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Operators Commuting with a Discrete Subgroup of Translations

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ABSTRACT. We study the structure of operators from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ into the tempered distributions $\mathcal{S}'(\mathbb{R})^n$ that commute with a discrete subgroup of translations. The formalism leads to simple derivations of recent results about the frame operator of shift-invariant systems, Gabor, and wavelet frames.

1. Introduction and main result

We investigate shift-invariant operators, i.e., operators that commute with a discrete group of translation operators. Special classes of such operators have received considerable attention in the context of approximation theory and shift-invariant systems, Gabor analysis, and wavelet frames; see Section 4. In the terminology of operator theory, we look at the commutant of a discrete group of translation operators. We work in the general context of continuous linear operators from the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ into the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. Since these operators are described by distributional kernels, shift-invariant operators can be treated as a special case of shift-invariant distributions. This strategy leads to a convenient formalism for shift-invariant operators and avoids many technicalities encountered in the literature.

Our study is motivated by the fact that in the theory of shift-invariant systems one encounters operators that are shift-invariant and admit a so-called Walnut series. The original examples are the Walnut representation of Gabor frame operators and a corresponding result for wavelet frames, see Section 4.2. Our main result, Theorem 1.2, shows that indeed an arbitrary shift-invariant operator can be expressed by a Walnut series.

For $x \in \mathbb{R}^n$, let $T_x\varphi(t) = \varphi(t-x)$ denote the translation of a function φ on $\mathbb{R}^n$; the notion extends to distributions. We use the following normalization for the Fourier transform $\mathcal{F}$,

$$\widehat{f}(s) = \int_{\mathbb{R}^n} f(t)e^{-2\pi i st}\,dt, \qquad s \in \mathbb{R}^n .$$

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Recall that the space of tempered distributions $S'(\mathbb{R}^n)$ is endowed with the weak*-topology, that is, $u_k \to u$ in $S'(\mathbb{R}^n)$ if $\langle u_k, \varphi\rangle \to \langle u, \varphi\rangle$ in $\mathbb{C}$, for all $\varphi \in S(\mathbb{R}^n)$. By $\mathcal{L}\big(S(\mathbb{R}^n), S'(\mathbb{R}^n)\big)$ we denote the space of continuous operators from $S(\mathbb{R}^n)$ into $S'(\mathbb{R}^n)$. It is equipped with the usual bounded convergence topology (compact-open topology), that is, $A_k \to A$ in $\mathcal{L}\big(S(\mathbb{R}^n), S'(\mathbb{R}^n)\big)$ if $A_k\varphi \to A\varphi$ in $S'(\mathbb{R}^n)$, for all $\varphi \in S(\mathbb{R}^n)$.

The following well-known result is a structure theorem for translation-invariant operators, i.e., operators that commute with all translations T_x, for $x \in \mathbb{R}^n$.

Theorem 1.1. ([15, 22]). *Let $A \in \mathcal{L}\big(S(\mathbb{R}^n), S'(\mathbb{R}^n)\big)$ satisfy $AT_x = T_x A$, for all $x \in \mathbb{R}^n$.*
(i) *Then A is a Fourier multiplier, i.e., $\widehat{A\varphi} = w \cdot \widehat{\varphi}$, for $\varphi \in S(\mathbb{R}^n)$, where $w \in S'(\mathbb{R}^n)$.*
(ii) *The mapping $A \mapsto w$ is continuous from $\mathcal{L}\big(S(\mathbb{R}^n), S'(\mathbb{R}^n)\big)$ into $S'(\mathbb{R}^n)$.*

Our main result is a structure theorem for shift-invariant operators, i.e., operators which commute only with a discrete subgroup of translations.

Theorem 1.2. *Given $a > 0$, suppose that the operator $A \in \mathcal{L}\big(S(\mathbb{R}^n), S'(\mathbb{R}^n)\big)$ satisfies $AT_{ak} = T_{ak}A$, for all $k \in \mathbb{Z}^n$.*
(i) *Then there exist uniquely defined $w_k \in S'(\mathbb{R}^n)$, for $k \in \mathbb{Z}^n$, such that A can be written as*

$$\widehat{A\varphi} = \sum_{k\in\mathbb{Z}^n} w_k \cdot \big(T_{k/a}\widehat{\varphi}\big), \qquad \varphi \in S(\mathbb{R}^n)\ , \tag{1.1}$$

with unconditional convergence of the series in $S'(\mathbb{R}^n)$.
(ii) *For each $k \in \mathbb{Z}^n$, the mapping $A \mapsto w_k$ is continuous from $\mathcal{L}\big(S(\mathbb{R}^n), S'(\mathbb{R}^n)\big)$ into $S'(\mathbb{R}^n)$.*

The proof is given in Section 3. We mention that Theorem 1.1 can be obtained by using Theorem 1.2 with, say, $a = 1$ and $a = \sqrt{2}$ at the same time.

Remark 1.3.
(i) The distributions $w_k \in S'(\mathbb{R}^n)$, $k \in \mathbb{Z}^n$, are obtained by inspecting the Fourier transform of the distributional kernel of A, see the proof of the theorem below.
(ii) For example, the identity operator $A = \mathrm{id}_{S\to S'}$ is characterized by the "multipliers"

$$w_k = \begin{cases} 1, & \text{for} \quad k = 0\ , \\ 0, & \text{for} \quad k \in \mathbb{Z}^n \setminus \{0\}, \end{cases} \qquad \text{in} \quad S'(\mathbb{R}^n)\ . \tag{1.2}$$

(iii) In operator notation we have $\mathcal{F}A\mathcal{F}^{-1} = \sum_{k\in\mathbb{Z}^n} w_k\, T_{k/a}$.
(iv) Theorem 1.2 says that if A commutes with all shifts contained in the lattice $\Lambda = a\mathbb{Z}^n$, then A can be expressed as a series of weighted shifts with respect to the dual lattice $\Lambda^\perp = \frac{1}{a}\mathbb{Z}^n$. By an obvious modification the result can be formulated for general lattices $\Lambda = L\mathbb{Z}^n$, given by $L \in \mathrm{GL}(n, \mathbb{R})$; note that $\Lambda^\perp = (L^{-1})^T\mathbb{Z}^n$ in this case.
(v) A similar expansion as in Theorem 1.2 (i) was found in [1] in the context of linear systems theory. Our proof is different and simpler by using Poisson's summation formula at a crucial point.

By using Theorem 1.2 we obtain a conceptually simple approach to important known results in the theory of shift-invariant systems, Gabor and wavelet frames.

The article is arranged as follows. Section 2 contains preliminary results on periodic distributions. Section 3 elaborates on these preliminary results and includes the proof of Theorem 1.2.

Finally, in Section 4 we illustrate the main result by examples, that is, the result is applied to shift-invariant systems, Gabor frames, and quasi-affine systems.

2. Preliminary results on invariant distributions

In this section, we describe the structure of distributions that are invariant under certain classes of translations.

In our work, the bracket $\langle u, \varphi \rangle$ denotes the duality between $\mathrm{S}'(\mathbb{R}^n)$ and $\mathrm{S}(\mathbb{R}^n)$, e.g., if u is a bounded function,

$$\langle u, \varphi \rangle = \int_{\mathbb{R}^n} u(x)\varphi(x)\, dx \ . \tag{2.1}$$

While this duality is not strictly preserved by the Fourier transform, we will frequently use the fact that

$$\langle u, \overline{\varphi} \rangle = \left\langle \widehat{u}, \overline{\widehat{\varphi}} \right\rangle , \tag{2.2}$$

where the bar denotes complex conjugation. For functions f, g, we use the tensor product notation $(f \otimes g)(x, y) = f(x)g(y)$. The same notation is also used for distributions, see [23, Section 5.1].

Given an invertible real $n \times n$ matrix M, define the dilation of a function f on $\mathbb{R}^n$ by

$$D_M f(x) = |\det(M)|^{-1/2} f\big(M^{-1}x\big), \qquad x \in \mathbb{R}^n \ .$$

The notion extends to distributions as usual by duality, i.e., $\langle D_M u, \varphi \rangle := \langle u, D_{M^{-1}} \varphi \rangle$, for $u \in \mathrm{S}'(\mathbb{R}^n)$, $\varphi \in \mathrm{S}(\mathbb{R}^n)$.

A usual form of the Poisson summation formula (PSF) states that for $a > 0$ and $f \in \mathrm{S}(\mathbb{R}^n)$, we have

$$\sum_{k \in \mathbb{Z}^n} f(ak) = a^{-n} \sum_{k \in \mathbb{Z}^n} \widehat{f}(k/a) \ , \tag{2.3}$$

with absolute convergence of both series. A more general version of the PSF is formulated for lattices $\Lambda = L\mathbb{Z}^n \subset \mathbb{R}^n$, where L is an invertible real $n \times n$-matrix. If $\Lambda^\perp = (L^{-1})^T \mathbb{Z}^n$ denotes the dual lattice and $|\Lambda| = |\det L|$ the lattice volume, then the PSF for Λ is

$$\sum_{\lambda \in \Lambda} f(\lambda) = |\Lambda|^{-1} \sum_{\lambda \in \Lambda^\perp} \widehat{f}(\lambda) \ .$$

It is less well known that the PSF can, in fact, be formulated for more general subgroups. For an arbitrary closed subgroup $H \subseteq \mathbb{R}^n$ with Haar measure dh the PSF works as follows: Let

$$H^\perp = \left\{ h' \in \mathbb{R}^n \colon \langle h, h' \rangle \in \mathbb{Z}, \ \text{ for all } \ h \in H \right\}$$

be the orthogonal subgroup with Haar measure dh', then

$$\int_H f(h)\, dh = c_H \cdot \int_{H^\perp} \widehat{f}\big(h'\big)\, dh' \ , \tag{2.4}$$

where the constant c_H is obtained from the Haar modulus of H [21, (31.46)]. In the following we are concerned with the subgroups

$$H_1 = \left\{ (ak, 0) \colon k \in \mathbb{Z}^m \right\} \subset \mathbb{R}^n, \qquad m \le n \ , \quad \text{and}$$
$$H_2 = \left\{ (ak, ak) \colon k \in \mathbb{Z}^m \right\} \subset \mathbb{R}^n, \qquad n = 2m \ .$$

The corresponding orthogonal subgroups are

$$H_1^{\perp} = \{(k/a, t) \colon k \in \mathbb{Z}^m,\ t \in \mathbb{R}^{n-m}\} \subset \mathbb{R}^n, \quad \text{and}$$
$$H_2^{\perp} = \{(t, k/a - t) \colon k \in \mathbb{Z}^m,\ t \in \mathbb{R}^m\} \subset \mathbb{R}^n .$$

These are discrete subgroups, but not full-rank lattices, so the orthogonal subgroups are not lattices. The PSF (2.4) for H_1 and H_2 reads as follows.

Lemma 2.1.

(i) *Let* $a > 0$ *and* $m \le n$. *Then for* $f \in \mathrm{S}(\mathbb{R}^n)$, *we have*

$$\sum_{k \in \mathbb{Z}^m} f(ak, 0) = a^{-m} \sum_{k \in \mathbb{Z}^m} \int_{\mathbb{R}^{n-m}} \widehat{f}(k/a, t)\, dt . \tag{2.5}$$

(ii) *Let* $a > 0$ *and* $n = 2m$. *Then for* $f \in \mathrm{S}(\mathbb{R}^n)$, *we have*

$$\sum_{k \in \mathbb{Z}^m} f(ak, ak) = a^{-m} \sum_{k \in \mathbb{Z}^m} \int_{\mathbb{R}^m} \widehat{f}(t, k/a - t)\, dt . \tag{2.6}$$

We note that the statements for H_1 and for H_2 are equivalent by a suitable coordinate transform applied to f. Indeed, let $M = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and note that $H_2 = MH_1$. Then for $f_1 := D_{M^{-1}} f$, we have $f_1(ak, 0) = f(ak, ak)$ and

$$\begin{aligned} \widehat{f_1}(k/a, t) &= \big(D_{M^{-1}} f\big)^{\wedge}(k/a, t) \\ &= \big(D_{M^t} \widehat{f}\big)(k/a, t) = \widehat{f}(t, k/a - t), \qquad k \in \mathbb{Z}^m,\ t \in \mathbb{R}^m , \end{aligned} \tag{2.7}$$

so (i) and (ii) are equivalent.

Generally, the PSF can be seen as an identity of distributions. Given H as above, let μ_H denote the (suitably normalized) Haar measure of H, identified as a tempered distribution on $\mathbb{R}^n$, i.e., $\langle \mu_H, f \rangle = \int_H f(h)\, dh$. Then

$$\widehat{\mu_H} = c_H \cdot \mu_{H^{\perp}} \tag{2.8}$$

and the original form (2.4) of the PSF is equivalent by (2.2). Now based on (2.8) and standard relations of the Fourier transform we have

$$\widehat{\mu_H * f} = \widehat{\mu_H} \cdot \widehat{f} = c_H \cdot \mu_{H^{\perp}} \cdot \widehat{f}, \qquad f \in \mathrm{S}(\mathbb{R}^n)$$

Thus, given $f \in \mathrm{S}(\mathbb{R}^n)$, the periodization $u := \mu_H * f$ along H can be expressed by the formula

$$\begin{aligned} \langle u, \overline{\varphi} \rangle = \left\langle \widehat{\mu_H * f}, \overline{\widehat{\varphi}} \right\rangle &= c_H \left\langle \mu_{H^{\perp}} \cdot \widehat{f}, \overline{\widehat{\varphi}} \right\rangle \\ &= c_H \left\langle \mu_{H^{\perp}}, \widehat{f} \cdot \overline{\widehat{\varphi}} \right\rangle = c_H \int_{H^{\perp}} \widehat{f}\left(h'\right) \overline{\widehat{\varphi}(h')}\, dh' . \end{aligned} \tag{2.9}$$

With $H = H_1, H_2$ as above, (2.9) yields the following identities.

Lemma 2.2.

(i) *Let* $a > 0$ *and* $m \le n$. *For* $f \in \mathrm{S}(\mathbb{R}^n)$, *we define*

$$u(x, y) = \sum_{k \in \mathbb{Z}^m} f(x - ak, y), \qquad x \in \mathbb{R}^m,\ y \in \mathbb{R}^{n-m} .$$

Then we have

$$\langle u, \overline{\varphi} \rangle = \sum_{k\in\mathbb{Z}^m} \int_{\mathbb{R}^m} w_k(t)\, \overline{\widehat{\varphi}(k/a, t)}\, dt\,, \qquad \varphi \in \mathrm{S}(\mathbb{R}^n)\,, \tag{2.10}$$

where $w_k(t) = a^{-m} \widehat{f}(k/a, t)$.

(ii) *Let* $a > 0$ *and* $n = 2m$. *For* $f \in \mathrm{S}(\mathbb{R}^n)$, *we define*

$$u(x, y) = \sum_{k\in\mathbb{Z}^m} f(x - ak, y - ak), \qquad x, y \in \mathbb{R}^m\,.$$

Then we have

$$\langle u, \overline{\varphi} \rangle = \sum_{k\in\mathbb{Z}^m} \int_{\mathbb{R}^m} w_k(t)\, \overline{\widehat{\varphi}(t, k/a - t)}\, dt\,, \qquad \varphi \in \mathrm{S}(\mathbb{R}^n)\,, \tag{2.11}$$

where $w_k(t) = a^{-m} \widehat{f}(t, k/a - t)$.

While this is a result for distributions given explicitly in the form of a periodization along H, our next result holds for arbitrary H-periodic distributions.

Proposition 2.3.
(i) *Let* $u \in \mathrm{S}'(\mathbb{R}^n)$ *satisfy* $u = T_{(ak,0)}u$, *for all* $k \in \mathbb{Z}^m$, *where* $a > 0$ *and* $m \le n$. *Then there exist uniquely defined* $w_k \in \mathrm{S}'(\mathbb{R}^{n-m})$, *for* $k \in \mathbb{Z}^m$, *such that*

$$\langle u, \overline{\varphi} \rangle = \sum_{k\in\mathbb{Z}^m} \left\langle w_k, \overline{R_k\widehat{\varphi}} \right\rangle, \qquad \varphi \in \mathrm{S}(\mathbb{R}^n)\,, \tag{2.12}$$

where $R_k\widehat{\varphi}(t) = \widehat{\varphi}(k/a, t)$, *for* $t \in \mathbb{R}^{n-m}$, *with unconditional convergence of the series. Moreover, the mapping* $u \mapsto w_k$ *is continuous from* $\mathrm{S}'(\mathbb{R}^n)$ *into* $\mathrm{S}'(\mathbb{R}^{n-m})$, *for each* $k \in \mathbb{Z}^m$.

(ii) *Let* $n = 2m$. *Let* $u \in \mathrm{S}'(\mathbb{R}^n)$ *satisfy* $u = T_{(ak,ak)}u$, *for all* $k \in \mathbb{Z}^m$, *where* $a > 0$. *Then there exist uniquely defined* $w_k \in \mathrm{S}'(\mathbb{R}^m)$, *for* $k \in \mathbb{Z}^m$, *such that*

$$\langle u, \overline{\varphi} \rangle = \sum_{k\in\mathbb{Z}^m} \left\langle w_k, \overline{Q_k\widehat{\varphi}} \right\rangle, \qquad \varphi \in \mathrm{S}(\mathbb{R}^n)\,, \tag{2.13}$$

where $Q_k\widehat{\varphi}(t) = \widehat{\varphi}(t, k/a - t)$, *for* $t \in \mathbb{R}^m$, *with unconditional convergence of the series. Moreover, the mapping* $u \mapsto w_k$ *is continuous from* $\mathrm{S}'(\mathbb{R}^n)$ *into* $\mathrm{S}'(\mathbb{R}^m)$, *for each* $k \in \mathbb{Z}^m$.

The proof will provide an explicit construction of the distributions w_k.

Proof.
(i) Let $\psi \in \mathrm{S}(\mathbb{R}^m)$ be such that $\sum_{k\in\mathbb{Z}^m} T_{ak}\psi = 1$ or, equivalently,

$$\widehat{\psi}(k/a) = \begin{cases} 1, & k = 0\,, \\ 0, & k \in \mathbb{Z}^m \setminus \{0\}\,. \end{cases}$$

First, for $k \in \mathbb{Z}^m$, define a distribution $w_k \in \mathrm{S}'(\mathbb{R}^{n-m})$ by

$$\langle w_k, \varphi \rangle = a^{-m} \left\langle \widehat{u}, \left(T_{k/a}\overline{\overline{\widehat{\psi}}}\right) \otimes \varphi \right\rangle, \qquad \varphi \in \mathrm{S}(\mathbb{R}^{n-m})\,. \tag{2.14}$$

Using the definition of w_k, we will verify the series expansion (2.12) for $\widehat{u}$. Given $\varphi \in S(\mathbb{R}^n)$, let

$$\Phi(x, y) = \psi(x) \sum_{k \in \mathbb{Z}^m} \varphi(x - ak, y), \qquad x \in \mathbb{R}^m,\ y \in \mathbb{R}^{n-m} .$$

Then the series defining Φ converges unconditionally in $S(\mathbb{R}^n)$, cf. [23, p. 178]. Using the assumption on ψ and the fact that $u = T_{(ak,0)}u$ we have that

$$\begin{aligned} \langle u, \varphi \rangle &= \left\langle u, \left(\sum_{k \in \mathbb{Z}^m} T_{(ak,0)} \psi \otimes 1 \right) \cdot \varphi \right\rangle \\ &= \sum_{k \in \mathbb{Z}^m} \langle u, (T_{(ak,0)} \psi \otimes 1) \cdot \varphi) \rangle \\ &= \sum_{k \in \mathbb{Z}^m} \left\langle u, T_{(ak,0)} \big[(\psi \otimes 1) \cdot T_{(-ak,0)} \varphi \big] \right\rangle \\ &= \sum_{k \in \mathbb{Z}^m} \langle u, (\psi \otimes 1) \cdot T_{(-ak,0)} \varphi \rangle = \langle u, \Phi \rangle . \end{aligned} \tag{2.15}$$

Next, let $\varphi_y(x) := \varphi(x, y)$ and

$$\Phi_y(x) := \psi(x) \sum_{k \in \mathbb{Z}^m} \varphi_y(x - ak), \qquad x \in \mathbb{R}^m,\ y \in \mathbb{R}^{n-m} .$$

The standard PSF (2.3) implies

$$\sum_{k \in \mathbb{Z}^m} T_{ak} \varphi_y(x) = a^{-m} \sum_{k \in \mathbb{Z}^m} \widehat{\varphi_y}(k/a) e^{2\pi i k x/a}, \qquad x \in \mathbb{R}^m .$$

Hence,

$$\begin{aligned} \widehat{\Phi_y}(s) &= \left(\psi \cdot \sum_{k \in \mathbb{Z}^m} T_{ak} \varphi_y \right)^{\wedge} (s) \\ &= a^{-m} \left(\psi \cdot \sum_{k \in \mathbb{Z}^m} \widehat{\varphi_y}(k/a) e^{2\pi i k \cdot /a} \right)^{\wedge} (s) \\ &= a^{-m} \sum_{k \in \mathbb{Z}^m} \widehat{\psi}(s - k/a)\, \widehat{\varphi_y}(k/a), \qquad s \in \mathbb{R}^m,\ y \in \mathbb{R}^{n-m} , \end{aligned}$$

and thus we obtain

$$\begin{aligned} \widehat{\Phi}(s, t) &= a^{-m} \sum_{k \in \mathbb{Z}^m} \widehat{\psi}(s - k/a)\, \widehat{\varphi}(k/a, t) \\ &= a^{-m} \sum_{k \in \mathbb{Z}^m} \big(T_{k/a} \widehat{\psi} \otimes R_k \widehat{\varphi} \big)(s, t), \qquad s \in \mathbb{R}^m,\ t \in \mathbb{R}^{n-m} . \end{aligned} \tag{2.16}$$

Since for fixed $t \in \mathbb{R}^{n-m}$, the function $\widehat{\varphi}(k/a, t)$ decays rapidly in k and the Schwartz seminorms of $T_{k/a}\widehat{\psi}$ grow only polynomially in k, it is easily verified that the series in (2.16) converges unconditionally in $S(\mathbb{R}^n)$; see [17, Lemma 11.2.1 or Proposition 11.2.4] for a similar argument.

Now using (2.16) and (2.14), we calculate

$$\begin{aligned}\left\langle \widehat{u}, \overline{\widehat{\Phi}} \right\rangle &= a^{-m} \left\langle \widehat{u}, \sum_{k\in\mathbb{Z}^m} \overline{T_{k/a}\widehat{\psi} \otimes R_k\widehat{\varphi}} \right\rangle \\ &= a^{-m} \sum_{k\in\mathbb{Z}^m} \left\langle \widehat{u}, \overline{T_{k/a}\widehat{\psi} \otimes R_k\widehat{\varphi}} \right\rangle \\ &= \sum_{k\in\mathbb{Z}^m} \left\langle w_k, \overline{R_k\widehat{\varphi}} \right\rangle .\end{aligned} \tag{2.17}$$

Finally, combining (2.15) and (2.17), we obtain

$$\langle u, \overline{\varphi}\rangle = \left\langle u, \overline{\Phi} \right\rangle = \left\langle \widehat{u}, \overline{\widehat{\Phi}} \right\rangle = \sum_{k\in\mathbb{Z}^m} \left\langle w_k, \overline{R_k\widehat{\varphi}} \right\rangle .$$

To show the uniqueness of the distributions w_k, for $k \in \mathbb{Z}^m$, assume that $\sum_{k\in\mathbb{Z}^n} \delta_{k/a} \otimes w_k = \sum_{k\in\mathbb{Z}^n} \delta_{k/a} \otimes w'_k$ for some other sequence of $w'_k \in S'(\mathbb{R}^{n-m})$. This means that for all $\varphi_1 \in S'(\mathbb{R}^m)$ and all $\varphi_2 \in S'(\mathbb{R}^{n-m})$ we have

$$\left\langle \sum_{k\in\mathbb{Z}^n} \delta_{k/a} \otimes w_k, \varphi_1 \otimes \varphi_2 \right\rangle = \sum_{k\in\mathbb{Z}^n} \varphi_1(k/a)\, \langle w_k, \varphi_2\rangle = \sum_{k\in\mathbb{Z}^n} \varphi_1(k/a) \left\langle w'_k, \varphi_2 \right\rangle .$$

If for each $k \in \mathbb{Z}^n$ we choose some $\varphi_1 \in S(\mathbb{R}^m)$ such that $\varphi_1(l/a) = \delta_{k,l}$, then we obtain $\langle w_k, \varphi_2\rangle = \langle w'_k, \varphi_2\rangle$, for all $\varphi_2 \in S(\mathbb{R}^{n-m})$, and thus $w_k = w'_k$ for $k \in \mathbb{Z}^n$.

The uniqueness also implies that the definition of the w_k is independent of the auxiliary function ψ in (2.14).

The continuity of the mapping $u \mapsto w_k$ from $S'(\mathbb{R}^n)$ into $S'(\mathbb{R}^{n-m})$, for each $k \in \mathbb{Z}^m$, follows immediately from (2.14).

(ii) Recall that $H_2 = MH_1$ for $M = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Define $u_1 := D_{M^{-1}}u \in S'(\mathbb{R}^n)$. Since $T_x D_{M^{-1}} = D_{M^{-1}} T_{Mx}$ and u is H_2-invariant, u_1 is H_1-invariant. Hence, (ii) follows from (i) by setting $Q_k = R_k D_{M^t}$. Then $Q_k\varphi(t) = \varphi(t, k/a - t)$ as in (2.7) and

$$\begin{aligned}\langle u, \overline{\varphi}\rangle = \langle D_M u_1, \overline{\varphi}\rangle &= \left\langle u_1, \overline{D_{M^{-1}}\varphi} \right\rangle \\ &= \sum_{k\in\mathbb{Z}^m} \left\langle w_k, \overline{R_k \left(D_{M^{-1}}\varphi\right)^\wedge} \right\rangle \\ &= \sum_{k\in\mathbb{Z}^m} \left\langle w_k, \overline{R_k D_{M^t}\widehat{\varphi}} \right\rangle = \sum_{k\in\mathbb{Z}^m} \left\langle w_k, \overline{Q_k\widehat{\varphi}} \right\rangle, \qquad \varphi \in S(\mathbb{R}^n) . \quad \square\end{aligned}$$

Note that Lemma 2.2 (i), (ii) is a special case of Proposition 2.3 (i), (ii) where the distributions w_k are indeed Schwartz functions and they are given explicitly. In fact, the continuous dependence of the w_k on u allows us to determine the w_k explicitly also in certain other cases, such as discussed next.

First, the conclusions of Lemma 2.2 also hold under the more general assumption that $f \in L^1(\mathbb{R}^n)$ (instead of $f \in S(\mathbb{R}^n)$). Secondly, in Section 4 we will need the expansion (2.11) for the case when f is a tensor product, as follows.

Lemma 2.4. *Let $n = 2m$ and $f = g \otimes h \in L^2(\mathbb{R}^n)$, for given $g, h \in L^2(\mathbb{R}^m)$. Let*

$$u = \sum_{k\in\mathbb{Z}^m} T_{(ak,ak)}(g \otimes h) = \sum_{k\in\mathbb{Z}^m} T_{ak} g \otimes T_{ak} h .$$

Then $u \in \mathrm{S}'(\mathbb{R}^n)$ *and*

$$\langle u, \overline{\varphi}\rangle = \sum_{k\in\mathbb{Z}^m} \int_{\mathbb{R}^m} w_k(t)\, \overline{\widehat{\varphi}(t, k/a - t)}\, dt, \qquad \varphi \in \mathrm{S}(\mathbb{R}^n)\ , \tag{2.18}$$

with $w_k(t) = a^{-m}\, \widehat{g}(t)\, \widehat{h}(k/a - t) \in L^1(\mathbb{R}^m)$.

Proof. First, let $Q = [0, a]^n$. Using the Cauchy-Schwarz inequality we have for $r, s \in \mathbb{Z}^m$,

$$\begin{aligned}
\iint_{(ar,as)+Q} |u(x, y)|^2\, dx\, dy &= \iint_{(ar,as)+Q} \Big|\sum_{k\in\mathbb{Z}^m} g(x - ak)\, h(y - ak)\Big|^2 dx\, dy \\
&= \iint_{Q} \Big|\sum_{k\in\mathbb{Z}^m} g(x - ak)\, h(y - ak)\Big|^2 dx\, dy \\
&\le \int_{[0,a]^m} \sum_{k\in\mathbb{Z}^m} |g(x - ak)|^2 dx \int_{[0,a]^m} \sum_{k\in\mathbb{Z}^m} |h(y - ak)|^2 dy \\
&= \|g\|_{L^2}^2\, \|h\|_{L^2}^2\ .
\end{aligned} \tag{2.19}$$

Thus, u is square integrable on any cube $Q_{r,s} = (ar, as) + Q$ and the local norm is bounded independently of the position $(r, s) \in \mathbb{Z}^n$. Therefore u is a tempered distribution and the mapping $(g, h) \mapsto u$ is continuous from $L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m)$ into $\mathrm{S}'(\mathbb{R}^n)$.

Next Proposition 2.3 asserts that

$$\langle u, \overline{\varphi}\rangle = \sum_{k\in\mathbb{Z}^m} \big\langle w_k, \overline{Q_k\widehat{\varphi}}\big\rangle, \qquad \varphi \in \mathrm{S}(\mathbb{R}^n)\ ,$$

for some $w_k \in \mathrm{S}'(\mathbb{R}^{n-m})$. Using the density of $\mathrm{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, we choose two sequences $g_r, h_r \in \mathrm{S}(\mathbb{R}^m)$ such that $g_r \otimes h_r$ converges to $f = g \otimes h$ in $L^2(\mathbb{R}^n)$. Set

$$u_r = \sum_{k\in\mathbb{Z}^m} T_{(ak,ak)}(g_r \otimes h_r)\ .$$

Then by Lemma 2.2 (ii),

$$\langle u_r, \overline{\varphi}\rangle = \sum_{k\in\mathbb{Z}^m} \big\langle w_k^{(r)}, \overline{Q_k\widehat{\varphi}}\big\rangle, \qquad \varphi \in \mathrm{S}(\mathbb{R}^n)\ ,$$

with the explicit formula

$$w_k^{(r)}(t) = a^{-m}\, \widehat{g_r}(t)\, \widehat{h_r}(k/a - t) \in L^1(\mathbb{R}^m)\ .$$

Since $u_r \to u$ as a consequence of (2.19), Proposition 2.3 implies that $w_k^{(r)} \to w_k$ in $\mathrm{S}'(\mathbb{R}^m)$, for all $k \in \mathbb{Z}^m$. On the other hand,

$$w_k^{(r)}(t) \to a^{-m}\widehat{g}(t)\, \widehat{h}(k/a - t) \quad \text{in} \quad L^1(\mathbb{R}^m)\ ,$$

and so $w_k(t) = a^{-m}\widehat{g}(t)\, \widehat{h}(k/a - t)$, as claimed. □

3. Proof of Theorem 1.2

The Schwartz kernel theorem states that there exists a topological isomorphism between the operators $A \in \mathcal{L}\big(\mathrm{S}(\mathbb{R}^n), \mathrm{S}'(\mathbb{R}^n)\big)$ and the distributions $\kappa \in \mathrm{S}'(\mathbb{R}^{2n})$. The distribution κ is called the kernel of A and the correspondence is given by $\langle A\varphi, \psi\rangle = \langle \kappa, \psi \otimes \varphi\rangle$, for $\varphi, \psi \in \mathrm{S}(\mathbb{R}^n)$. The kernel theorem together with Proposition 2.3 (ii) now allows us to prove the main theorem.

Proof of Theorem 1.2. Let $\kappa \in \mathrm{S}'(\mathbb{R}^{2n})$ denote the Schwartz kernel of the operator A, that is, $\langle A\varphi, \psi\rangle = \langle \kappa, \psi \otimes \varphi\rangle$, for $\varphi, \psi \in \mathrm{S}(\mathbb{R}^n)$. For $k \in \mathbb{Z}^n$, we note that

$$\begin{aligned}\langle T_{ak} A T_{-ak}\varphi, \psi\rangle &= \langle A T_{-ak}\varphi, T_{-ak}\psi\rangle \\ &= \langle \kappa, T_{-ak}\psi \otimes T_{-ak}\varphi\rangle \\ &= \langle \kappa, T_{(-ak,-ak)}(\psi \otimes \varphi)\rangle \\ &= \langle T_{(ak,ak)}\kappa, \psi \otimes \varphi\rangle, \qquad \varphi, \psi \in \mathrm{S}(\mathbb{R}^n)\ .\end{aligned}$$

Therefore, since $A = T_{ak} A T_{-ak}$ and the Schwartz kernel $\kappa \in \mathrm{S}'(\mathbb{R}^{2n})$ of an operator is unique, we have that $\kappa = T_{(ak,ak)}\kappa$ for $k \in \mathbb{Z}^n$. We now apply Proposition 2.3 (ii) to the invariant distribution κ. Since $\overline{\overline{\widehat{\varphi}}}(x) = \widehat{\varphi}(-x)$, we conclude that

$$\begin{aligned}\big\langle \widehat{A\varphi}, \overline{\widehat{\psi}}\,\big\rangle = \big\langle A\varphi, \overline{\psi}\,\big\rangle = \big\langle \kappa, \overline{\psi \otimes \overline{\varphi}}\,\big\rangle &= \sum_{k\in\mathbb{Z}^m} \big\langle w_k, Q_k\big(\overline{\widehat{\psi} \otimes \overline{\widehat{\varphi}}}\big)\big\rangle \\ &= \sum_{k\in\mathbb{Z}^m} \big\langle w_k, \overline{\widehat{\psi}} \cdot T_{k/a}\widehat{\varphi}\big\rangle \\ &= \sum_{k\in\mathbb{Z}^m} \big\langle w_k \cdot T_{k/a}\widehat{\varphi}, \overline{\widehat{\psi}}\big\rangle, \qquad \varphi, \psi \in \mathrm{S}(\mathbb{R}^n)\ .\end{aligned}$$

The continuity statement also follows from Proposition 2.3 (ii). □

4. Applications to shift-invariant systems

We next show how the structure theorem for operators that commute with a discrete group of translations can be used in the analysis of shift-invariant systems. Note that in the following arguments, we abide by the bilinear definition of the bracket $\langle\cdot,\cdot\rangle$, even though shift-invariant systems are commonly studied in the L^2-context.

4.1. Shift-invariant systems

Let $a > 0$ and $g_j \in L^2(\mathbb{R}^n)$, for $j \in I$, where I denotes a finite or countable index set. A family of the form

$$G = \big\{T_{ak}g_j : k \in \mathbb{Z}^n,\ j \in I\big\}\ ,$$

is called a shift-invariant system. By construction, the set G is invariant under all translations T_{ak} for $k \in \mathbb{Z}^n$. The closed span of G in some L^p is a shift-invariant space. For shift-invariant systems, see [2, 4, 18, 30]. On a theoretical level, the main objectives are to understand the spanning and stability properties of G. These are encoded in the spectrum of the frame operator associated to G. More generally, given the shift-invariant systems G and $H = \{T_{ak}h_j : k \in \mathbb{Z}^n,\ j \in I\}$, we define the frame type operator $S = S_{G,H}$ by

$$Sf = \sum_{j\in I}\sum_{k\in\mathbb{Z}^n} \langle f, T_{ak}\overline{g_j}\rangle T_{ak}h_j, \qquad f \in \mathrm{S}(\mathbb{R}^n)\ . \tag{4.1}$$

Remark 4.1. If $\varphi \in \mathrm{S}(\mathbb{R}^n)$, it is easy to see that the sequence $c_k = \langle \varphi, T_{ak}\overline{g}\rangle$ belongs to $\ell^2(\mathbb{Z}^n)$. If I is finite, then $\langle S\varphi, \psi\rangle = \sum_{k\in\mathbb{Z}^n}\sum_{j\in I}\langle \varphi, T_{ak}\overline{g_j}\rangle\langle \psi, T_{ak}h_j\rangle$ converges absolutely for $\varphi, \psi \in \mathrm{S}(\mathbb{R}^n)$ and, as a consequence, S is continuous from $\mathrm{S}(\mathbb{R}^n)$ into $\mathrm{S}'(\mathbb{R}^n)$. Hence, unconditional convergence of (4.1) is equivalent to the unconditional convergence of the outer sum.

The following structure theorem for S is an immediate consequence of Theorem 1.2. The result provides a simple approach to important results about so-called reproducing systems. In particular, we can easily characterize when the pair (G, H) is a reproducing system, that is, when $S = \mathrm{id}_{\mathrm{S}\to\mathrm{S}'}$.

Corollary 4.2. *Suppose that the frame type operator S, given in* (4.1)*, is continuous from* $\mathrm{S}(\mathbb{R}^n)$ *into* $\mathrm{S}'(\mathbb{R}^n)$*, and the defining series over I converges w^*-unconditionally. Then S is of the form*

$$\widehat{S\varphi} = \sum_{k\in\mathbb{Z}^n} w_k \cdot T_{k/a}\widehat{\varphi}, \qquad \varphi \in \mathrm{S}(\mathbb{R}^n)\ , \tag{4.2}$$

where

$$w_k = a^{-n}\sum_{j\in I} T_{k/a}\widehat{\overline{g_j}} \cdot \widehat{h}_j, \qquad \textit{for} \quad k \in \mathbb{Z}^n\ , \tag{4.3}$$

with unconditional convergence of the series (4.2) *in* $\mathrm{S}'(\mathbb{R}^n)$*.*

Proof. Given $k \in \mathbb{Z}^n$ and $j \in I$, the rank-one operator $S_{k,j}f = \langle f, T_{ak}g_j\rangle T_{ak}h_j$ has the kernel

$$T_{ak}h_j \otimes T_{ak}\overline{g_j} = T_{(ak,ak)}(h_j \otimes \overline{g_j})\ .$$

Hence, S has the kernel

$$\kappa = \sum_{j\in I}\sum_{k\in\mathbb{Z}^n} T_{(ak,ak)}(h_j \otimes \overline{g_j})\ .$$

Thus, the result follows from Theorem 1.2, and the explicit form of w_k is obtained in Example 2.4 (ii), observing that $\widehat{\overline{g_j}}(x) = \widehat{g}_j(-x)$. □

Remark 4.3. Recall that G is called a Bessel sequence if $S_{G,G}$ is bounded on $L^2(\mathbb{R}^n)$. We note that if $S_{G,H}$ is bounded on $L^2(\mathbb{R}^n)$, then G, H need not be Bessel sequences, cf. [14, p. 143 and p. 150, Remark (c)]. However, if G is a Bessel sequence, then the following equivalences hold.

(i) G is a frame and H is a dual frame $\Leftrightarrow$ $S_{G,H} = \mathrm{id}$.

(ii) G is a normalized tight frame $\Leftrightarrow$ $S_{G,G} = \mathrm{id}$.

(iii) G is an orthonormal basis $\Leftrightarrow$ the elements of G have norm one and G is a normalized tight frame.

We note that the same argument as in Remark 4.1 shows for Bessel sequences G, H that $S_{G,H}$ is continuous from $\mathrm{S}(\mathbb{R}^n)$ to $\mathrm{S}'(\mathbb{R}^n)$, and that the sum over I in (4.1) converges unconditionally.

The representation (4.2) was obtained in [30, 33, 27] under the additional hypothesis that S is bounded on $L^2(\mathbb{R}^n)$. Using the distributional version Theorem 1.2 makes the derivation of these important results conceptually simple and in addition avoids the technicalities connected with the L^2-boundedness. For a unified treatment of reproducing systems, see [19, 28, 30].

4.2. Gabor systems

Here we apply Corollary 4.2 to Gabor systems. Define the modulation operator

$$M_s f(t) = e^{2\pi i st} f(t), \qquad s, t \in \mathbb{R}^n .$$

Given $a, b > 0$ and $g, h \in L^2(\mathbb{R}^n)$, the families

$$G = \{T_{ak} M_{bl} g \colon k, l \in \mathbb{Z}^n\}$$

and $H = \{T_{ak} M_{bl} h\}$ are called Gabor systems, see [12, 13, 17, 24, 33]. The corresponding Gabor frame type operator

$$Sf = \sum_{k,l\in\mathbb{Z}^n} \langle f, T_{ak} M_{bl} \overline{g}\rangle T_{ak} M_{bl} h, \qquad f \in \mathrm{S}(\mathbb{R}^n) , \tag{4.4}$$

commutes with the translations T_{ak}, for $k \in \mathbb{Z}^n$, and it follows from [14, Corollary 3.3.3 (iii)] that S is continuous from $\mathrm{S}(\mathbb{R}^n)$ into $\mathrm{S}'(\mathbb{R}^n)$.

Example 4.4.
(i) According to Theorem 1.2 the Gabor frame type operator S in (4.4) can be represented in the form (4.2) with

$$w_k = a^{-n} \sum_{l\in\mathbb{Z}^n} T_{bl}\big(T_{k/a} \widehat{\overline{g}} \cdot \widehat{h}\big), \qquad \text{for} \quad k \in \mathbb{Z}^n . \tag{4.5}$$

This is the so-called Walnut representation and was first obtained under more restrictive conditions in [34].
(ii) As a consequence of (1.2), we have $S = \mathrm{id}_{\mathrm{S}\to\mathrm{S}'}$ if and only if $w_0 = 1$ and $w_k = 0$ for $k \neq 0$. In the case of Gabor frames this is equivalent to the Wexler-Raz conditions

$$\langle h, T_{l/b} M_{k/a} \overline{g}\rangle = \begin{cases} a^n b^n, & (k, l) = (0, 0) , \\ 0, & (k, l) \in \mathbb{Z}^2 \setminus \{(0, 0)\} . \end{cases} \tag{4.6}$$

Proof. For $g, h \in L^2(\mathbb{R}^n)$ the series in (4.5) converges unconditionally to a b-periodic function $w_k \in L^1([0, b]^n)$. The equivalence of (1.2) and (4.6) follows by calculating the Fourier coefficients of w_k,

$$\begin{aligned} \delta_{k,0}\delta_{m,0} &= \widehat{w_k}(m) = b^{-n} \int_{[0,b]^d} w_k(x) e^{-2\pi i m\cdot x/b}\, dx \\ &= (ab)^{-n} \int_{[0,b]^d} \sum_{l\in\mathbb{Z}^n} T_{bl}\big(T_{k/a}\widehat{\overline{g}} \cdot \widehat{h}\big)(x)\, e^{-2\pi i m\cdot x/b}\, dx \\ &= (ab)^{-n} \int_{\mathbb{R}^n} T_{k/a}\widehat{\overline{g}}(x) \widehat{h}(x)\, e^{-2\pi i m\cdot x/b}\, dx \\ &= (ab)^{-n} \big\langle \widehat{h}, M_{-m/b} T_{k/a} \widehat{\overline{g}}\big\rangle \\ &= (ab)^{-n} \big\langle h, T_{m/b} M_{k/a} \overline{g}\big\rangle . \end{aligned}$$

□

Remark 4.5.

(i) The Walnut representation of the Gabor frame operator is usually formulated without using the Fourier transform. We note that both descriptions are equivalent, since a Gabor frame operator conjugated with the Fourier transform is again a Gabor frame operator.
(ii) The ordering $T_{ak}M_{bl}$ as compared to $M_{bl}T_{ak}$ often used in the literature ensures that the Gabor system is shift-invariant. The commutation relation $T_{ak}M_{bl} = e^{-2\pi i abl\cdot k} M_{bl}T_{ak}$ shows that spanning, Bessel-sequence, or orthonormal basis properties of Gabor systems are preserved if we exchange the ordering.

By Example 4.4 we have obtained general versions of important results in Gabor analysis. Indeed, suppose that G and H are Bessel sequences. Then (4.2) with w_k given by (4.5) is the Walnut representation, found in [34], see [17, Sections 6.3 and 7.1], or [5, 27]. Secondly, the condition (1.2) with w_k given by (4.5) is a characterization of dual windows [6, Exercise 8.9]. Third, (4.6) are the Wexler-Raz conditions, found in [36], see [17, Section 7.3], [10, 26].

An alternative approach that also works for non-product time-frequency lattices was developed in [11, 14].

4.3. Affine systems

Finally, we apply Corollary 4.2 to affine systems. In dimension $n = 1$ the dilation by a is $D_a f(t) = |a|^{-1/2} f(a^{-1}t)$, for $t \in \mathbb{R}$. Given $a, b > 0$ and $g, h \in L^2(\mathbb{R})$, the families

$$G = \left\{ D_{a^j} T_{bk} g : k \in \mathbb{Z},\ j \in \mathbb{Z} \right\}$$

and $H = \{D_{a^j} T_{bk} h\}$ are called affine systems. Such systems are investigated in the theory of wavelet frames and bases, see, e.g., [7, 9, 25, 29]. In the following we assume that a is an integer > 1. Although, G and H are not shift-invariant, it is possible to associate a shift-invariant system $\widetilde{G}$ with identical spanning and stability properties, namely the *quasi-affine system* defined by

$$\begin{aligned} \widetilde{G} &= \left\{ D_{a^j} T_{bk} g : k \in \mathbb{Z},\ j \le 0 \right\} \cup \left\{ a^{-j/2} T_{bk} D_{a^j} g : k \in \mathbb{Z},\ j > 0 \right\} \\ &= \left\{ T_{bk} g_p : k \in \mathbb{Z},\ p \in I \right\}, \end{aligned} \tag{4.7}$$

where

$$I = \left\{ (j, l) : j, l \in \mathbb{Z},\ 0 \le l < a^{-j} \right\} \tag{4.8}$$

and

$$g_p = \min\left(1, a^{-j/2}\right) D_{a^j} T_{bl} g, \qquad p = (j, l) \in I\ . \tag{4.9}$$

The original source for the equivalence of affine and quasi-affine systems is [32] and it is extended in [3, 8, 16, 20, 31]. By $\widetilde{H}$ we denote the quasi-affine system corresponding to H. The quasi-affine frame type operator $S = S_{\widetilde{G},\widetilde{H}}$ is given by

$$Sf = \sum_{p \in I} \sum_{k \in \mathbb{Z}} \langle f, T_{bk} g_p \rangle T_{bk} h_p\ , \qquad f \in \mathrm{S}(\mathbb{R}). \tag{4.10}$$

We note that if G and H are Bessel systems, then so are $\widetilde{G}$, $\widetilde{H}$; in particular, under this assumption S is continuous from $\mathrm{S}(\mathbb{R})$ into $\mathrm{S}'(\mathbb{R})$, and the sum over I converges w^*-unconditionally.

Example 4.6. Assume that the quasi-affine frame type operator S, given in (4.10), is continuous from $\mathrm{S}(\mathbb{R})$ into $\mathrm{S}'(\mathbb{R})$, and that the sum over I converges w^*-unconditionally.
(i) Then S is of the form (4.2) with

$$\begin{aligned} w_k &= b^{-1} \sum_{j \geq 0} a^{-j} \big(T_{k/b} D_{a^{-j}} \overline{\widehat{g}}\big) \cdot \big(D_{a^{-j}} \widehat{h}\big) \\ &\quad + b^{-1} \sum_{j<0} \sum_{l=0}^{a^{-j}-1} \big(T_{k/b} D_{a^{-j}} \overline{M_{-bl} \widehat{g}}\big) \cdot \big(D_{a^{-j}} M_{-bl} \widehat{h}\big) \end{aligned} \tag{4.11}$$

and w^*-convergence in $\mathrm{S}'(\mathbb{R}^n)$.
(ii) Conditions (1.2) for $S = \mathrm{id}_{\mathrm{S}\to\mathrm{S}'}$ are equivalent to the Calderón condition for dual wavelet frames:

$$\sum_{j \in \mathbb{Z}} \overline{\widehat{g}}\big(a^j t\big)\, \widehat{h}\big(a^j t\big) = b \quad \text{a.e.}\,, \tag{4.12}$$

$$\sum_{j \geq 0} \overline{\widehat{g}}\big(a^j (t - m/b)\big)\, \widehat{h}\big(a^j t\big) = 0 \quad \text{a.e.}\,, \qquad \text{for all} \quad m \in \mathbb{Z} \setminus a\mathbb{Z}\,. \tag{4.13}$$

Compared to other approaches, the derivation of these conditions based on Theorem 1.2 is fairly easy.

Proof. Again the result follows from Corollary 4.2, where T_{ak} reads T_{bk}, for $k \in \mathbb{Z}$, and $\{g_j\}_{j \in I}$ takes the special form $\{g_p\}_{p \in I}$, described in (4.8), (4.9). The identity (4.11) is obtained by substituting the quasi-affine system (4.7) into the definition of w_k in (4.3). To verify the Calderón conditions, let

$$w_k^+ = \frac{1}{a^j b} \sum_{j \geq 0} \big(T_{k/b} D_{a^{-j}} \overline{\widehat{g}}\big) \cdot D_{a^{-j}} \widehat{h}$$

and $w_k^- = w_k - w_k^+$. Then

$$\begin{aligned} w_k^+(t) &= \frac{1}{a^j b} \sum_{j \geq 0} \Big(a^{j/2} \overline{\widehat{g}}\big(a^j (t - k/b)\big)\Big) \Big(a^{j/2} \widehat{h}\big(a^j t\big)\Big) \\ &= \frac{1}{b} \sum_{j \geq 0} \overline{\widehat{g}}\big(a^j (t - k/b)\big)\, \widehat{h}\big(a^j t\big)\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} w_k^-(t) &= \frac{a^j}{b} \sum_{j<0} \sum_{l=0}^{a^{-j}-1} \Big(e^{2\pi i\, b\, l\, a^j\, (t-k/b)}\, \overline{\widehat{g}}\big(a^j (t - k/b)\big)\Big)\Big(e^{-2\pi i\, b\, l\, a^j\, t} \widehat{h}\big(a^j t\big)\Big) \\ &= \frac{1}{b} \sum_{j<0} \left(a^j \sum_{l=0}^{a^{-j}-1} e^{-2\pi i\, l\, a^j\, k}\right) \overline{\widehat{g}}\big(a^j (t - k/b)\big)\, \widehat{h}\big(a^j t\big) \\ &= \begin{cases} \dfrac{1}{b} \displaystyle\sum_{j<0} \overline{\widehat{g}}\big(a^j (t - k/b)\big)\, \widehat{h}\big(a^j t\big), & \text{if} \quad k \in a^{|j|}\mathbb{Z}\,, \\ 0, & \text{otherwise}\,, \end{cases} \qquad k \in \mathbb{Z},\ t \in \mathbb{R}\,. \end{aligned}$$

In particular, the condition $w_0 = 1$ is seen to be equivalent to (4.12). Moreover, we find that $w_m^- = 0$ whenever $m \in \mathbb{Z} \setminus a\mathbb{Z}$. Hence, $w_m = 0$ for all $m \in \mathbb{Z} \setminus a\mathbb{Z}$ is equivalent to (4.13). It thus remains to show that these relations imply $w_k = 0$ for all nonzero k. For this purpose let $k \in \mathbb{Z} \setminus 0$, then $k = a^{-j_0} m$ for some $j_0 \leq 0$ and $m \in \mathbb{Z} \setminus a\mathbb{Z}$. We then compute

$$\begin{aligned} w_k(t) &= w_k^+(t) + \frac{1}{b} \sum_{j=j_0}^{-1} \overline{\widehat{g}}\big(a^j(t-k/b)\big)\,\widehat{h}\big(a^j t\big) \\ &= \frac{1}{b} \sum_{l=0}^{\infty} \overline{\widehat{g}}\big(a^l(s-m/b)\big)\,\widehat{h}\big(a^l s\big) \ , \end{aligned}$$

where we used the substitutions $l = j - j_0$ and $s = a^{j_0} t$. Hence,

$$w_k(t) = w_m\big(a^{j_0} t\big) \ ,$$

and $w_m = 0$ implies $w_k = 0$. □

Thus, we have obtained a general version of an important result in wavelet theory, the Calderón condition for wavelet frames and bases.

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