n - R \cdot 7)/5 \to \infty$.

The next theorem is our main result.

Theorem 2.3. *Let $\lambda, \mu = 1, 2, \dots$ such that $\gcd(\lambda,\mu) = 1$, and let*

$$a \in \mathbb{C}^p, \quad b \in \mathbb{C}^q, \quad c \in \mathbb{C}^{p'}, \quad d \in \mathbb{C}^{q'}, \quad b_i, d_i \notin \{0, -1, -2, \dots\}.$$

Let P_n be the polynomials in the expansion

$$ {}_pF_q\left[\begin{matrix} a \\ b \end{matrix} \,\middle|\, xt^\lambda\right] \cdot {}_{p'}F_{q'}\left[\begin{matrix} c \\ d \end{matrix} \,\middle|\, yt^\mu\right] = \sum_{n=0}^{\infty} P_n(x,y)t^n, $$

that is,

$$ P_n(x,y) = \sum_{\substack{j,\,k\,=\,0,1,\dots \\ \lambda j + \mu k = n}} \frac{(a)_j}{(b)_j}\,\frac{(c)_k}{(d)_k} \cdot \frac{x^j}{j!}\,\frac{y^k}{k!}. $$

We assume that c does not contain any $c_i \in \{0,-1,-2,\dots\}$. Then the following hold, for

$$ (R,K) = \mathcal{N}_{\lambda,\mu}(n). $$

If $K \le -1$, then $P_n(x,y) = 0$.
If $K \ge 0$, then for

$$ \begin{aligned} \mathcal{Q} &= \Big(\Delta_\lambda\big(-K+1-(d,1)\big), \Delta_\mu(R+a)\Big), \\ \mathcal{R} &= \Big(\Delta_\lambda(-K+1-c), \Delta_\mu\big(R+(b,1)\big)\Big), \end{aligned} \qquad \mathcal{M} = \frac{(-\lambda)^{\lambda(1+q'-p')}}{\mu^{\mu(1+q-p)}}, $$

we have

$$ P_n(x,y) = \frac{(a)_R}{(b)_R}\,\frac{(c)_K}{(d)_K} \cdot {}_{p_1}F_{q_1}\left[\begin{matrix} \mathcal{Q} \\ \mathcal{R}\setminus(1) \end{matrix} \,\middle|\, \mathcal{M}\,\frac{x^\mu}{y^\lambda}\right] \cdot \frac{x^R}{R!}\,\frac{y^K}{K!}, $$

where $p_1 = \lambda(q'+1) + \mu p$ and $q_1 = \lambda p' + \mu(q+1) - 1$.

The theorem is proved in §4, the examples are in §5.

Remark 2.4. (i) For $\{\lambda,\mu\} = \{1,2\}$, the expansion in the theorem is obtained by Brychkov [5, equation (3)], note the duality in the next item, (ii) of this remark. For $\lambda = \mu = 1$, the result reduces to the classic product formula, see [5, equation (1)].

(ii) The theorem contains an implicit dual result, obtained by swapping the factors before using the theorem. In the result of the theorem it amounts to swapping

$$ \begin{matrix} (a,b,x,\lambda) \\ (R,K) = \mathcal{N}_{\lambda,\mu}(n) \end{matrix} \qquad \leftrightarrow \qquad \begin{matrix} (c,d,y,\mu) \\ (R,K) = \mathcal{N}_{\mu,\lambda}(n). \end{matrix} \tag{4} $$

The restriction on c in the theorem can be avoided by swapping the factors, unless both factors are polynomials, or by the results of §7 and §8. The options are summarized in §9.

(iii) It is implicit in the proof of the theorem and follows by Lemma 3.6 that the upper parameters $\mathcal{Q}$ contain the nonpositive integer $-\lfloor K/\lambda \rfloor$. Hence, the hypergeometric function in the theorem is indeed a polynomial. It is explicit in the proof and follows by Lemma 3.6 that the lower parameters $\mathcal{R}$ contain 1. Hence, the expression $\mathcal{R}\setminus(1)$ in the theorem, which removes 1 from $\mathcal{R}$, is well-defined.

(iv) In the hypergeometric polynomial of Theorem 2.3 the Delta symbol occurs with two different parameters (λ, μ), similar to the Mellin transform of the product of two Meijer G-functions with one argument a rational power [32, p. 80]. Results for hypergeometric functions of this or more general type are for example in [2, 17, 25–29, 39].

3. Preliminary results

The next result is a Cauchy product formula for Laurent series with power arguments. It is a preparatory result for Lemma 3.3 below for power series with power arguments.

Lemma 3.1. *Let $\lambda, \mu = 1, 2, \dots$ such that $\gcd(\lambda, \mu) = 1$. Let $\lambda', \mu' \in \mathbb{Z}$ satisfy the Bézout identity*

$$\lambda'\lambda + \mu'\mu = 1,$$

such as (λ', μ') obtained from (λ, μ) by the extended Euclidean algorithm. Let $n = 0, 1, 2, \dots$.

(i) *Let $\mathcal{U}_n$ denote the set*

$$\mathcal{U}_n = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \ \alpha, \beta \in \mathbb{Z}, \ \alpha\lambda + \beta\mu = n \right\}.$$

Then $\mathcal{U}_n$ is of the form

$$\mathcal{U}_n = \left\{ n \cdot \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} + \gamma \cdot \begin{pmatrix} \mu \\ -\lambda \end{pmatrix} : \ \gamma \in \mathbb{Z} \right\}.$$

(ii) *For two formal Laurent series g, h,*

$$g(u) = \sum_{j \in \mathbb{Z}} g_j u^j, \quad h(u) = \sum_{j \in \mathbb{Z}} h_j u^j,$$

let p_n denote the coefficient with index n in the expansion

$$g(t^\lambda) \cdot h(t^\mu) = \sum_{n \in \mathbb{Z}} p_n t^n, \qquad \text{that is,} \qquad p_n = \sum_{\substack{j, k \in \mathbb{Z} \\ \lambda j + \mu k = n}} g_j h_k.$$

Then p_n is of the form

$$p_n = \sum_{\gamma \in \mathbb{Z}} g_{\lambda' n + \gamma\mu} \cdot h_{\mu' n - \gamma\lambda}.$$

We state two proofs of (i) and one proof of (ii).

Proof. (i) Let $\mathcal{U}_n'$ and $\mathcal{U}_n''$ denote the first and second descriptions of $\mathcal{U}_n$ in the lemma, respectively.

I. Arithmetic proof. Observe that

$$\begin{aligned} \mathcal{U}_n' &= \{ v(\alpha) : \ \alpha \in \mathbb{Z}, \ \alpha\lambda \equiv n \pmod{\mu} \}, \\ \mathcal{U}_n'' &= \{ v(\alpha) : \ \alpha \in \mathbb{Z}, \ \alpha \equiv \lambda' n \pmod{\mu} \}, \end{aligned} \qquad v(\alpha) = \begin{pmatrix} \alpha \\ (n - \alpha\lambda)/\mu \end{pmatrix}, \tag{5}$$

where the expression for $\mathcal{U}''$ in (5) is verified by using the identity

$$\lambda'\lambda n + \mu'\mu n = n, \tag{6}$$

which follows from the Bézout identity. Since λ' is an inverse of λ modulo μ, we have the equivalence

$$\alpha\lambda \equiv n \pmod{\mu} \quad \Leftrightarrow \quad \alpha \equiv \lambda' n \pmod{\mu}, \tag{7}$$

and thus, by (5) and (7), we obtain $\mathcal{U}_n' = \mathcal{U}_n''$.

II. Geometric proof. Let

$$U' = \begin{pmatrix} \lambda & \mu \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U'' = \begin{pmatrix} 1 & 0 \\ \lambda' & \mu \\ \mu' & -\lambda \end{pmatrix}, \tag{8}$$

and let $\Lambda(U')$ and $\Lambda(U'')$ denote the lattices generated by the columns of U' and U'', respectively. We observe that

$$\mathcal{U}_n' = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \ \begin{pmatrix} n \\ \alpha \\ \beta \end{pmatrix} \in \Lambda(U') \right\}, \qquad \mathcal{U}_n'' = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \ \begin{pmatrix} n \\ \alpha \\ \beta \end{pmatrix} \in \Lambda(U'') \right\}. \tag{9}$$

Next, we use the Bézout identity in the computation of a matrix product and a determinant,

$$U' \cdot Q = U'', \quad \text{for} \quad Q = \begin{pmatrix} \lambda' & \mu \\ \mu' & -\lambda \end{pmatrix}, \qquad \text{and} \qquad \det(Q) = -1. \tag{10}$$

Thus $U' \mapsto U''$ is a unimodular lattice basis transform and hence

$$\Lambda(U') = \Lambda(U''). \tag{11}$$

From (9) and (11) we conclude that $\mathcal{U}'_n = \mathcal{U}''_n$.

(ii) Write

$$p_n = \sum_{(\alpha,\beta)\,\in\,\mathcal{U}_n} g_\alpha \cdot h_\beta, \tag{12}$$

and apply (i). ☐

Remark 3.2. In the geometric proof of Lemma 3.1(i) the conclusion that U' and U'' generate the same lattice also follows by showing that their column-style Hermite normal forms coincide,

$$\mathcal{H}(U') = \mathcal{H}(U'') = \begin{pmatrix} 1 & 0 \\ \lambda'_0 & \mu \\ \mu'_0 & -\lambda \end{pmatrix}, \tag{13}$$

where (λ'_0, μ'_0) is selected among the pairs (λ', μ') that satisfy the Bézout identity by the condition $\lambda'_0 \in \{0, 1, \ldots, \mu - 1\}$, that is, $\lambda'_0 = \mathrm{mod}(\lambda', \mu)$ and $\mu'_0 = (1 - \lambda'_0\lambda)/\mu$.

The next lemma is a Cauchy product formula for power series with power arguments.

Lemma 3.3. *Let* $\lambda, \mu = 1, 2, \ldots$ *such that* $\gcd(\lambda, \mu) = 1$. *For* $n = 0, 1, 2, \ldots,$ *using* Definition 2.1 *let*

$$(R, K) = \mathcal{N}_{\lambda,\mu}(n).$$

Then the following hold.

(i) *Let* $\mathcal{V}_n$ *denote the set*

$$\mathcal{V}_n = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \ \alpha, \beta = 0, 1, \ldots; \ \alpha\lambda + \beta\mu = n \right\}.$$

Then $\mathcal{V}_n$ *is of the form*

$$\mathcal{V}_n = \begin{cases} \varnothing, & K \le -1, \\ \mathcal{V}^{R,K}, & K \ge 0, \end{cases} \qquad \textit{where} \quad \mathcal{V}^{R,K} = \left\{ \begin{pmatrix} R \\ K \end{pmatrix} + \gamma \cdot \begin{pmatrix} \mu \\ -\lambda \end{pmatrix} : \ \gamma = 0, 1, \ldots, \lfloor K/\lambda \rfloor \right\}.$$

(ii) *For two formal power series* g, h,

$$g(u) = \sum_{j=0}^{\infty} g_j u^j, \quad h(u) = \sum_{j=0}^{\infty} h_j u^j,$$

let p_n *denote the coefficient with index* n *in the expansion*

$$g(t^\lambda) \cdot h(t^\mu) = \sum_{n=0}^{\infty} p_n t^n, \qquad \textit{that is,} \quad p_n = \sum_{\substack{j,k = 0,1,\ldots \\ \lambda j + \mu k = n}} g_j h_k.$$

Then p_n *is of the form*

$$p_n = \begin{cases} 0, & K \le -1, \\ p^{R,K}, & K \ge 0, \end{cases} \qquad \textit{where} \quad p^{R,K} = \sum_{\gamma=0}^{\lfloor K/\lambda \rfloor} g_{R+\gamma\mu} \cdot h_{K-\gamma\lambda}.$$

Proof. (i) Let λ', μ' satisfy the Bézout identity $\lambda'\lambda + \mu'\mu = 1$. Then Lemma 3.1(i) implies

$$\left\{\begin{pmatrix}\alpha\\ \beta\end{pmatrix} :\ \alpha,\beta \in \mathbb{Z},\ \alpha\lambda + \beta\mu = n\right\} = \left\{n \cdot \begin{pmatrix}\lambda'\\ \mu'\end{pmatrix} + \gamma \cdot \begin{pmatrix}\mu\\ -\lambda\end{pmatrix} :\ \gamma \in \mathbb{Z}\right\}. \tag{14}$$

Next, since $R = \operatorname{mod}(\lambda' n, \mu)$ and $K = (n - R\lambda)/\mu$, we observe that

$$\left\{\begin{pmatrix}\lambda' n\\ \mu' n\end{pmatrix} + \gamma \cdot \begin{pmatrix}\mu\\ -\lambda\end{pmatrix} :\ \gamma \in \mathbb{Z}\right\} = \left\{\begin{pmatrix}R\\ K\end{pmatrix} + \gamma \cdot \begin{pmatrix}\mu\\ -\lambda\end{pmatrix} :\ \gamma \in \mathbb{Z}\right\}. \tag{15}$$

Combining (14) and (15) we obtain

$$\left\{\begin{pmatrix}\alpha\\ \beta\end{pmatrix} :\ \alpha,\beta \in \mathbb{Z},\ \alpha\lambda + \beta\mu = n\right\} = \left\{\begin{pmatrix}R\\ K\end{pmatrix} + \gamma \cdot \begin{pmatrix}\mu\\ -\lambda\end{pmatrix} :\ \gamma \in \mathbb{Z}\right\}. \tag{16}$$

Let $\mathcal{V}_n'$ and $\mathcal{V}_n''$ denote the first and second descriptions of $\mathcal{V}_n$ in the lemma, respectively. Adding the non-negativity condition we obtain from (16) that

$$\begin{aligned}
\mathcal{V}_n' &= \left\{\begin{pmatrix}\alpha\\ \beta\end{pmatrix} :\ \alpha,\beta \in \mathbb{Z},\ \alpha\lambda + \beta\mu = n,\ \alpha,\beta \geq 0\right\}\\
&= \left\{\begin{pmatrix}R\\ K\end{pmatrix} + \gamma \cdot \begin{pmatrix}\mu\\ -\lambda\end{pmatrix} :\ \gamma \in \mathbb{Z},\ R + \gamma\mu \geq 0,\ K - \gamma\lambda \geq 0\right\}\\
&= \left\{\begin{pmatrix}R\\ K\end{pmatrix} + \gamma \cdot \begin{pmatrix}\mu\\ -\lambda\end{pmatrix} :\ \gamma \in \mathbb{Z},\ \lceil -R/\mu\rceil \leq \gamma \leq \lfloor K/\lambda\rfloor\right\}\\
&= \left\{\begin{pmatrix}R\\ K\end{pmatrix} + \gamma \cdot \begin{pmatrix}\mu\\ -\lambda\end{pmatrix} :\ \gamma \in \mathbb{Z},\ 0 \leq \gamma \leq \lfloor K/\lambda\rfloor\right\} = \mathcal{V}_n'',
\end{aligned} \tag{17}$$

noting that $R \in \{0, 1, \ldots, \mu - 1\}$ implies $\lceil -R/\mu\rceil = 0$, and that in the last line of (17) the set is empty if and only if K is negative.

(ii) Write

$$p_n = \sum_{(\alpha,\beta)\,\in\,\mathcal{V}_n} g_\alpha \cdot h_\beta, \tag{18}$$

and apply (i). $\square$

Remark 3.4. (i) An equivalent form of Lemma 3.3(ii), obtained for $x, y \in \mathbb{C}\backslash\{0\}$ by the modification

$$g(t^\lambda) \cdot h(t^\mu) = \sum_{n=0}^{\infty} p_n\, t^n \quad \mapsto \quad g(xt^\lambda) \cdot h(yt^\mu) = \sum_{n=0}^{\infty} P_n(x, y) t^n, \tag{19}$$

follows by letting

$$g_j \mapsto x^j \cdot g_j \quad \text{and} \quad h_j \mapsto y^j \cdot h_j. \tag{20}$$

(ii) The special case $h_j = 1/j!$, which means $h(u) = \exp(u)$, is discussed in [44, Theorem 4], and if in this case $\lambda = 1$, then Lemma 3.3 reduces to [35, equation (2.3)], [44, Theorem 1].

(iii) For $(\lambda, \mu) = (1, 1)$, the lemma reduces to the usual Cauchy product formula.

The next lemma are multi-parameter versions of two identities for the Pochhammer symbol.

Lemma 3.5. *For $a \in \mathbb{C}^p$, the following hold.*

(i) *For $K = 0, 1, 2, \ldots,$ if a does not contain any $a_i \in \{0, -1, -2, \ldots, -K + 1\}$, then*

$$(a)_{K-j} = (-1)^{pj} \cdot \frac{(a)_K}{(1 - a - K)_j}, \qquad j = 0, 1, \ldots, K.$$

(ii) *For* $\lambda = 1, 2, \ldots,$

$$\big(\Delta_\lambda(a)\big)_j = \frac{(a)_{\lambda j}}{\lambda^{p\lambda j}}, \qquad j = 0, 1, 2, \ldots.$$

Proof. Since each of the two identities is multiplicative in the parameters, they follow from the case $p = 1$, which is [31, p. 9, equations (9) and (10)]. □

The next lemma are properties of the Delta symbol.

Lemma 3.6. (i) *If* $a \in \mathbb{C}$ *is not an integer, then* $\Delta_\lambda(a)$ *does not contain an integer.*
(ii) *If* n *is an integer, then* $\Delta_\lambda(n)$ *contains exactly one integer,* $\lceil n/\lambda \rceil$.
(iii) *If* $n \in \{1, 2, \ldots, \lambda\}$, *then* $\Delta_\lambda(n)$ *contains* 1.

Proof. (i) We prove indirectly. Suppose that $\Delta_\lambda(a)$ contains an integer N. Then

$$N = (a + k_0)/\lambda, \tag{21}$$

with $k_0 \in \{0, 1, \ldots, \lambda - 1\}$. Thus $a = N\lambda - k_0$ is also an integer.
(ii) For $n \in \mathbb{Z}$, we have

$$\lceil n/\lambda \rceil = (n + k_0)/\lambda, \tag{22}$$

with $k_0 = \mathrm{mod}(-n, \lambda) \in \{0, 1, \ldots, \lambda - 1\}$. Thus $\lceil n/\lambda \rceil$ belongs to $\Delta_\lambda(n)$.
The set $\Delta_\lambda(a)$, for $a \in \mathbb{C}$, consists of λ points in the complex plane that lie on a line segment of length strictly smaller than 1. Hence, $\Delta_\lambda(a)$ cannot contain two or more integers.
(iii) Apply (ii). □

In the next lemma we formulate power series manipulations for hypergeometric functions. Manipulations of this type correspond to transformations of hypergeometric series, such as [37, equation (12)], or Meijer G-functions [32, p.6, Property 1.2.4; p.7, Property 1.2.4 and Property 1.2.5].

Lemma 3.7. *For* $\rho, \rho' \in \mathbb{C}$, *write two hypergeometric functions with arguments* $\rho\, u$ *and* $\rho' u$, *respectively, as power series in* u,

$${}_pF_q\left[\begin{matrix} v \\ w \end{matrix} \;\Big|\; \rho\, u\right] = \sum_{j=0}^{\infty} g_j u^j, \qquad v \in \mathbb{C}^p, \quad w \in \mathbb{C}^q, \quad w_i \notin \{0, -1, -2, \ldots\},$$

$${}_{p'}F_{q'}\left[\begin{matrix} v' \\ w' \end{matrix} \;\Big|\; \rho' u\right] = \sum_{j=0}^{\infty} h_j u^j, \qquad v' \in \mathbb{C}^{p'}, \quad w' \in \mathbb{C}^{q'}, \quad w'_i \notin \{0, -1, -2, \ldots\},$$

obtained by letting $g_j = (v)_j/(w, 1)_j \cdot \rho^j$ *and* $h_j = (v')_j/(w', 1)_j \cdot (\rho')^j$. *Then the following hold.*
(i) *For* $R = 0, 1, \ldots,$ *we have*

$$\sum_{j=0}^{\infty} g_{R+j}\, u^j = \rho^R \cdot \frac{(v)_R}{(w, 1)_R} \cdot {}_pF_q\left[\begin{matrix} R + v, 1 \\ R + (w, 1) \end{matrix} \;\Big|\; \rho\, u\right],$$

and the series terminates at $j = N - R$ *(or vanishes) if the original series terminates at* $j = N$, *for* $R \le N$ *(or* $R > N$, *respectively).*
(ii) *For* $K = 0, 1, \ldots,$ *if* v *does not contain any* $v_i \in \{0, -1, \ldots, -K + 1\}$ *or equivalently, if the original series does not terminate at* $j = 0, 1, \ldots, K - 1$, *then*

$$\sum_{j=0}^{K} g_{K-j}\, u^j = \rho^K \cdot \frac{(v)_K}{(w, 1)_K} \cdot {}_pF_q\left[\begin{matrix} -K + 1 - (w, 1), 1 \\ -K + 1 - v \end{matrix} \;\Big|\; \frac{(-1)^{p+q+1}}{\rho} \cdot u\right].$$

(iii) *For* $\lambda = 1, 2, \ldots,$ *we have*

$$\sum_{j=0}^{\infty} g_{\lambda j}\, u^j = {}_{\lambda p}F_{\lambda(q+1)-1}\left[\begin{matrix}\Delta_\lambda(v)\\ \Delta_\lambda(w,1)\setminus(1)\end{matrix}\;\middle|\; \frac{\rho^\lambda}{\lambda^{\lambda(1+q-p)}}\cdot u\right],$$

and the series terminates at $j = \lfloor N/\lambda \rfloor$ *if the original series terminates at* $j = N$.

(iv) *We have*

$$\sum_{j=0}^{\infty} g_j\, h_j\, u^j = {}_{p+p'}F_{q+q'+1}\left[\begin{matrix}v, v'\\ w, w', 1\end{matrix}\;\middle|\; \rho\,\rho'\, u\right],$$

and the series terminates at $j = \min(N, N')$, *if the original series terminate at* $j = N, N'$.

Proof. The identities follow from identities for the Pochhammer symbol. The observations for the terminating series follow by assuming $g_j = 0$ and $h_{j'} = 0$, for $j \geq N+1$ and $j' \geq N'+1$, at the left-hand side of each identity. We also give the details for the right-hand side of each identity.

(i) Use $(v)_{R+j} = (v)_R \cdot (v+R)_j$. Suppose that the original series terminates at $j = N$, that is, v contains the nonpositive integer $-N$. For $R \leq N$ it implies the upper parameters $R + v$ contain the nonpositive integer $-(N-R)$. For $R > N$ it implies $(v)_R = 0$.

(ii) Use Lemma 3.5(i). The assumptions on w imply that the upper parameters

$$\big(-K+1-(w,1),1\big) = \big(-K, -K+1-w, 1\big) \tag{23}$$

contain $-K$ and no greater nonpositive integer. Thus the hypergeometric series indeed terminates and is a polynomial of degree K.

(iii) Use Lemma 3.5(ii). The removal of the parameter 1 from $\Delta_\lambda(w,1)$ is well-defined, since $\Delta_\lambda(1)$ contains 1 by Lemma 3.6(iii). If the original series terminates at $j = N$, then v contains the integer $-N$ and no greater nonpositive integer. Hence, Lemma 3.6(ii) implies that $\Delta_\lambda(v)$ contains the integer $\lceil -N/\lambda \rceil = -\lfloor N/\lambda \rfloor$ and no greater nonpositive integer. Thus the new series terminates at $j = \lfloor N/\lambda \rfloor$.

(iv) Combine $(v)_j (v')_j = (v, v')_j$. If v and v' contain $-N$ and $-N'$ but no greater nonpositive integer, respectively, then the union of v and v' contains $-\min(N, N')$ and no greater nonpositive integer. $\square$

4. Proof of Theorem 2.3

Note the factorial $j!$ can be written as the Pochhammer symbol $(1)_j$, for example,

$$\frac{(a)_j}{(b,1)_j} = \frac{(a)_j}{(b)_j}\cdot\frac{1}{j!}. \tag{24}$$

Proof. Write the two factors as power series as in (1),

$${}_pF_q\left[\begin{matrix}a\\ b\end{matrix}\;\middle|\; u\right] = \sum_{j=0}^{\infty} g_j\, u^j, \qquad g_j = \frac{(a)_j}{(b,1)_j}, \qquad \text{and} \tag{25}$$

$${}_{p'}F_{q'}\left[\begin{matrix}c\\ d\end{matrix}\;\middle|\; u\right] = \sum_{j=0}^{\infty} h_j\, u^j, \qquad h_j = \frac{(c)_j}{(d,1)_j}. \tag{26}$$

By Lemma 3.3 the polynomial P_n in the expansion

$${}_pF_q\left[\begin{matrix}a\\ b\end{matrix}\;\middle|\; xt^\lambda\right] {}_{p'}F_{q'}\left[\begin{matrix}c\\ d\end{matrix}\;\middle|\; yt^\mu\right] = \left(\sum_{j=0}^{\infty} g_j x^j t^{\lambda j}\right)\left(\sum_{j=0}^{\infty} h_j y^j t^{\mu j}\right) = \sum_{n=0}^{\infty} P_n(x,y) t^n \tag{27}$$

is of the form

$$P_n(x,y) = \begin{cases} 0, & K \le -1, \\ p^{R,K}(x,y), & K \ge 0, \end{cases} \tag{28}$$

where

$$\begin{aligned} p^{R,K}(x,y) &= \sum_{j=0}^{\lfloor K/\lambda \rfloor} \Big(g_{R+\mu j}\, x^{R+\mu j} \Big) \cdot \Big(h_{K-\lambda j}\, y^{K-\lambda j} \Big) \\ &= \Big(\sum_{j=0}^{\lfloor K/\lambda \rfloor} g_{R+\mu j} \cdot h_{K-\lambda j} \cdot \Big(\frac{x^\mu}{y^\lambda} \Big)^j \Big) \cdot x^R \cdot y^K. \end{aligned} \tag{29}$$

Thus the case $K \le -1$ is complete and we assume $K \ge 0$.

By making use of Lemma 3.7(i) we deduce from (25) that

$$\sum_{j=0}^{\infty} g_{R+j}\, u^j = \frac{(a)_R}{(b,1)_R} \cdot {}_{p+1}F_{q+1}\left[\begin{matrix} R+a, 1 \\ R+(b,1) \end{matrix} \,\Big|\, u \right], \tag{30}$$

and by Lemma 3.7(iii) we thus obtain

$$\begin{aligned} \sum_{j=0}^{\infty} g_{R+\mu j}\, u^j &= \frac{(a)_R}{(b,1)_R} \cdot {}_{\mu(p+1)}F_{\mu(q+2)-1}\left[\begin{matrix} \Delta_\mu(R+a,1) \\ \Delta_\mu(R+(b,1),1) \setminus (1) \end{matrix} \,\Big|\, \frac{u}{\mu^{\mu(1+q-p)}} \right] \\ &= \frac{(a)_R}{(b,1)_R} \cdot {}_{\mu p+1}F_{\mu(q+1)}\left[\begin{matrix} \Delta_\mu(R+a), 1 \\ \Delta_\mu(R+(b,1)) \end{matrix} \,\Big|\, \frac{u}{\mu^{\mu(1+q-p)}} \right], \end{aligned} \tag{31}$$

where in the last step we canceled out the parameters $\Delta_\mu(1) \setminus (1)$ that occurred both in the upper and in the lower parameters.

The assumption in Theorem 2.3 that c does not contain any $c_i \in \{0, -1, -2, \dots\}$ allows us to make use of Lemma 3.7(ii) to deduce from (26) that

$$\sum_{j=0}^{K} h_{K-j}\, u^j = \frac{(c)_K}{(d,1)_K} \cdot {}_{q'+2}F_{p'}\left[\begin{matrix} -K+1-(d,1), 1 \\ -K+1-c \end{matrix} \,\Big|\, (-1)^{p'+q'+1}\, u \right], \tag{32}$$

and by Lemma 3.7(iii) we thus obtain

$$\begin{aligned} \sum_{j=0}^{\lfloor K/\lambda \rfloor} h_{K-\lambda j}\, u^j &= \frac{(c)_K}{(d,1)_K} \cdot {}_{\lambda(q'+2)}F_{\lambda(p'+1)-1}\left[\begin{matrix} \Delta_\lambda(-K+1-(d,1),1) \\ \Delta_\lambda(-K+1-c,1) \setminus (1) \end{matrix} \,\Big|\, \frac{(-1)^{\lambda(p'+q'+1)}}{\lambda^{\lambda(1+p'-(q'+2))}} \cdot u \right] \\ &= \frac{(c)_K}{(d,1)_K} \cdot {}_{\lambda(q'+1)+1}F_{\lambda p'}\left[\begin{matrix} \Delta_\lambda(-K+1-(d,1)), 1 \\ \Delta_\lambda(-K+1-c) \end{matrix} \,\Big|\, (-\lambda)^{\lambda(1+q'-p')} \cdot u \right], \end{aligned} \tag{33}$$

where in the last step we canceled out the parameters $\Delta_\lambda(1) \setminus (1)$ that occurred both in the upper and in the lower parameters.

Thus with $p_1 = \lambda(q'+1) + \mu p$ and $q_1 = \lambda p' + \mu(q+1) - 1$ from the theorem and with

$$\mathcal{C} = \frac{(a)_R\, (c)_K}{(b,1)_R\, (d,1)_K}, \qquad \mathcal{M} = \frac{(-\lambda)^{\lambda(1+q'-p')}}{\mu^{\mu(1+q-p)}}, \tag{34}$$

we obtain by (31), (33) and Lemma 3.7(iv) that

$$\begin{aligned}
&\sum_{j=0}^{\lfloor K/\lambda \rfloor} g_{R+\mu j} \cdot h_{K-\lambda j} \cdot u^j \\
&= \mathcal{C} \cdot {}_{p_1+2}F_{q_1+2}\left[\begin{matrix} \Delta_\lambda(1-(d,1)-K), 1, \Delta_\mu(R+a), 1 \\ \Delta_\lambda(1-c-K,1), \Delta_\mu(R+(b,1),1), 1 \end{matrix} \,\middle|\, \mathcal{M}\,u\right] \qquad (35)\\
&= \mathcal{C} \cdot {}_{p_1+2}F_{q_1+2}\left[\begin{matrix} \mathcal{Q}, 1, 1 \\ \mathcal{R}, 1 \end{matrix} \,\middle|\, \mathcal{M}\,u\right] = \mathcal{C} \cdot {}_{p_1+1}F_{q_1+1}\left[\begin{matrix} \mathcal{Q}, 1 \\ \mathcal{R} \end{matrix} \,\middle|\, \mathcal{M}\,u\right].
\end{aligned}$$

Next, since $R \in \{0, \dots, \mu-1\}$ we have by Lemma 3.6(iii) that 1 belongs to $\Delta_\mu(R+1)$, hence 1 belongs to $\mathcal{R}$ and thus

$$ {}_{p_1+1}F_{q_1+1}\left[\begin{matrix} \mathcal{Q}, 1 \\ \mathcal{R} \end{matrix} \,\middle|\, \mathcal{M}\,u\right] = {}_{p_1}F_{q_1}\left[\begin{matrix} \mathcal{Q} \\ \mathcal{R}\setminus(1) \end{matrix} \,\middle|\, \mathcal{M}\,u\right]. \qquad (36)$$

By combining (28), (29), (35) and (36) with $u = x^\mu/y^\lambda$, we obtain for $K \geq 0$,

$$P_n(x,y) = p^{R,K}(x,y) = \mathcal{C} \cdot {}_{p_1}F_{q_1}\left[\begin{matrix} \mathcal{Q} \\ \mathcal{R}\setminus(1) \end{matrix} \,\middle|\, \mathcal{M}\,\frac{x^\mu}{y^\lambda}\right] \cdot x^R \cdot y^K. \qquad (37)$$

□

5. Examples for Theorem 2.3

The examples in this section illustrate Theorem 2.3 by using

$$(1-u)^{-a} = {}_1F_0\left[\begin{matrix} a \\ - \end{matrix} \,\middle|\, u\right], \qquad \exp(u) = {}_0F_0\left[\begin{matrix} - \\ - \end{matrix} \,\middle|\, u\right]. \qquad (38)$$

Let $\lambda, \mu = 1, 2, \dots$ such that $\gcd(\lambda, \mu) = 1$, and let $x, y \in \mathbb{C} \setminus \{0\}$.

Example 5.1. The expansion

$$ {}_pF_q\left[\begin{matrix} a \\ b \end{matrix} \,\middle|\, xt^\lambda\right] \cdot \exp(yt^\mu) = \sum_{n=0}^{\infty} P_n(x,y)t^n, \qquad \begin{matrix} a \in \mathbb{C}^p,\ b \in \mathbb{C}^q, \\ b_i \notin \{0,-1,-2,\dots\}, \end{matrix}$$

generates the polynomials

$$P_n(x,y) = \sum_{\substack{j,k=0,1,\dots \\ \lambda j + \mu k = n}} \frac{(a)_j}{(b)_j} \cdot \frac{x^j}{j!} \frac{y^k}{k!},$$

and for $n = 0, 1, 2, \dots$, with

$$\mathcal{M} = \frac{(-\lambda)^\lambda}{\mu^{\mu(1+q-p)}} \qquad \text{and} \qquad \mathcal{M}' = \frac{(-\mu)^{\mu(1+q-p)}}{\lambda^\lambda},$$

we have the following.

(i) Let $(R,K) = \mathcal{N}_{\lambda,\mu}(n)$. If $K \leq -1$, then $P_n(x,y) = 0$. If $K \geq 0$, then

$$P_n(x,y) = \frac{(a)_R}{(b)_R} \cdot {}_{\mu+\lambda p}F_{\lambda(q+1)-1}\left[\begin{matrix} \Delta_\lambda(-K), \Delta_\mu(R+a) \\ \Delta_\mu(R+(b,1))\setminus(1) \end{matrix} \,\middle|\, \mathcal{M}\,\frac{x^\mu}{y^\lambda}\right] \cdot \frac{x^R}{R!} \frac{y^K}{K!}.$$

(ii) Let $(R,K) = \mathcal{N}_{\mu,\lambda}(n)$. If $K \leq -1$, then $P_n(x,y) = 0$. If $K \geq 0$ and if a does not contain any $a_i \in \{0,-1,-2,\dots\}$, then

$$P_n(x,y) = \frac{(a)_K}{(b)_K} \cdot {}_{\mu(q+1)}F_{\mu p+\lambda-1}\left[\begin{matrix} \Delta_\mu(-K+1-(b,1)) \\ \Delta_\mu(-K+1-a), \Delta_\lambda(R+1)\setminus(1) \end{matrix} \,\middle|\, \mathcal{M}'\,\frac{y^\lambda}{x^\mu}\right] \cdot \frac{x^K}{K!} \frac{y^R}{R!}.$$

Proof. Note (38), apply Theorem 2.3 and Remark 2.4(ii). □

Remark 5.2. (i) For $\mu = 1$ in (i), we have $(R, K) = (0, n)$ and obtain the formula by Srivastava [35, equation (2.5)], [36, equation (3.9)], mentioned in §1. For $(\lambda, \mu) = (2, 1)$, it reduces to [4, equation (26)]. The case $(\lambda, \mu) = (1, 1)$ reduces to [4, equation (27)], [12, p. 267, formula (25)], [40, p. 166, problem 11].

(ii) In this example the polynomials $S_n(u) = P_n(1, u)$ satisfy $S_n'(u) = S_{n-\mu}(u)$, for $n \geq \mu$. For $\mu = 1$, the family $\{n! \cdot S_n\}$ thus extends the list of Appell sequences formed by hypergeometric polynomials [6, p. 552, Examples 12-15].

Example 5.3. The expansion

$$(1 - xt^\lambda)^{-a} \cdot (1 - yt^\mu)^{-c} = \sum_{n=0}^{\infty} P_n(x, y) t^n, \qquad a, c \in \mathbb{C},$$

generates the two-variable Erkuş-Srivastava polynomials

$$P_n(x, y) = \sum_{\substack{j,k=0,1,\ldots \\ \lambda j + \mu k = n}} (a)_j\, (c)_k \cdot \frac{x^j}{j!} \frac{y^k}{k!},$$

and for $n = 0, 1, 2, \ldots$, we have the following.

(i) Let $(R, K) = \mathcal{N}_{\lambda,\mu}(n)$. If $K \leq -1$, then $P_n(x, y) = 0$. If $K \geq 0$ and if $c \neq 0, -1, -2, \ldots$, then

$$P_n(x, y) = (a)_R\, (c)_K \cdot {}_{\lambda+\mu p}F_{\lambda p'+\mu-1}\left[\begin{matrix} \Delta_\lambda(-K), \Delta_\mu(R + a) \\ \Delta_\lambda(-K + 1 - c), \Delta_\mu(R + 1) \setminus (1) \end{matrix} \,\middle|\, \frac{x^\mu}{y^\lambda}\right] \cdot \frac{x^R}{R!} \frac{y^K}{K!}.$$

(ii) Let $(R, K) = \mathcal{N}_{\mu,\lambda}(n)$. If $K \leq -1$, then $P_n(x, y) = 0$. If $K \geq 0$ and if $a \neq 0, -1, -2, \ldots$, then

$$P_n(x, y) = (a)_K\, (c)_R \cdot {}_{\mu+\lambda p'}F_{\mu p+\lambda-1}\left[\begin{matrix} \Delta_\mu(-K), \Delta_\lambda(R + c) \\ \Delta_\mu(-K + 1 - a), \Delta_\lambda(R + 1) \setminus (1) \end{matrix} \,\middle|\, \frac{y^\lambda}{x^\mu}\right] \cdot \frac{x^K}{K!} \frac{y^R}{R!}.$$

Proof. Note (38), apply Theorem 2.3 and Remark 2.4(ii). □

Remark 5.4. The Erkuş-Srivastava polynomials [16] include as special cases the polynomials of the example and various other classes, see [1, 7, 13–15, 18, 30, 38, 41], [12, p. 247, formulas (16); p. 267, formulas (1) and (2)], [40, p. 442, equation (17)].

The next example is a special case of Example 5.1.

Example 5.5. The expansion

$$(1 - xt^\lambda)^{-a} \cdot \exp(yt^\mu) = \sum_{n=0}^{\infty} P_n(x, y) t^n, \qquad a \in \mathbb{C},$$

generates the polynomials

$$P_n(x, y) = \sum_{\substack{j,k=0,1,\ldots \\ \lambda j + \mu k = n}} (a)_j \cdot \frac{x^j}{j!} \frac{y^k}{k!},$$

and for $n = 0, 1, 2, \ldots$, we have the following.

(i) Let $(R, K) = \mathcal{N}_{\lambda,\mu}(n)$. If $K \leq -1$, then $P_n(x, y) = 0$. If $K \geq 0$, then

$$P_n(x, y) = (a)_R \cdot {}_{\lambda+\mu p}F_{\mu-1}\left[\begin{matrix} \Delta_\lambda(-K), \Delta_\mu(R + a) \\ \Delta_\mu(R + 1) \setminus (1) \end{matrix} \,\middle|\, (-\lambda)^\lambda \frac{x^\mu}{y^\lambda}\right] \cdot \frac{x^R}{R!} \frac{y^K}{K!}.$$

(ii) Let $(R,K)=\mathcal{N}_{\mu,\lambda}(n)$. If $K\le -1$, then $P_n(x,y)=0$. If $K\ge 0$ and if $a\ne 0,-1,-2,\dots$, then

$$P_n(x,y)=(a)_K\cdot {}_{\mu}F_{\mu p+\lambda-1}\left[\begin{matrix}\Delta_\mu(-K)\\ \Delta_\mu(-K+1-a),\Delta_\lambda(R+1)\setminus(1)\end{matrix}\,\middle|\,\frac{1}{\lambda^\lambda}\,\frac{y^\lambda}{x^\mu}\right]\cdot\frac{x^K}{K!}\,\frac{y^R}{R!}.$$

Proof. Note (38), apply Theorem 2.3 and Remark 2.4(ii). Alternatively, remove the parameter b from Example 5.1. ☐

Remark 5.6. For $\mu=1$, the class of polynomials in this example is a special case of generalized Lagrange-based Apostol-type polynomials [41, Definition 6 and equation (42)]. For $\lambda=\mu=1$, it reduces to the Charlier-type Laguerre polynomials $L_n^{(-n-\alpha)}$, see [11, p. 189, formula (19)].

Example 5.7. The expansion

$$\exp(xt^\lambda+yt^\mu)=\sum_{n=0}^{\infty}P_n(x,y)t^n,$$

generates the two-variable ordinary-type complete Bell polynomials, normalized as

$$P_n(x,y)=\sum_{\substack{j,k=0,1,\dots\\ \lambda j+\mu k=n}}\frac{x^j}{j!}\,\frac{y^k}{k!},$$

and for $n=0,1,2,\dots$, we have the following.

(i) Let $(R,K)=\mathcal{N}_{\lambda,\mu}(n)$. If $K\le -1$, then $P_n(x,y)=0$. If $K\ge 0$, then

$$P_n(x,y)={}_{\lambda}F_{\mu-1}\left[\begin{matrix}\Delta_\lambda(-K)\\ \Delta_\mu(R+1)\setminus(1)\end{matrix}\,\middle|\,\frac{(-\lambda)^\lambda}{\mu^\mu}\,\frac{x^\mu}{y^\lambda}\right]\cdot\frac{x^R}{R!}\,\frac{y^K}{K!}.$$

(ii) Let $(R,K)=\mathcal{N}_{\mu,\lambda}(n)$. If $K\le -1$, then $P_n(x,y)=0$. If $K\ge 0$, then

$$P_n(x,y)={}_{\mu}F_{\lambda-1}\left[\begin{matrix}\Delta_\mu(-K)\\ \Delta_\lambda(R+1)\setminus(1)\end{matrix}\,\middle|\,\frac{(-\mu)^\mu}{\lambda^\lambda}\,\frac{y^\lambda}{x^\mu}\right]\cdot\frac{x^K}{K!}\,\frac{y^R}{R!}.$$

Proof. Note (38), apply Theorem 2.3 and Remark 2.4(ii). In the result it amounts to remove the parameter a from Example 5.5. ☐

Remark 5.8. The polynomials in this example are solutions of the (λ,μ)-heat equation,

$$\frac{\partial^\lambda}{\partial y^\lambda}P_n(x,y)=\frac{\partial^\mu}{\partial x^\mu}P_n(x,y). \tag{39}$$

They are expressed by the usual (exponential-type) complete Bell polynomials Y_n as

$$P_n(x,y)=\frac{1}{n!}\cdot Y_n(\xi_1,\xi_2\dots),\qquad \xi_j=\begin{cases}x\cdot\lambda!, & j=\lambda,\\ y\cdot\mu!, & j=\mu,\\ 0, & \text{otherwise.}\end{cases} \tag{40}$$

If λ or μ equals 1, they reduce to the Gould-Hopper polynomials, such as for $\{\lambda,\mu\}=\{1,2\}$ the two-variable Hermite polynomials called Kampé de Fériet polynomials or heat polynomials, which can be expressed by the usual Hermite polynomials. We refer to [8, 9, 19, 21–25, 28, 34, 43], [11, p. 194, formula (19)], [40, p. 76], [42, p. 170, equation (5)].

6. A duality

By the next result we notice a duality that holds for the polynomials P_n in Theorem 2.3 and in the examples in §5.

Theorem 6.1. *For a formal power series $f(u,v)$, let P_n denote the polynomial in the expansion*

$$f(xt^\lambda, yt^\mu) = \sum_{n=0}^{\infty} P_n(x,y)\, t^n.$$

Then the following hold.

(i) *The polynomial P_n satisfies*

$$P_n(x^\lambda, y^\mu) = P_n(y^{-\mu}, x^{-\lambda})\, x^n\, y^n,$$

(ii) *The polynomials $Q_1(u) = P_n(u,1)$ and $Q_2(u) = P_n(1,u)$ satisfy*

$$Q_1(x^\lambda) = Q_2(x^{-\lambda})\, x^n \quad \text{and} \quad Q_2(x^\mu) = Q_1(x^{-\mu})\, x^n.$$

Proof. (i) Using the expansion with $(x,y,t) \mapsto (x^\lambda, y^\mu, t)$ yields

$$f(x^\lambda t^\lambda, y^\mu t^\mu) = \sum_{n=0}^{\infty} P_n(x^\lambda, y^\mu)\, t^n, \tag{41}$$

and using the expansion with $(x,y,t) \mapsto (y^{-\lambda}, x^{-\mu}, xyt)$ yields

$$f(x^\lambda t^\lambda, y^\mu t^\mu) = \sum_{n=0}^{\infty} P_n(y^{-\mu}, x^{-\lambda})\, x^n y^n t^n. \tag{42}$$

Since the generating functions in (41) and (42) coincide, also the series coefficients coincide.

(ii) Apply (i). $\square$

Remark 6.2. (i) For a direct verification that the duality described in Theorem 6.1 is indeed satisfied by P_n in Theorem 2.3, note that the identity $n = R\lambda + K\mu$ implies

$$(y^{-\lambda})^R\, (x^{-\mu})^K \cdot x^n\, y^n = y^{-R\lambda}\, x^{-K\mu} \cdot x^{R\lambda + K\mu}\, y^{R\lambda+K\mu} = (x^\lambda)^R\, (y^\mu)^K. \tag{43}$$

(ii) For λ or μ equal to 1, the duality of the theorem is in [17, equation (1.4)]. For $(\lambda,\mu) = (1,1)$, it reduces to the result that in this case the polynomials Q_1 and Q_2 are the reversed (or reciprocal) polynomials of each other [3, p. 32, Lemma 9.6].

7. First complement to Theorem 2.3

In this section by Theorem 7.3 we remove the restriction in Theorem 2.3 that c does not contain any $c_i \in \{0,-1,-2,\dots\}$.

For $(R,K) = \mathcal{N}_{\lambda,\mu}(n)$ and $N = 0,1,2,\dots$, we define the modified pair $(\widehat{R}, \widehat{K})$ by

$$\begin{pmatrix} \widehat{R} \\ \widehat{K} \end{pmatrix} = \begin{pmatrix} R \\ K \end{pmatrix} + T \cdot \begin{pmatrix} \mu \\ -\lambda \end{pmatrix}, \qquad \text{where } \; T = \lceil (K-N)/\lambda \rceil. \tag{44}$$

Remark 7.1. (i) An equivalent definition is

$$\widehat{K} = N - \operatorname{mod}(N-K, \lambda) \quad \text{and} \quad \widehat{R} = (n - \widehat{K}\mu)/\lambda$$

or also equivalently, if $\mu' \in \mathbb{Z}$ is an inverse of μ modulo λ,

$$\widehat{K} = N - \operatorname{mod}(N - \mu' \cdot n, \lambda) \quad \text{and} \quad \widehat{R} = (n - \widehat{K}\mu)/\lambda.$$

(ii) The numbers $(\widehat{R}, \widehat{K})$ are the unique pair of integers that satisfy

$$\widehat{K} \in N - \{0, 1, \ldots, \lambda - 1\} \quad \text{and} \quad \widehat{R}\lambda + \widehat{K}\mu = n.$$

(iii) Comparing (R, K) and $(\widehat{R}, \widehat{K})$ as sequences in n, observe the reversed behavior,

$$R \text{ periodic}, \ K \to \infty, \qquad \widehat{R} \to \infty, \ \widehat{K} \text{ periodic}, \qquad \text{for } n \to \infty.$$

The next lemma complements Lemma 3.3(i) by adding the constraint $\beta \leq N$. It thus complements Lemma 3.3(ii) for the case that the power series g terminates.

Lemma 7.2. (i) *In* Lemma 3.3(i) *we can add: Let* $N = 0, 1, \ldots$ *and*

$$\mathcal{W}_n = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \ \alpha, \beta = 0, 1, \ldots; \ \alpha\lambda + \beta\mu = n, \ \beta \leq N \right\}.$$

Then with $\widehat{R}, \widehat{K}$ *from* (44) *we have*

$$\mathcal{W}_n = \begin{cases} \varnothing & K \leq -1, \\ \mathcal{V}^{R,K} & 0 \leq K \leq N, \\ \mathcal{V}^{\widehat{R},\widehat{K}} & K \geq N+1. \end{cases}$$

(ii) *In* Lemma 3.3(ii) *we can add: If the series* $g(u)$ *terminates and is a polynomial* $g(u) = \sum_{n=0}^{N} g_n u^n$ *of degree* N*, then with* $\widehat{R}, \widehat{K}$ *from* (44) *we also have*

$$p_n = \begin{cases} 0, & K \leq -1, \\ p^{R,K}, & 0 \leq K \leq N, \\ p^{\widehat{R},\widehat{K}}, & K \geq N+1. \end{cases}$$

Proof. (i) Add the constraint $\beta \leq N$ to Lemma 3.3(i). From the equivalence

$$\beta \leq N \quad \Leftrightarrow \quad K - \gamma\lambda \leq N \quad \Leftrightarrow \quad \gamma \geq (K - N)/\lambda \tag{45}$$

we see that for $K \leq N$, the condition $\beta \leq N$ is redundant, since it follows from $\gamma \geq 0$ in this case. Hence, we assume that $K \geq N + 1$. In this case we add the restriction on γ from (45) to the result of Lemma 3.3(i) and obtain for $T = \lceil (K - N)/\lambda \rceil$,

$$\begin{aligned} \mathcal{W}_n &= \left\{ \begin{pmatrix} R \\ K \end{pmatrix} + \gamma \cdot \begin{pmatrix} \mu \\ -\lambda \end{pmatrix} : \ \gamma = T, \ldots, \lfloor K/\lambda \rfloor \right\} \\ &= \left\{ \begin{pmatrix} R \\ K \end{pmatrix} + (T + \gamma) \cdot \begin{pmatrix} \mu \\ -\lambda \end{pmatrix} : \ \gamma = 0, 1, \ldots, \lfloor K/\lambda \rfloor - T \right\} \\ &= \left\{ \begin{pmatrix} R + T\mu \\ K - T\lambda \end{pmatrix} + \gamma \cdot \begin{pmatrix} \mu \\ -\lambda \end{pmatrix} : \ \gamma = 0, 1, \ldots, \lfloor (K - T\lambda)/\lambda \rfloor \right\} \\ &= \left\{ \begin{pmatrix} \widehat{R} \\ \widehat{K} \end{pmatrix} + \gamma \cdot \begin{pmatrix} \mu \\ -\lambda \end{pmatrix} : \ \gamma = 0, 1, \ldots, \lfloor \widehat{K}/\lambda \rfloor \right\} = \mathcal{V}^{\widehat{R},\widehat{K}}. \end{aligned} \tag{46}$$

(ii) Repeat the proof of Lemma 3.3(ii) but instead of Lemma 3.3(i) use Lemma 7.2(i). $\square$

The next theorem is the goal of this section.

Theorem 7.3. *In Theorem 2.3 we can add: If* $Z = c \cap \{0, -1, -2, \dots\} \neq \varnothing$*, such that the second factor is a polynomial of degree* $N = \min|Z|$*, then with* $\widehat{R}, \widehat{K}$ *from (44) we have*

$$P_n = \begin{cases} 0 & K \leq -1, \\ P^{R,K} & 0 \leq K \leq N, \\ P^{\widehat{R},\widehat{K}} & K \geq N+1, \end{cases}$$

where $(R, K) \mapsto P^{R,K}$ *denotes the construction of* P_n *in Theorem 2.3, depending on* (R, K)*.*

Proof. Repeat the proof of Theorem 2.3 but instead of Lemma 3.3(ii) use Lemma 7.2(ii), such that for $K \geq N+1$, we have that K and R are replaced by $\widehat{K}$ and $\widehat{R}$, respectively. Only the argument for (36), which states that 1 belongs to $\mathcal{R}$, has to be adjusted, since the argument that $\Delta_\mu(R+1)$ contains 1 does not work for R replaced by $\widehat{R}$. The adjusted argument is that $\Delta_\lambda(-\widehat{K}+1-c)$ contains 1. In fact, since Remark 7.1 implies

$$N - \widehat{K} \in \{0, 1, \dots, \lambda - 1\}, \tag{47}$$

we have by Lemma 3.6(iii) that 1 belongs to $\Delta_\mu(N - \widehat{K} + 1)$. Since $-N$ belongs to c by assumption, it implies that 1 belongs to $\Delta_\mu(-\widehat{K} - c + 1)$ and thus 1 belongs to $\mathcal{R}$. □

Remark 7.4. (i) The parameters of $P^{\widehat{R},\widehat{K}}$ can be obtained from the parameters of $P^{R,K}$ by the modification $\mathcal{Q} \mapsto \mathcal{Q} + T$ and $\mathcal{R} \mapsto \mathcal{R} + T$.

(ii) In Lemma 7.2 and Theorem 7.3 the complementary conditions $K \leq N$ and $K \geq N+1$ can be replaced by overlapping conditions $K \leq N$ and $K \geq N$, since both expressions coincide for $K = N$.

(iii) Theorem 7.3 applies if the second factor is a polynomial, regardless of whether the first factor is also a polynomial. We notice if $Z' = a \cap \{0, -1, -2, \dots\} \neq \varnothing$, such that the first factor is also a polynomial and its degree is $N' = \min|Z'|$, then $P_n = P^{\widehat{R},\widehat{K}} = 0$, for $\widehat{R} \geq N' + 1$. This observation can be obtained by adding the constraint $\alpha \leq N'$ in Lemma 7.2.

8. Second complement to Theorem 2.3

The restriction in Theorem 2.3 that

$$c \text{ does not contain any } c_i \in \{0, -1, -2, \dots\} \tag{48}$$

was removed in the previous section by Theorem 7.3 that works by separating $K \leq N$ from $K \geq N+1$. In the present section we point out an alternative result that works with the dual restriction that

$$c \text{ does not contain any } c_i \in \{1, 2, 3, \dots\}, \tag{49}$$

such that the separation is not always necessary. The alternative result, Theorem 8.4 below is obtained by using the regularized version of a hypergeometric function [33, p. 405, Definition 16.2.5], [5, p. 1375],

$$ {}_p\widetilde{F}_q\left[\begin{matrix} a \\ b \end{matrix} \,\middle|\, x\right] = \sum_{k=0}^{\infty} \frac{(a)_k}{\Gamma(b+k)} \frac{x^k}{k!}, \quad a \in \mathbb{C}^p, \quad b \in \mathbb{C}^q, \tag{50}$$

where $\Gamma(a) = \Gamma(a_1) \cdots \Gamma(a_p)$, for $a = (a_1, \dots, a_p)$.

Remark 8.1. In the definition (50) and in other results below where the Gamma function appears in the denominator of a fraction, it is to be considered as a part of the reciprocal Gamma function.

While $\Gamma(x)$ is undefined for $x \in \{0,-1,-2,\dots\}$, its reciprocal $1/\Gamma(x)$ is an entire function by letting

$$1/\Gamma(x) = 0, \qquad \text{for} \;\; x \in \{0,-1,-2,\dots\}. \tag{51}$$

Definition 8.2. For $\lambda = 1,2,\dots$, define

$$\sigma_\lambda(a) = \sqrt{\frac{(2\pi)^{(\lambda-1)p}}{\lambda^{2(a_1+\dots+a_p)-p}}}, \qquad a=(a_1,\dots,a_p) \in \mathbb{C}^p.$$

The next lemma is a multivariable Gauss multiplication formula.

Lemma 8.3. *Let $\lambda = 1,2,\dots$. Then*

$$\frac{1}{\Gamma\big(\Delta_\lambda(a)\big)} = \frac{1}{\sigma_\lambda(a)} \cdot \frac{1}{\Gamma(a)}, \qquad a \in \mathbb{C}^p.$$

for $\sigma_\lambda(a)$ from Definition 8.2.

Proof. Since the formula is multiplicative in the parameters, we can assume $p=1$.

For $a \neq 0,-1,-2,\dots$, the identity expresses the Gauss multiplication formula [33, p. 138, formula 5.5.6], [32, p. 7, formula (1.2.6)] in reciprocal form.

For $a = 0,-1,-2,\dots$, we have by Lemma 3.6(ii) that also $\Delta_\lambda(a) \in \{0,-1,-2,\dots\}$. Hence by Remark 8.1 the identity states that $0 = (1/\sigma_\lambda(a)) \cdot 0$, which is true. ☐

The next theorem is the goal of this section.

Theorem 8.4. *In Theorem 2.3, if a does not contain any $c_i \in \{0,1,2,\dots\}$, then for*

$$\mathcal{C}_1 = (-1)^{p'K} \cdot \frac{(a)_R\, \Gamma(b)\, \Gamma(1-c)}{(d)_K\, K!} \cdot \sigma_\lambda(-K+1-c)\, \sigma_\mu(R+(b,1)),$$

we have

$$P_n(x,y) = \mathcal{C}_1 \cdot {}_{p_1}\widetilde{F}_{q_1}\left[\begin{matrix}\mathcal{Q}\\ \mathcal{R}\setminus(1)\end{matrix}\;\middle|\; \mathcal{M}\,\frac{y^\lambda}{x^\mu}\right] \cdot x^R y^K.$$

Proof. Case I. Suppose that c does not contain any $c_i \in \{0,-1,-2,\dots\}$. Then for

$$\mathcal{C} = \frac{(a)_K\,(c)_R}{(b,1)_K\,(d,1)_R} = \frac{(a)_K\,(c)_R}{(b)_K\,(d)_R} \cdot \frac{1}{R!}\,\frac{1}{K!}, \tag{52}$$

we have by Theorem 2.3 that

$$P_n(x,y) = \mathcal{C} \cdot {}_{p_1}F_{q_1}\left[\begin{matrix}\mathcal{Q}\\ \mathcal{R}\setminus(1)\end{matrix}\;\middle|\; \mathcal{M}\,\frac{y^\lambda}{x^\mu}\right] \cdot x^R y^K. \tag{53}$$

The general assumption from (1) that b,d do not contain any $b_i, d_i \in \{0,-1,-2,\dots\}$ and the assumption in Theorem 8.4 that c does not contain any $c_i \in \{1,2,\dots\}$, imply that $\Gamma(b)$ and $\Gamma(1-c)$ exist and that $(d)_K$ does not contain 0, so we conclude that $\mathcal{C}_1$ is well-defined.

For the next computation, recall Remark 8.1 on the reciprocal Gamma function. By using Lemma 3.5(i) and Lemma 8.3 we compute

$$\begin{aligned}
\mathcal{C} &= \frac{(a)_R\,(c)_K}{(b,1)_R\,(d,1)_K} = (-1)^{p'K}\frac{(a)_R\,(-K+1-c)_K}{(b,1)_R\,(d,1)_K}\\
&= (-1)^{p'K}\frac{(a)_R\,\Gamma(b,1)\,\Gamma(1-c)}{(d,1)_K} \cdot \frac{1}{\Gamma\big(-K+1-c\big)\cdot\Gamma(R+(b,1))}\\
&= (-1)^{p'K}\frac{(a)_R\,\Gamma(b,1-c)}{(d,1)_K} \cdot \frac{\sigma_\lambda\big(-K+1-c\big)\cdot\sigma_\mu(R+(b,1))}{\Gamma\big(\Delta_\lambda(-K+1-c)\big)\cdot\Gamma\big(\Delta_\mu(R+(b,1))\big)} = \frac{\mathcal{C}_1}{\Gamma(\mathcal{R})}.
\end{aligned} \tag{54}$$

Combining (53) and (54) we conclude that

$$
\begin{aligned}
P_n(x,y) &= \mathcal{C} \cdot {}_{p_1}F_{q_1}\left[\begin{matrix}\mathcal{Q}\\ \mathcal{R}\setminus(1)\end{matrix}\,\middle|\,\mathcal{M}\,\frac{y^\lambda}{x^\mu}\right]\cdot x^R y^K \\
&= \mathcal{C}_1 \cdot \left(\frac{1}{\Gamma(\mathcal{R})}\cdot {}_{p_1}F_{q_1}\left[\begin{matrix}\mathcal{Q}\\ \mathcal{R}\setminus(1)\end{matrix}\,\middle|\,\mathcal{M}\,\frac{y^\lambda}{x^\mu}\right]\right)\cdot x^R y^K \\
&= \mathcal{C}_1 \cdot {}_{p_1}\widetilde{F}_{q_1}\left[\begin{matrix}\mathcal{Q}\\ \mathcal{R}\setminus(1)\end{matrix}\,\middle|\,\mathcal{M}\,\frac{y^\lambda}{x^\mu}\right]\cdot x^R y^K,
\end{aligned} \tag{55}
$$

where in the last step we used the expression for ${}_{p_1}\widetilde{F}_{q_1}$ by ${}_{p_1}F_{q_1}$ from [33, p. 405, equation 16.2.5]. Note $\Gamma(\mathcal{R}\setminus(1)) = \Gamma(\mathcal{R})$, since $\Gamma(1) = 1$.

Case II. Suppose that c contains one or more parameters $c_i \in \{0,-1,-2,\dots\}$. In this case $\mathcal{C}\cdot {}_{p_1}F_{q_1} = 0\cdot\infty =$ undefined, for large K. But $\mathcal{C}_1$ and ${}_{p_1}\widetilde{F}_{q_1}$ are well-defined and thus Case II follows from Case I by the continuity of the mappings

$$
c \;\mapsto\; \mathcal{C}_1 \quad\text{and}\quad c \;\mapsto\; {}_{p_1}\widetilde{F}_{q_1}\left[\begin{matrix}\mathcal{Q}\\ \mathcal{R}\setminus(1)\end{matrix}\,\middle|\,\mathcal{M}\,\frac{y^\lambda}{x^\mu}\right]. \tag{56}
$$

Note that while c is varying, the function ${}_{p_1}\widetilde{F}_{q_1}$ remains a polynomial of degree at most $k_0 = \lfloor K/\lambda\rfloor$ by the argument discussed for ${}_{p_1}F_{q_1}$ in Remark 2.4(iii). □

9. Final remarks

The next remark summarizes the scope of Theorem 2.3, Theorem 7.3 and Theorem 8.4. For the swapping of the factors that is mentioned in the remark, see Remark 2.4(ii).

Remark 9.1. (i) If c does not contain any $c_i \in \{0,-1,-2,\dots\}$ or equivalently, if the second factor is not a polynomial, then Theorem 2.3 can be used.

(ii) If a does not contain any $a_i \in \{0,-1,-2,\dots\}$ or equivalently, if the first factor is not a polynomial, then Theorem 2.3 can be used after swapping the factors.

(iii) If c contains one or more $c_i \in \{0,-1,-2,\dots\}$, then Theorem 7.3 can be used.

(iv) If a contains one or more $a_i \in \{0,-1,-2,\dots\}$, then Theorem 7.3 can be used after swapping the factors.

(v) If c does not contain any $c_i \in \{1,2,3,\dots\}$, then also Theorem 8.4 can be used.

(vi) If a does not contain any $a_i \in \{1,2,3,\dots\}$, then also Theorem 8.4 can be used after swapping the factors.

Observe that the combination of (i) and (iii), (or (ii) and (iv)), covers the general case of arbitrary a and c.

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