Note on B-Splines, Wavelet Scaling Functions, and Gabor Frames

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***Abstract*—Let g be a continuous, compactly supported function on $\mathbb{R}$ such that the integer translates of g constitute a partition of unity. We show that the Gabor system (g, a, b), with window g and time-shift and frequency-shift parameters $a, b > 0$ has no lower frame bound larger than 0 if $b = 2, 3, \ldots$ and $a > 0$. In particular, (g, a, b) is not a Gabor frame if g is a continuous, compactly supported wavelet scaling function and if $b = 2, 3, \ldots$ and $a > 0$. We give an example for our result for the case that $g = B_1$, the triangle function supported by $[-1, 1]$, by showing pictures of the canonical dual corresponding to (g, a, b) where $ab = 1/4$ and b crosses the lines $N = 2, 3, \ldots$.**

***Index Terms*—B-splines, Gabor frame, partition of unity, Ron–Shen condition, wavelet scaling function.**

I. Introduction

Let $a > 0$, $b > 0$, and $g \in L^2(\mathbb{R})$. We call the function system

$$\left(g_{na,mb}\right)_{n,m\in\mathbb{Z}} \equiv (g, a, b) \tag{1}$$

a Gabor frame if there are $A > 0$, $B < \infty$ such that for all $f \in L^2(\mathbb{R})$

$$A\|f\|^2 \le \sum_{n,m} \left|\left\langle f, g_{na,mb}\right\rangle\right|^2 \le B\|f\|^2. \tag{2}$$

Here $g_{x,y}$ denotes for $x, y \in \mathbb{R}$ the time–frequency-shifted function

$$g_{x,y}(t) = e^{2\pi i y t} g(t - x), \qquad t \in \mathbb{R}. \tag{3}$$

The numbers A and B that appear in (2) are called the lower and the upper frame bound, respectively. It is well known that (g, a, b) can be a Gabor frame only if $ab \le 1$; also, if $ab = 1$ and (g, a, b) is a Gabor frame, then g cannot be continuous and compactly supported. We refer for basic information about (Gabor) frames to [1, Sec. 3.4, 3.5, 4.1, and 4.2]; a comprehensive and recent treatment of Gabor systems and frames can be found in [2, Ch. 5–9, 11–13]. We shall use here the following criterion, due to Ron and Shen [3], for being a Gabor frame, see [2, p. 117, Proposition 6.3.4]; for convenience, we restrict ourselves to continuous, compactly supported windows g. The system (g, a, b) is a Gabor frame with frame bounds $A > 0$, $B < \infty$ if and only if for all $c \in l^2(\mathbb{Z})$ and all $t \in \mathbb{R}$

$$A\|c\|^2 \le \frac{1}{b} \sum_{n=-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty} g(t - na - k/b) c_k \right|^2 \le B\|c\|^2. \tag{4}$$

For $g \in L^2(\mathbb{R})$, we denote by F_g the set

$$F_g = \Big\{ (a, b) \mid a > 0,\ b > 0,\ (g, a, b) \text{ is a Gabor frame} \Big\}. \tag{5}$$

It is often quite hard to determine the set F_g for a given window $g \in L^2(\mathbb{R})$. In certain cases, for instance when one has restrictions on the supporting set of g, the Ron–Shen criterion can be of great help in telling whether (a, b) belongs to F_g. In [4, Sec. 3], a considerable effort has been made to determine F_g for the case that $g = B_0 = \chi_{[0,1)}$; the result is a complicated subset of $\{ (a, b) \mid a > 0,\ b > 0,\ ab \le 1\}$ where (ir)rationality of ab plays a key role. Furthermore, only in a few cases of well-behaved windows g it has been shown that $F_g = \{ (a, b) \mid a > 0,\ b > 0,\ ab < 1\}$; among these g are the Gaussians [5], [6], hyperbolic secants [7] and two-sided exponentials [8, Sec. 5].

In this correspondence, we ask the question whether for certain standard windows g from approximation theory and wavelet theory the set F_g consists of all (a, b) with $a > 0$, $b > 0$, and $ab < 1$ as well. Unlike the example $g = B_0$ given earlier, the windows of this type are smooth and well decaying, which implies that the sets F_g are open sets [9]. We shall show in Section II that for any continuous, compactly supported g satisfying the partition-of-unity identity

$$\sum_{k=-\infty}^{\infty} g(t - k) = 1, \qquad t \in \mathbb{R} \tag{6}$$

no lower frame bound $A > 0$ for the Gabor system (g, a, b) exists when $a > 0$ and $b = 2, 3, \ldots$. Condition (6) can be shown to hold for large classes of windows; in particular, it is satisfied for some commonly used windows in signal processing, such as the raised cosine

$$\mathrm{RC}\,(t) = \begin{cases} (1 + \cos \pi t)/2, & \text{when } |t| \le 1 \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

and the trapezoidal function, for $0 < \delta < 1/2$ defined as

$$\mathrm{T}(t) = \begin{cases} 1, & |t| \le 1/2 - \delta \\ (2\delta)^{-1}(1/2 + \delta - |t|), & \Big| |t| - 1/2 \Big| < \delta \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Further, condition (6) on g is satisfied when g is a compactly supported, continuous scaling function from the theory of wavelets [1, Remark 5 on pp. 144–145 and Note 9 on p. 165], or when g is a B-spline with knots at the integers [10]. Hence, in these cases, the set F_g consists apparently of countably many open sets, separated from one another by the horizontal lines $b = N = 2, 3, \ldots$.

This result came as a big surprise to us, because it is generally assumed that any "nice" window g and "reasonable" choice of $a, b > 0$ yield Gabor frames. Our observation demonstrates that even for window functions that are perfectly natural in approximation theory and wavelet theory, a and b have to be chosen extremely carefully.

In Section III, we consider the linear B-spline or triangle function $g(t) = B_1(t) = \max(0, 1 - |t|)$ as an example. We show pictures of the canonical dual function corresponding to the Gabor system (g, a, b) with $ab = 1/4$ and b close to 2 and 3. It is conjectured that F_g consists

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of all (a, b) with $0 < a < 2$, $b > 0$, $ab < 1$, and $b \neq 2, 3, \ldots$. The pictures in Section III support this conjecture.

II. Proof of the Main Result

We assume that g is continuous and compactly supported (later we comment on weakening those conditions), and that g satisfies (6). Also, we let $a > 0$ and $b = 2, 3, \ldots$. We shall show that the Gabor system (g, a, b) has no positive lower frame bound (the Ron–Shen criterion (4) implies that (g, a, b) has a finite upper frame bound). To that end, we shall display $c^K \in l^2(\mathbb{Z})$, $K = 1, 2, \ldots$, such that for all $t \in \mathbb{R}$

$$\frac{1}{\|c^K\|^2} \sum_{n=-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty} g(t - na - k/N)\, c_k^K \right|^2 \to 0, \qquad K \to \infty\,. \tag{9}$$

Let $b = N = 2, 3, \ldots$ and set

$$c_k = e^{2\pi i k r/N}, \qquad k \in \mathbb{Z}, \tag{10}$$

where $r = 1, \ldots, N-1$. Then for all $t \in \mathbb{R}$, $n \in \mathbb{Z}$, we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty} & g(t - na - k/N)\, c_k \\ &= \sum_{k=0}^{N-1} \sum_{l=-\infty}^{\infty} g\Big(t - na - \tfrac{k}{N} - l\Big) e^{2\pi i r(\frac{k}{N}+l)} \\ &= \sum_{k=0}^{N-1} e^{2\pi i k r/N} = 0.\end{aligned} \tag{11}$$

For $K = 1, 2, \ldots$ we define the sequence $c^K \in l^2(\mathbb{Z})$ by

$$c_k^K = \begin{cases} c_k, & |k| \le K, \\ 0, & |k| > K, \end{cases} \quad k \in \mathbb{Z} \tag{12}$$

and the subsets of indexes

$$\begin{aligned} V_k &= \Big\{ l \in \mathbb{Z} \mid g(t - na - \tfrac{k}{N} - l) \neq 0 \Big\} \\ W_k &= \Big\{ l \in \mathbb{Z} \mid c_{k+lN}^K \neq 0 \Big\}. \end{aligned} \tag{13}$$

Then the equality $\sum_k g(t - na - \frac{k}{N})\, c_k^K = 0$ holds for all $t \in \mathbb{R}$, $n \in \mathbb{Z}$ whenever either

$$V_k \subset W_k, \qquad k = 0, \ldots, N-1 \tag{14}$$

or

$$V_k \cap W_k = \emptyset, \qquad k = 0, \ldots, N-1\,. \tag{15}$$

Conversely, for $t \in \mathbb{R}$, $n \in \mathbb{Z}$ both (14) and (15) fail to be true if and only if $W_k \setminus V_k \neq \emptyset$, i.e., if there are $j_1, j_2 \in \mathbb{Z}$ such that

$$g(t - na - \tfrac{j_1}{N}) \neq 0 \neq g(t - na - \tfrac{j_2}{N}), \qquad c_{j_1}^K = 0 \neq c_{j_2}^K\,. \tag{16}$$

Let $|\mathrm{I}|$ be the length of a supporting interval I of g. Then the number of $n \in \mathbb{Z}$ such that both (14) and (15) fail to hold is at most $2(|\mathrm{I}|/a + 1)$ for any $t \in \mathbb{R}$. Hence,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} & \left| \sum_{k=-\infty}^{\infty} g(t - na - k/N) c_k^K \right|^2 \\ &\le 2(|\mathrm{I}|/a + 1) \sum_{k=-\infty}^{\infty} |g(t - na - k/N)|^2 \\ &\le 2(|\mathrm{I}|/a + 1)(MN(|\mathrm{I}| + 1))^2 \end{aligned} \tag{17}$$

where M is an upper bound for $|g|$. Since (17) is independent of K and since $\|c^K\| = (2K+1)^{1/2}$, we obtain (9) for any $t \in \mathbb{R}$.

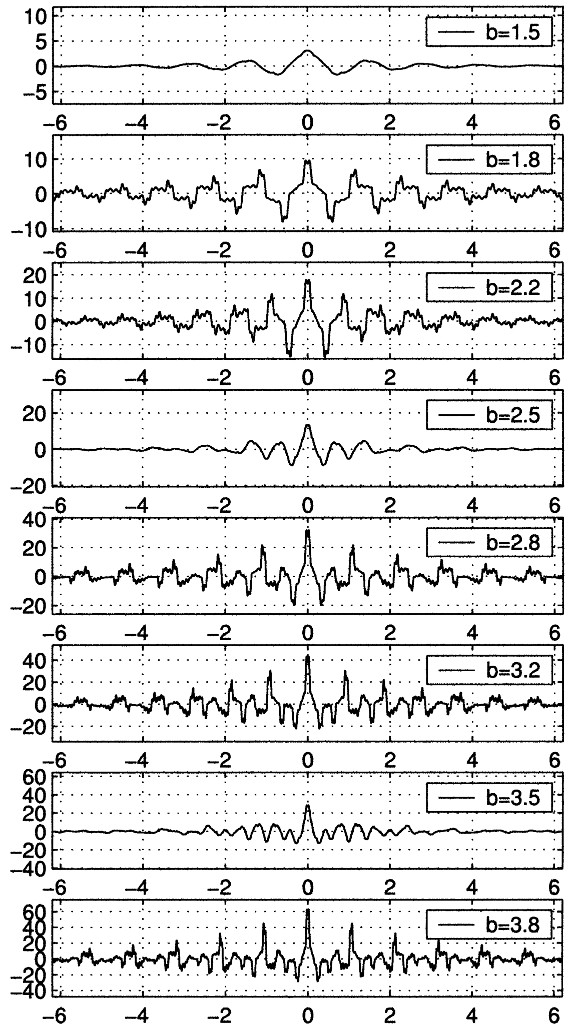

Fig. 1. The canonical Gabor dual $\gamma_{a,b} = S_{a,b}^{-1} g$ of the triangle function $g = B_1$, see (19), for $ab = 1/4$, $b = 1.5, 1.8, 2.2, 2.5, 2.8, 3.2, 3.5, 3.8$.

Remark

The property (11) with c_k given in (10) can be phrased as $(Zg)(s, r/N) = 0$ for $s \in \mathbb{R}$, $r = 1, \ldots, N-1$. Here

$$(Zg)(s, \nu) = N^{-1/2} \sum_{k=-\infty}^{\infty} g\left(\frac{s-k}{N}\right) e^{2\pi i k \nu}, \qquad s, \nu \in \mathbb{R} \tag{18}$$

is a Zak transform of g, see [11, Sec. 1.5]. Using Gabor frame operator theory in the Zak transform domain, one can get the main result under considerably weaker conditions on g than the ones made previously.

III. Example

In this section, we consider the choice

$$g(t) = B_1(t) = \max(0, 1 - |t|), \qquad t \in \mathbb{R} \tag{19}$$

and we display the canonical dual $\gamma_{a,b} = S_{a,b}^{-1} g$, see [2, Sec. 7.6] for the role of the canonical dual in Gabor analysis, for some values of $a > 0$, $b > 0$, with $ab = 1/4$ and b near 2 or 3 (Fig. 1). Here $S_{a,b}$ is the frame operator corresponding to (g, a, b), which is invertible when (g, a, b) is a frame. One easily sees from the Ron–Shen criterion in (4) or [2, Theorem 6.4.1] that (g, a, b) is a frame when $a < 2 \le 1/b$. For the other cases that are shown in the figure, we have only numerical evidence that the lower frame bound A for (g, a, b) is positive. The dual windows were approximated by considering sampled Gabor systems for $l^2(\mathbb{Z})$ and their dual systems, see [12], [13] for details, with sample rates taken so high that further increasing them produced no visible changes in the figure. As can be seen, the $\gamma_{a,b}$ obtained in that way turns from a well-behaved function for the values $b = 1.5, 2.5$ into a quite irregularly behaved one when b approaches 2 or 3.

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Sparse Representations in Unions of Bases

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***Abstract*—The purpose of this correspondence is to generalize a result by Donoho and Huo and Elad and Bruckstein on sparse representations of signals in a union of two orthonormal bases for $\mathbb{R}^N$. We consider general (redundant) dictionaries for $\mathbb{R}^N$, and derive sufficient conditions for having unique sparse representations of signals in such dictionaries. The special case where the dictionary is given by the union of $L \geq 2$ orthonormal bases for $\mathbb{R}^N$ is studied in more detail. In particular, it is proved that the result of Donoho and Huo, concerning the replacement of the ℓ^0 optimization problem with a linear programming problem when searching for sparse representations, has an analog for dictionaries that may be highly redundant.**

***Index Terms*—Dictionaries, Grassmannian frames, linear programming, mutually incoherent bases, nonlinear approximation, sparse representations.**

I. INTRODUCTION

We consider vectors (also referred to as signals) in $\mathcal{H} = \mathbb{R}^N$ (resp., $\mathcal{H} = \mathbb{C}^N$). The goal is to find an efficient representation of a signal $s \in \mathcal{H}$. One well-known way to do this is to take an orthonormal basis $\Phi = \{\phi_1, \ldots, \phi_N\}$ for $\mathcal{H}$ and use the Fourier coefficients $\{\langle s, \phi_k\rangle\}_{k=1}^N$ to represent s. This approach is simple and works reasonably well in many cases. However, one can also consider a more general type of expansion where the orthonormal basis is replaced by a so-called dictionary for $\mathcal{H}$.

Definition 1: A **dictionary** in $\mathcal{H} = \mathbb{R}^N$ (resp., $\mathcal{H} = \mathbb{C}^N$) is a family of $K \geq N$ unit (column) vectors $\{g_k\}$ that spans $\mathcal{H}$. We will use the matrix notation $\boldsymbol{D} = [g_1, \ldots, g_K]$ for a dictionary.

By a representation of s in $\boldsymbol{D}$ we mean a (column) vector $\alpha = (\alpha_k) \in \mathbb{R}^K$ (resp., in $\mathbb{C}^K$) such that $s = \boldsymbol{D}\alpha$. We notice that when $K > N$, the vectors of $\boldsymbol{D}$ are no longer linearly independent and the representation of s is not unique. The hope is that among all possible representations of s there is a *very sparse* representation, i.e., a representation with few nonzero coefficients. The tradeoff is that we have to *search* all possible representations of s to find the sparse representations, and then determine whether there is a unique sparsest representation. Following [1] and [2], we will measure the sparsity of a representation $s = \boldsymbol{D}\alpha$ by two quantities: the ℓ^0 and the ℓ^1 norm of α, respectively (the ℓ^0-norm simply counts the number of nonzero entries of a vector). This leads to the following two minimization problems to determine the sparsest representation of s:

$$\text{minimize} \quad \|\alpha\|_0 \quad \text{subject to } s = \boldsymbol{D}\alpha \tag{1}$$

and

$$\text{minimize} \quad \|\alpha\|_1 \quad \text{subject to } s = \boldsymbol{D}\alpha. \tag{2}$$

It turns out that the optimization problem (2) is much easier to handle than (1) through the use of linear programming (LP), so it is important to know the relationship between the solution(s) of (1) and (2), and to determine sufficient conditions for the two problems to have the same unique solution. This problem has been studied in detail in [1] and later has been refined in [2] in the special case where the dictionary $\boldsymbol{D}$ is the union of *two* orthonormal bases. In what follows, we generalize the results of [1] and [2] to arbitrary dictionaries.[1] The case where $\boldsymbol{D}$ is the union of $L \geq 2$ orthonormal bases for $\mathcal{H}$ is studied in detail. This leads to a natural generalization of the recent results from [2] valid for $L = 2$.

In Section II, we provide conditions for a solution α of the problem

$$\text{minimize} \quad \|\alpha\|_\tau \quad \text{subject to } s = \boldsymbol{D}\alpha \tag{3}$$

to be indeed the unique solution, with $0 \leq \tau \leq 1$ and an arbitrary dictionary $\boldsymbol{D}$. We put a special emphasis on sufficient conditions of the type $\|\alpha\|_0 < f(\boldsymbol{D})$ and prove a sufficient condition for $\tau \in \{0, 1\}$ with $f(\boldsymbol{D}) = (1 + 1/M(\boldsymbol{D}))/2$ where

$$M(\boldsymbol{D}) := \max_{k \neq k'} |\langle g_k, g_{k'}\rangle| \tag{4}$$

is the **coherence** of the dictionary. The special case where $\boldsymbol{D}$ is the union of $L \geq 2$ bases is studied in Section III, leading to explicit sufficient conditions for $\tau = 0$ with

$$f(\boldsymbol{D}) = \left(1/2 + \frac{1}{2(L-1)}\right) / M(\boldsymbol{D})$$

and for $\tau \in \{0, 1\}$ with

$$f(\boldsymbol{D}) = \left(\sqrt{2} - 1 + \frac{1}{2(L-1)}\right) / M(\boldsymbol{D}).$$

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[1] Parallel work done independently by Donoho and Elad [3] also addresses the question of generalizing previous results to general dictionaries. Though there are some similarities between this work to the work in [3], a somewhat different perspective on the problem is adopted and the proofs use different techniques.